Kinetics of weakly turbulent wave fields

V. I. Erofeev and V. M. Malkin

Nuclear Physics Institute, Siberian Division, USSR Academy of Sciences
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Only the first nonvanishing term in the expansion of the collision integral in powers of the wave energy is usually considered in the kinetic equation for waves. Yet the succeeding-approximation terms can become leading even at energies for which the turbulence remains weak. Although uniformly small, these terms can be essential if they violate any of the first-approximation conservation laws. Attempts to calculate the higher nonlinear terms of the kinetic wave equations have led to divergences. No effective method of eliminating these divergences has been devised, despite the abundance of general renormalization schemes proposed in the literature. The problem is solved by using the Wyld diagram technique. The structure of the collision integral is elucidated for any order of the expansion in powers of the wave energy, and the conditions for the validity of the weak-turbulence theory are made more precise. Moreover, the cubic collisional term, which is of greatest practical interest, is calculated completely for waves with a decay dispersion law.

1. INTRODUCTION

Weak-turbulence theory is based (see, e.g., Refs. 1 and 2) on the concept of long-lived excitations of a dispersive medium—quasiparticles. The time derivative of the quasiparticle distribution function at a particular instant is expressed in terms of the values of this function at that instant with the aid of a function called the collision integral. This integral is calculated by expansion in terms of one or several parameters proportional to the turbulence energy. It is assumed that each term of the expansion corresponds to a certain interaction channel and can be interpreted in the language of the processes induced in this channel. If a homogeneous medium contains only one kind of quasiparticles that obey a dispersive decay law, the first nonvanishing term of the expansion of the collision integral is generated by a three-wave interaction and is given by[1)]

$$\mathrm{St}_{\mathbf{k}_1}^{(2)}=\int d^3\mathbf{k}_2\, d^3\mathbf{k}_3\, (-U_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}+U_{\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_3}+U_{\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_1}), \qquad (1.1)$$

where

$$U_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}=w_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\delta^3(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3})\times n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3}(-1/n_{\mathbf{k}_1}+1/n_{\mathbf{k}_2}+1/n_{\mathbf{k}_3}).$$

Here $n_{\mathbf{k}}$ is the quasiparticle distribution function in the space of the wave vectors $\mathbf{k}$, $U_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ is the contribution made to the collision integral by the decay of the wave $(\mathbf{k}_1,\omega_{\mathbf{k}_1})$ into waves $(\mathbf{k}_2,\omega_{\mathbf{k}_2})$ and $(\mathbf{k}_3,\omega_{\mathbf{k}_3})$, and by the inverse process. The delta functions contained in the equation for $U_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ ensure conservation of the quasiparticle momentum and energy. If the relations $\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3$ and $\omega_{\mathbf{k}_1}=\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}$ are incompatible, $\omega_{\mathbf{k}}$ obeys a so-called nondecay law, and the contribution of lowest order in the turbulence energy is made to the collision integral by four-wave interaction, particularly by scattering of two waves into two. The collision-integral expansion term corresponding to this process is given by

$$\mathrm{St}_{\mathbf{k}_1}^{(3)}=\int d^3\mathbf{k}_2\, d^3\mathbf{k}_3\, d^3\mathbf{k}_4\, \omega_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\times\delta(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_4})\times n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3}n_{\mathbf{k}_4}\left(\frac{1}{n_{\mathbf{k}_1}}+\frac{1}{n_{\mathbf{k}_2}}-\frac{1}{n_{\mathbf{k}_3}}-\frac{1}{n_{\mathbf{k}_4}}\right). \qquad (1.2)$$

Equations (1.1) and (1.2) can be obtained by classical perturbation theory supplemented by some hypotheses concerning the expression of higher correlation functions in terms of paired ones ("phase ergodicity"). Attempts at calculating or at least estimating by this method the higher terms of the expansion of the collision integral lead to divergences (see, e.g., Ref. 3). The necessary renormalizations have not yet been obtained, although the feasibility in principle "of constructing a more accurate theory for the analytic description of the wave fields, based for example on Wyld's diagram technique," was mentioned already in Ref. 3 and was even implemented in two cases of importance in the theory of Langmuir turbulence.[4,5] This situation can be attributed in part to "insufficient contact between the principal work on renormalization and turbulence in plasma physics, on one hand, and methods of other disciplines, particularly fluid dynamics, on the other" (Ref. 6). In addition, the discrepancy was sustained by the widely held opinion that processes of lowest order in wave energy play the dominant role in the entire region where weak-turbulence theory is applicable. This opinion is justified for problems containing only one small parameter. If, however, several parameters are present the first term of the collision-integral expansion can contain an additional smallness, and is exceeded by the succeeding terms even in the weak-turbulence region. (An illustrative case of this kind was investigated in Ref. 5 by using the Wyld diagram technique). An assessment of the role of nonlinear processes must be approached with caution also under conditions when they are weak but violate some of the conservation laws that hold in the first-order approximation, such as the conservation of the quasiparticle number in the process (1.2). Nonconservative small additions to the collision integral can come into play if the evolution time of the wave fields is long enough.

Our main purpose is to calculate correctly the collision integral of the quasiparticle to arbitrary order in their energy. Interaction between quasiparticles and particles is assumed to be immaterial. This constraint is due to technical differences between the description of wave–wave and wave–particle interactions.[2)] The difference between the smallness parameters of these interaction ensures the existence of an applicability region for the employed approximation.

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In the absence of wave–particle interaction one can use the canonical variant[8] of the Wyld diagram technique[9] and obtain a system of Dyson equations for the equal-time Green's function and the paired correlation function. The problem consists thus of reducing, for weakly turbulent wave fields, the Dyson equations to a kinetic equation that is local in time. A certain leeway in the choice of the "quasiparticle distribution function" can be used for utmost simplification of the collision integral. The desired goal can be a kinetic equation in the form

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=\sum_{n=2}^{\infty} S_{\mathbf{k}}^{(n)},$$

$$S_{\mathbf{k}}^{(n)}=\sum_{\sigma_1,\ldots,\sigma_n=\pm 1}\int\prod_{i=1}^{n} d^3\mathbf{k}_i\delta^3\Big(\mathbf{k}-\sum_{j=1}^{n}\sigma_j\mathbf{k}_j\Big)\delta\Big(\omega_{\mathbf{k}}-\sum_{j=1}^{n}\sigma_j\omega_{\mathbf{k}_j}\Big)$$
$$\times W_{\mathbf{k}_1\ldots\mathbf{k}_n}^{\sigma_1\ldots\sigma_n} n_{\mathbf{k}} n_{\mathbf{k}_1}\ldots n_{\mathbf{k}_n}\Big(\frac{1}{n_{\mathbf{k}}}-\sum_{j=1}^{n}\frac{\sigma_j}{n_{\mathbf{k}_j}}\Big). \quad (1.3)$$

A heuristic derivation of (1.3) is based on quantization of the classical wave field, representation of the collision integral in the form of the difference between the rates of arrival of the quanta in the state $|\mathbf{k}\rangle$ and their departure from the state, and return to the classical limit (see Ref. 10). This roundabout approach is partially shortened by the effectiveness of quantum perturbation theory. For example, a "quantum" calculation of the combinations of quasiparticle distribution functions contained in (1.1) and (1.2) turns out to be more lucid than a calculation based on classical perturbation theory. A more rigorous treatment, however, shows that the approach described is not free of the need for renormalization and raises additional problems. Their simplest manifestation appears already in the theory, linear in the wave amplitude, of scattering by random inhomogeneities of a medium. The kinetic equation is in this case obviously linear in the field energy. Yet the rates of arrival of the waves in state $|\mathbf{k}\rangle$ and of the opposite process contain nonlinear terms, and to prove that they cancel one another it is necessary to resort to higher orders of quantum perturbation theory (see Ref. 11).

2. BASIC EQUATIONS

The wave system not interacting with the medium is Hamiltonian and is described by the equation

$$is\frac{\partial a^s(\mathbf{r},t)}{\partial t}=\frac{\delta H}{\delta a^{-s}(\mathbf{r},t)}. \quad (2.1)$$

We consider henceforth for simplicity only one mode of the oscillations. The Hamiltonian H depends then on two field variables, the wave amplitude a $(\mathbf{r},t)$ and its complex conjugate $a^*(\mathbf{r},t)$, labeled respectively by "plus" and "minus" subscripts s. We have

$$H=\sum_{n=2}^{\infty} H_n,$$

$$H_n=\frac{1}{n!}\sum_{s_1,\ldots,s_n}\int d^3\mathbf{r}_1\ldots d^3\mathbf{r}_n V^{s_1,\ldots,s_n}(\mathbf{r}_1,\ldots,\mathbf{r}_n)$$
$$\times a^{s_1}(\mathbf{r}_1,t)\ldots a^{s_n}(\mathbf{r}_n,t). \quad (2.2)$$

Since the medium is assumed to be stationary, H does not depend explicitly on the time. The absence of temporal dispersion follows automatically from the assumed absence of dissipation. The expansion coefficients $V_{1,\ldots,n}^{s,\ldots,s}(\mathbf{r}_1,\ldots,\mathbf{r}_n)$ of the Hamiltonian can be assumed, without loss of generality, to be symmetric with respect to all possible permutations of the pairs $(s_1,\mathbf{r}_1),\ldots,(s_n,\mathbf{r}_n)$. Reversal of the signs of all the subscripts $s_1,\ldots,s_n$ is equivalent to complex conjugation.

It is customary to add to the right-hand side of (2.1) a Gaussian random force $f^s(\mathbf{r},t)$, that describes a thermal-noise source. The kinetic equation for substantially epithermal oscillations is practically independent of the force f^s, so that its inclusion in (2.1) can also be regarded as a convenient formal device. Averaging over an ensemble of realizations of the random force f^s is designated below by angle brackets.

The main objects of the Wyld diagram technique is the pair correlation function

$$N^{s,s'}(\mathbf{r},t,\mathbf{r}',t')=\langle a^s(\mathbf{r},t)a^{-s'}(\mathbf{r}',t')\rangle \quad (2.3)$$

and the Green's function

$$G^{s,s'}(\mathbf{r},t,\mathbf{r}',t')=\Big\langle\frac{\delta a^s(\mathbf{r},t)}{\delta f^{s'}(\mathbf{r}',t')}\Big\rangle. \quad (2.4)$$

Complex conjugation of these functions is equivalent to interchange of the subscripts s and s'. Regarding the functions (2.3) and (2.4) as kernels of the corresponding operators in the coordinate–time representation, we can rewrite the Dyson equations that relate them in the form

$$\hat{N}=\hat{G}\hat{\Phi}\hat{G}^+,\quad \hat{G}={}^0\hat{G}+{}^0\hat{G}\hat{\Sigma}\hat{G}. \quad (2.5)$$

The kernel of the operator ${}^0\hat{G}$ is the Green's function of the linear problem [i.e., of Eq. (2.1) with Hamiltonian H_2], while the kernels of the operators $\hat{\Phi}$ and $\hat{\Sigma}$ are the so-called self-energy functions, for which the suitable renormalized perturbation-theory series no longer contain terms that diverge on weakly turbulent wave fields. A cross marks the Hermitian adjoint of an operator. The operators $\hat{N}$ and $\hat{\Phi}$ coincide with $\hat{N}^+$ and $\hat{\Phi}^+$, i.e., are Hermitian.

We assume below that the medium is spatially homogeneous and is stable to self-excitation of small oscillations. We can then assume, without loss of generality, the linear part of (2.1) to be diagonal in the Fourier representation:

$$is\frac{\partial a_{\mathbf{k}}^s(t)}{\partial t}=\omega_{s\mathbf{k}}a_{\mathbf{k}}^s(t)+(2\pi)^3\frac{\delta}{\delta a_{-\mathbf{k}}^{-s}}\sum_{n=3}^{\infty}H_n+f_{\mathbf{k}}^s,$$
$$a_{\mathbf{k}}^s(t)=\int d^3\mathbf{r}e^{-i\mathbf{k}\mathbf{r}}a^s(\mathbf{r},t), \quad (2.6)$$

and the natural frequencies $\omega_{\mathbf{k}}$ to be real. The Hamiltonian H_n $(n\geqslant 3)$ has in terms of the variables $a_{\mathbf{k}}^s(t)$ the form

$$H_n=\frac{1}{n!}\sum_{s_1,\ldots,s_n}\int\prod_{i=1}^{n}\frac{d^3\mathbf{k}_i}{(2\pi)^3}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{s_1,\ldots,s_n}a_{\mathbf{k}_1}^{s_1}\ldots a_{\mathbf{k}_n}^{s_n}. \quad (2.7)$$

The coefficient $V_{\mathbf{k}_1\ldots\mathbf{k}_n}^{s_1\ldots s_n}$ is symmetric with respect to permutations of any pair if its indices $(s_1,\mathbf{k}_1),\ldots,(s_n,\mathbf{k}_n)$ and contains a delta-function of the sum of all the wave vectors:

$$V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{s_1,\ldots,s_n}=(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\ldots+\mathbf{k}_n)U_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{s_1,\ldots,s_n}. \quad (2.8)$$

Canonical transformations that differ from identity ones only by increments small in terms of the wave field do not alter the linear part of Eq. (2.6) and can be used as simplifications of the structure of H_n ($n\geqslant 3$) (see, e.g., Ref. 3).

The Fourier transforms of the kernels of the operators $\hat G$, $\hat N$, etc. are defined by equations of the type

$$G^{s,s'}_{\mathbf{k},\mathbf{k}'}(t,t')=\int d^3\mathbf{r}\,d^3\mathbf{r}'\exp[i(\mathbf{kr}-\mathbf{k}'\mathbf{r}')]G^{s,s'}(\mathbf{r},t,\mathbf{r}',t').$$

If not only the medium but also the turbulence is spatially inhomogeneous, the kernels of all the operators in (2.5) depend only on the differences of their spatial arguments, so that the Dyson equations can be written in the form

$$\hat N_{\mathbf{k}}=\hat G_{\mathbf{k}}\hat\Phi_{\mathbf{k}}\hat G_{\mathbf{k}}^{+},\quad \hat G_{\mathbf{k}}={}^{0}\hat G_{\mathbf{k}}+{}^{0}\hat G_{\mathbf{k}}\hat\Sigma_{\mathbf{k}}\hat G_{\mathbf{k}}. \tag{2.9}$$

All the operators contained here act in a space of two-component functions of the time. The kernels of the new operators are connected with the initial ones by relations of the type

$$G^{s,s'}_{\mathbf{k},\mathbf{k}'}(t,t')=(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')G^{s,s'}_{\mathbf{k}}(t,t'). \tag{2.10}$$

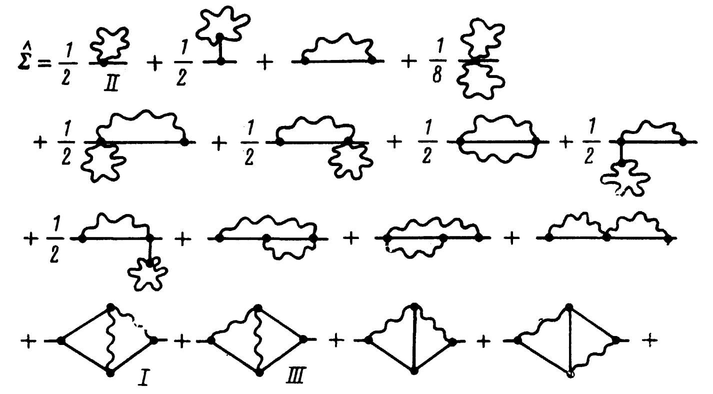
The sequence in which the Fourier transformation and Hermitian conjugation are carried out is immaterial in the calculation of the kernel of the operator $G_{\mathbf{k}}^{+}$, since

$$(\hat G^{+})^{s,s'}_{\mathbf{k}}(t,t')=G^{-s',-s}_{-\mathbf{k}}(t',t)=[(\hat G_{\mathbf{k}})^{+}]^{s,s'}(t,t').$$

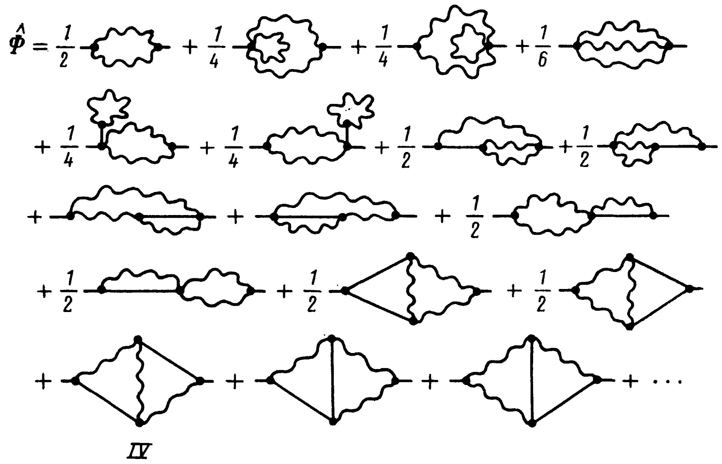
The kernels of the operators $\hat\Phi_{\mathbf{k}}$, $\hat\Sigma_{\mathbf{k}}$, etc. are in general not diagonal in the superscripts s and s'. The off-diagonal components of the function $N^{s,s'}_{\mathbf{k}}(t,t')$ have the meaning of the so-called anomalous correlators considered earlier in connection with the theory of parametric wave excitation.[12,13] Under weak turbulence conditions the anomalous correlators are expressed via normal successive approximations. By a canonical transformation of the variables $a^{s}_{\mathbf{k}}(t)$ (see Ref. 3) we can cause the expansion coefficients of the Hamiltonian $V^{s_1,\ldots,s_n}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}$ to vanish outside the resonance energy surfaces $\Sigma^{n}_{i=1}s_i\omega_{s_i}\mathbf{k}_i=0$ (more accurately, in the regions $|\Sigma^{n}_{i=1}s_i\omega_{s_i}\mathbf{k}_i|\gtrsim\delta\omega$, where $\delta\omega$ is large compared with the reciprocal wave-interaction time γ), and annihilate by the same token the anomalous correlators in approximately the first $\omega_{\mathbf{k}}/\delta\omega$ orders of the expansion in the wave energy. It is not advisable to make $\delta\omega$ smaller than the scale $\Delta\omega$ of the frequency variation of the spectral energy density of the waves, for this would increase the expansion parameter $\gamma/\Delta\omega+\gamma/\delta\omega$ used in the theory of weak turbulence. In view of these circumstances, one can get rid of the anomalous correlators painlessly only when the characteristic frequency $\omega_{\mathbf{k}}$ of the waves exceeds noticeably the width $\Delta\omega$ of their spectrum, e.g., if a large gap exists in the spectrum of the $\omega_{\mathbf{k}}$ dispersion law. For $\Delta\omega\sim\omega_{\mathbf{k}}$ the scalar variant of the Wyld diagram technique[8] is insufficient to describe homogeneous (and even stationary) turbulence even when only one oscillation mode is present. It is useful to note that the matrix variant of the Wyld technique, which we need to derive the kinetic equation, can yield also a more compact formulation of the theory of parametric wave excitation.

3. EXPANSIONS OF THE SELF-ENERGY FUNCTIONS

The Dyson equations (2.5) must be supplemented by equations for the self-energy operators $\hat\Sigma$ and $\hat\Phi$ in terms of the pair correlator $\hat N$ and the Green's operator $\hat G$. The required equations are derived by using the Wyld diagram technique in perfect analogy with the procedure used in Ref. 8 for the scalar model of homogeneous stationary turbulence. The diagram series for $\hat\Sigma$ and $\hat\Phi$ are of the form:

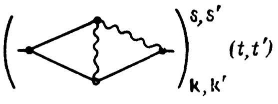


(3.1)

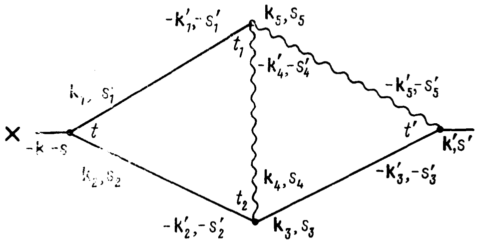


(3.2)

A wave segment denotes here the pair correlator $\hat{N}$, a straight-line segment the Green's operator $\hat{G}$, and a point joining n lines the convolution of the corresponding operators with n-index coefficient in the expansion of the Hamiltonian in the field amplitude. The rules for "reading" the diagrams are clear from Ref. 8 and from the examples therein. Thus, diagram I in (3.1) corresponds to an operator whose matrix elements in the $(t,\mathbf{k})$ representation are equal to



$$= \sum_{\substack{s_1,\dots,s_5\\ s_1',\dots,s_5'}} \int dt_1\, dt_2 \prod_{i=1}^{5} \frac{d^3\mathbf{k}_i\, d^3\mathbf{k}_i'}{(2\pi)^6}$$



(3.3′)

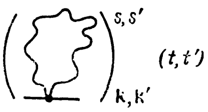
$$= \sum_{s_1,\dots,s_5} \int dt_1\, dt_2 \prod_{i=1}^{5} \frac{d^3\mathbf{k}_i\, d^3\mathbf{k}_i'}{(2\pi)^6}$$

$$\times\ V^{-s,s_1,s_2}_{-\mathbf{k},\mathbf{k}_1.\mathbf{k}_2} V^{-s_2',s_3,s_4}_{-\mathbf{k}_2',\mathbf{k}_3,\mathbf{k}_4} V^{-s_1',-s_4',s_5}_{-\mathbf{k}_1',-\mathbf{k}_4',\mathbf{k}_5} V^{-s_3'\ -s_5',s'}_{-\mathbf{k}_3',-\mathbf{k}_5',\mathbf{k}'}$$

$$\times G^{s_1,s_1'}_{\mathbf{k}_1,\mathbf{k}_1'}(t,t_1)\, G^{s_2\ s_2'}_{\mathbf{k}_2,\mathbf{k}_2'}(t,t_2)\, G^{s_3,s_3'}_{\mathbf{k}_3,\mathbf{k}_3'}(t_2,t')\, N^{s_4,s_4'}_{\mathbf{k}_4,\mathbf{k}_4'}(t_2,t_1)\, N^{s_5,s_5'}_{\mathbf{k}_5,\mathbf{k}_5'}(t_1,t'). \tag{3.3}$$

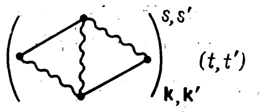
For convenience, we show here also the auxiliary diagram (3.3′) corresponding to the integrand of (3.3). Leading from the "entrance" (point t) to the "exit" (point t') of this diagram is a single path of straight-line segments, called in Ref. 8 the "backbone" of the diagram. The segments making up the backbone are read in the direction from the entrance into the diagram. The remaining straight-line segments ("ribs" of the diagram) are read in the direction away from the backbone (a direction uniquely defined for each segment). Wavy lines can be read in either direction, since the operator $\hat{N}$ is Hermitian, and two oppositely directed readings of any line correspond to operators that are Hermitian adjoints of one another.

The formulated reading rules are equally applicable to all diagrams of the series (3.1). If the backbone degenerates into a point, i.e., the entrance coincides with the exit, the matrix elements of the operator corresponding to the diagram contain a delta-function of the difference between the time-dependent arguments. For example, for diagram II of (3.1) we have



$$=\delta(t-t') \sum_{s_1,s_1'} \int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_1'}{(2\pi)^6} V^{-s,s_1,-s_1',s'}_{-\mathbf{k},\mathbf{k}_1,-\mathbf{k}_1',\mathbf{k}'} N^{s_1,s_1'}_{\mathbf{k}_1,\mathbf{k}_1'}(t,t). \tag{3.4}$$

Any diagram in the series (3.2) can be cut into two parts with only wavy lines (the so-called main section). Each vertex on the left (right) of the main section is connected with the entrance (exit) of the diagram by a single path made up of straight segments that must be read in the direction from the entrance (exit) of the diagram. For example, diagram IV [for the same arrangement of the times, wave vectors, and indices as on the auxiliary diagram (3.3′)] is read as follows:

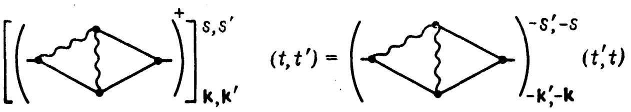


$$= \sum_{\substack{s_1,s_2,\dots,s_5\\ s_1',s_2',\dots,s_5'}} \int dt_1\, dt_2 \prod_{i=1}^{5} \frac{d^3\mathbf{k}_i\, d^3\mathbf{k}_i'}{(2\pi)^6} V^{-s,s_1,s_2}_{-\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$$

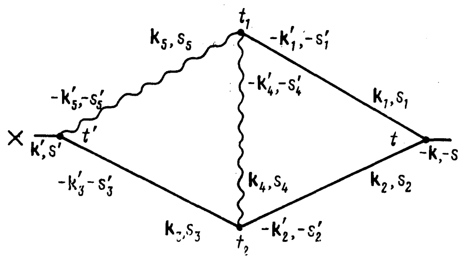
$$\times\ V^{-s_2',s_3,s_4}_{-\mathbf{k}_2',\mathbf{k}_3,\mathbf{k}_4} V^{-s_1',-s_4',s_5}_{-\mathbf{k}_1',\mathbf{k}_4',\mathbf{k}_6} V^{-s_3',-s_5's'}_{-\mathbf{k}_3',-\mathbf{k}_5',\mathbf{k}'} G^{s_1,s_1'}_{\mathbf{k}_1,\mathbf{k}_1'}(t,t_1)\, G^{-s_3',-s_3}_{-\mathbf{k}_3',-\mathbf{k}_3}(t',t_2)$$

$$\times\ N^{s_2,s_2'}_{\mathbf{k}_2,\mathbf{k}_2'}(t,t_2)\, N^{s_4,s_4'}_{\mathbf{k}_4,\mathbf{k}_4'}(t_2,t_1)\, N^{s_5,s_5'}_{\mathbf{k}_5,\mathbf{k}_5'}(t_1,t'). \tag{3.5}$$

The expansions (3.1) and (3.2) contain in addition to each diagram its reflection about a vertical plane. In the expansion (3.2), each pair of mirror-symmetry diagrams corresponds to a pair of Hermitian adjoint operators. It follows from this property, obviously, that the operator $\hat{\Phi}$ is Hermitian. The situation is somewhat more complicated with the diagrams of (3.1). The analytic expression [see diagram III of (3.1)]



$$= \sum_{\substack{s_1,\dots,s_5\\ s_1',\dots,s_5'}} \int dt_1\, dt_2 \prod_{i=1}^{5} \frac{d^3\mathbf{k}_i\, d^3\mathbf{k}_i'}{(2\pi)^6}$$



(3.6)

differs from (3.3) only in that the backbone Green operators are replaced by their Hermitian adjoints. The same symmetry is possessed by the remaining terms of the expansion (3.1). In particular, diagrams with coincident entrance and exit correspond to Hermitian operators. When the Hermitian ($\hat{\Sigma}^H = \frac{1}{2}(\hat{\Sigma} + \hat{\Sigma}^+)$) or anti-Hermitian ($\hat{\Sigma}^A = (1/2i)(\hat{\Sigma} - \hat{\Sigma}^+)$) parts of the operator $\hat{\Sigma}$ are calculated, this symmetry makes it possible to replace in each term of the expansion (3.1) the direct product of all the backbone Green function by the respective Hermitian or anti-Hermitian parts. This replacement is particularly useful in the calculation of $\hat{\Sigma}^A$, which is carried out by the same scheme as in Ref. 8. The anti-Hermitian part of the direct product of arbitrary operators satisfies the identity

$$(\hat{a}_1 \otimes \hat{a}_2 \otimes \cdots \otimes \hat{a}_n)^A = \hat{a}_1{}^A \otimes \hat{a}_2{}^+ \otimes \cdots \otimes \hat{a}_n{}^+ + \hat{a}_1 \otimes \hat{a}_2{}^A \otimes \hat{a}_3{}^+ \otimes \hat{a}_4{}^+ \otimes \cdots \otimes \hat{a}_n{}^+ + \cdots + \hat{a}_1 \otimes \hat{a}_2 \otimes \cdots \otimes \hat{a}_{n-1} \otimes \hat{a}_n{}^A. \tag{3.7}$$

With the aid of this identity, each term of the expansion of the operator $\hat{\Sigma}^A$, a term containing n backbone Green operators, is transformed into a sum of n new terms. To obtain their diagrams it suffices to "pluralize" (by successive separation of each line of its backbone) the diagram corresponding to the initial term into n new diagrams, and separate in correspondence to a chosen (say, marked by a cross) line the anti-Hermitian part of the Green operator, and read all the backbone lines located to the right of the chosen line in a direction opposite to that chosen initially, meaning from the exit point of the diagram, without changing the remaining reading rules.

The described procedure causes the expansion of the operator $\hat{\Sigma}^A$ to become similar to the expansion (3.2). Successive choice of each line of the main sections of the diagrams of the series (3.2) and its replacement by a crossed straight line leads to recalculation of all the multiplied diagrams for $\hat{\Sigma}^A$ (and only these diagrams). The number of times each diagram is repeated is exactly equal to the number of times by which the coefficient of the diagram exceeds the coefficient of the corresponding diagram in (3.2). Since the common parts of the mutually corresponding diagrams of $\hat{\Phi}$ and $\hat{\Sigma}^A$ are read in the same manner, an analytic equation for $\hat{\Sigma}^A$ can be obtained by successive separation of each of the "main" (i.e., represented by the lines of the main sections) pair correlators in the equation for $\hat{\Phi}$, and by replacement of the chosen correlator by the anti-Hermitian part of the Green operator. The Hermitian part of the Green operator is reconstructed from its anti-Hermitian part with the aid of the obvious relations

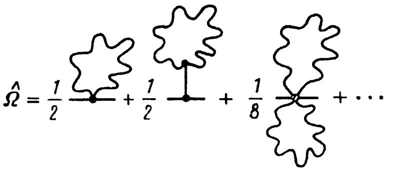
$$\Sigma^{s,s'}_{\mathbf{k},\mathbf{k}'}(t,t') = 0, \quad t < t';$$
$$(\hat{\Sigma}^+)^{s,s'}_{\mathbf{k},\mathbf{k}'}(t,t') = \Sigma^{s',s}_{\mathbf{k}',\mathbf{k}}(t',t)^* = 0, \quad t > t',$$

according to which

$$(\hat{\Sigma}^H)^{s,s'}_{\mathbf{k},\mathbf{k}'}(t+\tau/2, t-\tau/2) = \Omega^{s,s'}_{\mathbf{k},\mathbf{k}'}(t)\,\delta(\tau) + i[\Theta(\tau) - \Theta(-\tau)]\,(\hat{\Sigma}^A)^{s,s'}_{\mathbf{k},\mathbf{k}'}(t+\tau/2, t-\tau/2). \tag{3.8}$$

Here $\Theta(\tau)$ is the Heaviside step function: $\Theta(\tau) = 1, \tau > 0$; $\Theta(\tau) = 0, \tau < 0$.

The first term in the right-hand side of (3.8) is equal to the sum of all the terms of the expansion of the function $\Sigma^{s,s'}_{\mathbf{k},\mathbf{k}'}(t+\tau/2, t-\tau/2)$, which correspond to diagrams with degenerate backbones, and can be regarded as the matrix element of the operator



(3.9)

The Fourier transform of (3.8) with respect to τ,

$$A_\omega = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} A(\tau),$$

is the known Kramers-Kroning relation (see, e.g., Ref. 14):

$$(\hat{\Sigma}^H)^{s,s'}_{\mathbf{k},\mathbf{k}',\omega}(t) = \Omega^{s,s'}_{\mathbf{k},\mathbf{k}'}(t) + \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{d\xi}{\xi - \omega} (\hat{\Sigma}^A)^{s,s'}_{\mathbf{k},\mathbf{k}',\xi}(t). \tag{3.10}$$

The crossed integral sign means, as usual, integration in the sense of the principal value.

In the case of uniform turbulence, the analytic expressions for the self-energy functions can be simplified by substituting in them (2.8) and relations of type (2.10) for $\hat{N}$, $\hat{\Sigma}$, etc. In particular, Eq. (3.10) takes the form

$$(\hat{\Sigma}^H)^{s,s'}_{\mathbf{k},\omega}(t) = \Omega^{s,s'}_{\mathbf{k}}(t) + \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{d\xi}{\xi - \omega} (\hat{\Sigma}^A)^{s,s'}_{\mathbf{k},\xi}(t). \tag{3.11}$$

Similar simplifications occur when the Fourier transform with respect to time is taken in the case of stationary turbulence. In this case the coefficient of the direct product of all the principal pair correlators

$$N^{s_1,s_1'}_{\mathbf{k}_1\mathbf{k}_1',\omega_1} N^{s_2,s_2'}_{\mathbf{k}_2,\mathbf{k}_2',\omega_2} \cdots N^{s_n,s_n'}_{\mathbf{k}_n,\mathbf{k}_n',\omega_n}$$

in an arbitrary term of the expansion of the function $\Phi^{s,s'}_{\mathbf{k},\mathbf{k}',\omega}$ is proportional to $\delta(-\omega + \omega_1 + \ldots + \omega_n)$. Therefore, under the condition

$$(\hat{G}^A)^{s,s'}_{\mathbf{k},\mathbf{k}',\omega} = \beta\omega N^{s,s'}_{\mathbf{k},\mathbf{k}',\omega} \quad (\beta = \text{const}) \tag{3.12}$$

the operators $\hat{\Sigma}^A$ and $\hat{\Phi}$ are connected by the relation

$$(\hat{\Sigma}^A)^{s,s'}_{\mathbf{k},\mathbf{k}',\omega} = \beta\omega\Phi^{s,s'}_{\mathbf{k},\mathbf{k}',\omega}. \tag{3.13}$$

The inverse is also true: under the condition (3.13) the operators $\hat{G}^A$ and $\hat{N}$ are related by (3.12). This can be verified by representing $\hat{G}^A$ in a form similar to the first equation[3)] of (2.5):

$$\hat{G}^A = \hat{G}\hat{\Sigma}^A\hat{G}^+. \tag{3.14}$$

The condition (3.12) singles out thus a certain class of solutions of the set of equations (2.5), (3.1), and (3.2). This conclusion, which reinforces the analogous premise of Ref. 8, can be called a generalized fluctuation–dissipation theorem.

4. KINETIC EQUATION FOR WAVES

The Dyson equations (2.9) in conjunction with expansions (3.1) and (3.2) constitute a closed system of equations for the functions $G^{s,s'}_{\mathbf{k}}(t,t') N^{s,s'}_{\mathbf{k}}(t,t')$, $\Sigma^{s,s'}_{\mathbf{k}}(t,t')$ and $\Phi^{s,s'}_{\mathbf{k}}(t,t')$. When the energy density of the turbulence is decreased without limit, the last three functions tend to zero, and the Green function tends to the finite limit

$${}^0G^{s,s'}_{\mathbf{k}}(t,t') = -is\delta_{s,s'}\Theta(t-t')\exp\{-is\omega_{s\mathbf{k}}(t-t')\}. \tag{4.1}$$

At a finite energy density of the turbulence, the Green function $G^{s,s'}_{\mathbf{k}}(t+\tau/2, t-\tau/2)$ not only oscillates, but is damped with τ. The characteristic damping time γ^{-1} has the meaning of the time of interaction of the quasiparticles, and increases without limit as the turbulence energy tends to zero. The pair correlation function $N_{\mathbf{k}}(t+\tau/2, t-\tau/2)$ is damped in $|\tau|$ at the same characteristic time γ^{-1}. It will become clear below that the damping time with respect to τ of the self-energy functions $\Sigma^{s,s'}_{\mathbf{k}}(t+\tau/2, t-\tau/2)$ and $\Phi^{s,s'}_{\mathbf{k}}(t+\tau/2, t-\tau/2)$ remains as a rule finite when the turbulence energy is decreased without limit. The limiting value $(\Delta\omega)^{-1}$ of this time can be roughly interpreted as the time of phase mixing of noninteracting waves, and the quantity $\Delta\omega$

itself can be interpreted as the time scale of the equal-time correlator $N_{\mathbf{k}}^{+,+}(t,t)$ with respect to the variable $\omega_{\mathbf{k}}$. At sufficiently low turbulence energy, the following condition is usually met:

$$\gamma \ll \Delta\omega. \tag{4.2}$$

It is just this condition which allows us to reduce the system (2.9), (3.1), and (3.2) to a kinetic equation, local in time, for the waves.

As the first step in this direction we must express the Green operator $\hat{G}_{\mathbf{k}}$ in terms of the self-energy operator $\hat{\Sigma}_{\mathbf{k}}$ with the aid of the second equation of (2.9). This is equivalent to inverting the operator

$$\hat{G}_{\mathbf{k}}^{-1} = {}^0\hat{G}_{\mathbf{k}}^{-1} - \hat{\Sigma}_{\mathbf{k}} \equiv \hat{L}_{\mathbf{k}},$$

whose matrix elements in the time representation are

$$L_{\mathbf{k}}^{s,s'}(t,t') = \left(is\frac{\partial}{\partial t} - \omega_{s\mathbf{k}}\right)\delta_{s,s'}\delta(t-t') - \Sigma_{\mathbf{k}}^{s,s'}(t,t'). \tag{4.3}$$

The first term of (4.3) is local in time, while the second is zero for $t<t'$ and is damped with decrement of order $\Delta\omega$ when the difference of its arguments $t-t'>0$ is increased. It is therefore possible to apply the operator $\hat{L}_{\mathbf{k}}$ to arbitrary two-component functions of time that do not increase too rapidly as $t\to-\infty$. The behavior of $G_{\mathbf{k}}^{s,s'}(t,t')$ in the asymptotic region $t-t' \gg (\Delta\omega)^{-1}$ is determined by slowly damped (in time) functions from the zero-space of the operator $L_{\mathbf{k}}$. They have the meaning of the natural oscillations of the medium and satisfy the equation

$$(\hat{L}_{\mathbf{k}}\Psi_{\mathbf{k}}^{\sigma})^s = \left(is\frac{\partial}{\partial t} - \omega_{s\mathbf{k}}\right)\Psi_{\mathbf{k}}^{s,\sigma} - \sum_{s_1}\int dt_1 \Sigma_{\mathbf{k}}^{s,s_1}(t,t_1)\Psi_{\mathbf{k}}^{s_1,\sigma}(t_1) = 0. \tag{4.4}$$

The first superscript of $\Psi_{\mathbf{k}}^{s,\sigma}(t)$ numbers the components of the vector eigenfunction $\Psi_{\mathbf{k}}^{\sigma}(t)$ that oscillates at a frequency close to $\sigma\omega_{\sigma\mathbf{k}}$. This function is determined by Eq. (4.4) apart from an arbitrary numerical factor. Substitution of the expression

$$\Psi_{\mathbf{k}}^{\sigma}(t) = \mathbf{e}_{\mathbf{k}}^{\sigma}(t)\exp\left[-i\int_{t_0}^{t} dt_1\lambda_{\mathbf{k}}^{\sigma}(t_1)\right] \tag{4.5}$$

in (4.4) leads to the equation

$$\left[is\frac{\partial}{\partial t} + s\lambda_{\mathbf{k}}^{\sigma}(t) - \omega_{s\mathbf{k}}\right]e_{\mathbf{k}}^{s,\sigma}(t) = \sum_{s_1}\int dt_1\Sigma_{\mathbf{k}}^{s,s_1}(t,t_1)e_{\mathbf{k}}^{s_1,\sigma}(t_1)\exp\left[i\int_{t_1}^{t} dt_2\lambda_{\mathbf{k}}^{\sigma}(t_2)\right]. \tag{4.6}$$

The supplementary conditions needed for a unique determination of $e_{\mathbf{k}}^{s,\sigma}(t)$ and $\lambda_{\mathbf{k}}^{\sigma}(t)$ will be formulated above, and furthermore in such a manner that they have the natural symmetry

$$e_{\mathbf{k}}^{s,\sigma}(t) = (e_{-\mathbf{k}}^{-s,-\sigma}(t))^*, \quad \lambda_{\mathbf{k}}^{\sigma}(t) = -(\lambda_{-\mathbf{k}}^{-\sigma}(t))^*, \tag{4.7}$$

and that they be independent of time in the stationary-turbulence limit. The second of the properties (4.7) makes it possible to represent the real and imaginary parts of $\lambda_{\mathbf{k}}^{\sigma}(t)$ in the form

$$\tilde{\omega}_{\mathbf{k}}^{\sigma}(t) \equiv \mathrm{Re}\,\lambda_{\mathbf{k}}^{\sigma}(t) = \sigma\tilde{\omega}_{\sigma\mathbf{k}}(t), \qquad \gamma_{\mathbf{k}}^{\sigma}(t) \equiv -\mathrm{Im}\,\lambda_{\mathbf{k}}^{\sigma}(t) = \gamma_{\sigma\mathbf{k}}(t). \tag{4.8}$$

The deviations of the functions $\lambda_{\mathbf{k}}^{\sigma}(t)$ and $e_{\mathbf{k}}^{s,\sigma}(t)$ from the values $\sigma\omega_{\sigma\mathbf{k}}$ and $\delta_{s,\sigma}$ given by the linear theory change within the same characteristic time T as the turbulence spectrum. For an essentially nonstationary turbulence, T can be estimated to equal the wave-interaction time γ^{-1}. To obtain a common description of all possible spectra, including the stationary one for which $T=\infty$, it is necessary to assume that T is an independent quantity from the interval

$$T \geqslant \gamma^{-1}.$$

The contribution of the natural oscillations $G_{\mathbf{k}}^{s,s'}(t,t')$ can be expanded in terms of the functions $\Psi_{\mathbf{k}}^{s,\sigma}(t)$:

$${}^cG_{\mathbf{k}}^{s,s'}(t,t') = \sum_{\sigma} e_{\mathbf{k}}^{s,\sigma}(t)\, g_{\mathbf{k}}^{\sigma}(t,t')\, c_{\mathbf{k}}^{s',\sigma}(t')^*,$$
$$g_{\mathbf{k}}^{\sigma}(t,t') = -i\sigma\Theta(t-t')\exp\left[-i\int_{t'}^{t} dt_1\lambda_{\mathbf{k}}^{\sigma}(t_1)\right]. \tag{4.9}$$

The coefficients $c_{\mathbf{k}}^{s,\sigma}(t)$ are chosen such that the contribution of the beats

$$\delta G_{\mathbf{k}}^{s,s'}(t,t') \equiv G_{\mathbf{k}}^{s,s'}(t,t') - {}^cG_{\mathbf{k}}^{s,s'}(t,t') \tag{4.10}$$

have no long-lived asymptotes, i.e., that it attenuate in the energy region $t-t' \gg (\Delta\omega)^{-1}$ with a decrement of order $\Delta\omega$. Applying to (4.10) from the left the operator $\hat{L}_{\mathbf{k}}$ we readily obtain an equation for the function $\delta G_{\mathbf{k}}^{s,s'}(t,t')$:

$$\hat{L}_{\mathbf{k}}\delta\hat{G}_{\mathbf{k}} = \hat{1} - \hat{L}_{\mathbf{k}}{}^c\hat{G}_{\mathbf{k}} \tag{4.11}$$

($\hat{1}$ is a unit operator). The condition for solvability of Eq. (4.11), on a class of functions $\delta G_{\mathbf{k}}^{s,s'}(t,t')$ that attenuate rapidly with increase of $t-t'$, is formulated in natural fashion in terms of rapidly growing (with time) functions $\bar{\Psi}_{\mathbf{k}}^{\sigma}(t)$ from the zero-space of the Hermitian adjoint $\hat{L}_{\mathbf{k}}^{+}$ of the operator $\hat{L}_{\mathbf{k}}$. The equation for these functions

$$0 = (\hat{L}_{\mathbf{k}}^{+}\bar{\Psi}_{\mathbf{k}}^{\sigma})^s(t) \equiv \left(is\frac{\partial}{\partial t} - \omega_{s\mathbf{k}}\right)\bar{\Psi}_{\mathbf{k}}^{s,\sigma}(t) - \sum_{s_1}\int dt_1\Sigma_{\mathbf{k}}^{s_1,s}(t_1,t)^*\,\bar{\Psi}_{\mathbf{k}}^{s_1,\sigma}(t_1) \tag{4.12}$$

has two solutions that are exceeded as $t\to\infty$ by an exponential with exponent much smaller than $t\Delta\omega$. They can be written in the form

$$\bar{\Psi}_{\mathbf{k}}^{\sigma}(t) = \bar{\mathbf{e}}_{\mathbf{k}}^{\sigma}(t)\exp\left[-i\int_{t_0}^{t} d\tau\lambda_{\mathbf{k}}^{\sigma}(\tau)^*\right], \tag{4.13}$$

where the vector $\bar{\mathbf{e}}_{\mathbf{k}}^{\sigma}(t)$ no longer contains fast oscillations, and does not depend on the time at all in the stationary case. Given the function $\lambda_{\mathbf{k}}^{\sigma}(t)$, this vector is given, apart from an arbitrary numerical factor, by

$$\left[is\frac{\partial}{\partial t} + s\lambda_{\mathbf{k}}^{\sigma}(t)^* - \omega_{s\mathbf{k}}\right]\bar{e}_{\mathbf{k}}^{s,\sigma}(t) = \sum_{s_1}\int dt_1\left\{\Sigma_{\mathbf{k}}^{s_1,s}(t_1,t)\exp\left[i\int_{t}^{t_1}\lambda_{\mathbf{k}}^{\sigma}(t_2)\,dt_2\right]\right\}^*\bar{e}_{\mathbf{k}}^{s_1,\sigma}(t_1). \tag{4.14}$$

To derive the conditions for the solvability of (4.11) we must multiply it scalarly from the left by the vector function

$\bar{\Psi}_{\mathbf{k}}^{\sigma}$. The left-hand side of the resultant equation vanishes:

$$(\bar{\Psi}_{\mathbf{k}}{}^{\sigma}, \hat{L}_{\mathbf{k}}\delta G_{\mathbf{k}})^{s'}(t') = \sum_{s} \int dt \bar{\Psi}_{\mathbf{k}}^{s,\sigma}(t)^* (\hat{L}_{\mathbf{k}}\delta\hat{G}_{\mathbf{k}})^{s,s'}(t,t')$$
$$= \sum_{s_1} \int dt_1 [(\hat{L}_{\mathbf{k}}{}^{+}\bar{\Psi}_{\mathbf{k}}{}^{\sigma})^{s_1}(t_1)]^* \delta G_{\mathbf{k}}^{s_1,s'}(t_1,t') = 0. \quad (4.15)$$

The integration by parts, carried out when the action of the operator $L_{\mathbf{k}}$ is transferred from $\delta\hat{G}_{\mathbf{k}}$ to $\bar{\Psi}_{\mathbf{k}}^{\sigma}$, is valid only if the function $\delta G_{\mathbf{k}}^{s,s'}(t,t')$ decreases rapidly enough with increase of $t - t'$. It is clear therefore that for $\delta G_{\mathbf{k}}^{s,s'}(t,t')$ to behave as required the right-hand side of the equality in question must vanish, i.e.,

$$\bar{\Psi}_{\mathbf{k}}^{s',\sigma}(t')^* = (\bar{\Psi}_{\mathbf{k}}{}^{\sigma}, \hat{L}_{\mathbf{k}}{}^{c}\hat{G}_{\mathbf{k}})^{s'}(t'). \quad (4.16)$$

The latter can be rewritten in the form

$$\bar{e}_{\mathbf{k}}^{s',\sigma}(t')^* = \sum_{\sigma'} \mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t')\, c_{\mathbf{k}}^{s',\sigma'}(t')^*, \quad (4.17)$$

$$\mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t') \equiv \sigma' \Big\{ \sum_{s} s\bar{e}_{\mathbf{k}}^{s,\sigma}(t')^* e_{\mathbf{k}}^{s,\sigma'}(t')$$
$$- i \sum_{s,s_1} \int_{t'}^{\infty} dt \int_{-\infty}^{t'} dt_1 \bar{e}_{\mathbf{k}}^{s,\sigma}(t)^* \Sigma_{\mathbf{k}}^{s,s_1}(t,t_1)$$
$$\times e_{\mathbf{k}}^{s_1,\sigma'}(t_1) \exp\Big[i\int_{t'}^{t} dt_2 \lambda_{\mathbf{k}}{}^{\sigma}(t_2) + i\int_{t_1}^{t'} dt_2 \lambda_{\mathbf{k}}{}^{\sigma'}(t_2)\Big]\Big\}. \quad (4.18)$$

It is easy to verify by direct calculation that the function $\mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t)$ satisfies the identity

$$\Big[i\frac{\partial}{\partial t} - \lambda_{\mathbf{k}}{}^{\sigma}(t) + \lambda_{\mathbf{k}}{}^{\sigma'}(t)\Big] \mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t) = 0, \quad (4.19)$$

from which it follows at $\sigma = \sigma'$ that

$$\partial \mathscr{U}_{\mathbf{k}}^{\sigma,\sigma}(t)/\partial t = 0, \quad (4.20)$$

and at $\sigma' = -\sigma$, in view of the smooth time dependence of $\mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t)$,

$$\mathscr{U}_{\mathbf{k}}^{\sigma,-\sigma}(t) = 0. \quad (4.21)$$

By suitable choice of the numerical factor in the definition of each of the vectors $\bar{\mathbf{e}}_{\mathbf{k}}^{\sigma}(t)$ we can make $\mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t)$ a unit matrix:

$$\mathscr{U}_{\mathbf{k}}^{\sigma,\sigma'}(t) = \delta_{\sigma,\sigma'}. \quad (4.22)$$

By virtue of (4.17), the coefficient $c_{\mathbf{k}}^{s,\sigma}(t)$ coincides then with $\bar{e}_{\mathbf{k}}^{s,\sigma}(t)$ and has the natural symmetry:

$$c_{\mathbf{k}}^{s,\sigma}(t) = \bar{e}_{\mathbf{k}}^{s,\sigma}(t) = \bar{e}_{-\mathbf{k}}^{-s,-\sigma}(t)^* = c_{-\mathbf{k}}^{-s,-\sigma}(t)^*,$$

while Eq. (4.9) can be written in the form

$${}^{c}G_{\mathbf{k}}^{s,s'}(t,t') = -i\sum_{\sigma} \sigma\Theta(t-t')\, \Psi_{\mathbf{k}}^{s,\sigma}(t)\, \bar{\Psi}_{\mathbf{k}}^{s',\sigma}(t')^*. \quad (4.23)$$

According to (4.9), the transformation

$$G_{\mathbf{k}}^{s,s'}(t,t') = \sum_{\sigma,\sigma'} e_{\mathbf{k}}^{s,\sigma}(t)\, \tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')\, c_{\mathbf{k}}^{s',\sigma'}(t')^* \quad (4.24)$$

diagonalizes, with respect to the superscript, the contribution of the natural oscillations to the Green function

$${}^{c}\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') = g_{\mathbf{k}}{}^{\sigma}(t,t')\,\delta_{\sigma,\sigma'} \equiv g_{\mathbf{k}}^{\sigma,\sigma'}(t,t'). \quad (4.25)$$

Since the quantities $e_{\mathbf{k}}^{s,\sigma}(t)$ and $c_{\mathbf{k}}^{s,\sigma}(t)$ are practically independent of the random force $f_{\mathbf{k}}^{s}(t)$, Eq. (4.25) can be regarded as a consequence of the substitutions[4)]

$$a_{\mathbf{k}}{}^{s}(t) = \sum_{\sigma} e_{\mathbf{k}}^{s,\sigma}(t)\, \tilde{a}_{\mathbf{k}}{}^{\sigma}(t), \quad \tilde{f}_{\mathbf{k}}{}^{\sigma}(t) = \sum_{\sigma} c_{\mathbf{k}}^{s,\sigma}(t)^* f_{\mathbf{k}}{}^{s}(t).$$

The transformation, corresponding to the first of these substitutions, of the pair correlation function $\hat{N}_{\mathbf{k}}$ is of the form

$$N_{\mathbf{k}}^{s,s'}(t,t') = \sum_{\sigma,\sigma'} e_{\mathbf{k}}^{s,\sigma}(t)\, \tilde{N}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')\, e_{\mathbf{k}}^{s',\sigma'}(t')^*. \quad (4.26)$$

The modified pair correlator $\tilde{N}_{\mathbf{k}}$ satisfies the equation

$$\hat{\tilde{N}}_{\mathbf{k}} = \hat{\tilde{G}}_{\mathbf{k}} \hat{\tilde{\Phi}}_{\mathbf{k}} \hat{\tilde{G}}_{\mathbf{k}}{}^{+}, \quad (4.27)$$

where

$$\tilde{\Phi}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') = \sum_{s,s'} c_{\mathbf{k}}^{s,\sigma}(t)^*\, \Phi_{\mathbf{k}}^{s,s'}(t,t')\, c_{\mathbf{k}}^{s',\sigma'}(t'). \quad (4.28)$$

Equations similar to (4.28) should be used to determine the functions $\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')$ and $\tilde{\Omega}_{\mathbf{k}}^{\sigma,\sigma'}(t)$:

$$\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') = \sum_{s,s'} c_{\mathbf{k}}^{s,\sigma}(t)^*\, \Sigma_{\mathbf{k}}^{s,s'}(t,t')\, c_{\mathbf{k}}^{s',\sigma'}(t'),$$
$$\tilde{\Omega}_{\mathbf{k}}^{\sigma,\sigma'}(t) = \sum_{s,s'} c_{\mathbf{k}}^{s,\sigma}(t)^*\, \Omega_{\mathbf{k}}^{s,s'}(t)\, c_{\mathbf{k}}^{s',\sigma'}(t). \quad (4.29)$$

The anti-Hermitian part of the Green function should be transformed in accordance with the same law (4.26) as the pair correlator:

$$(\hat{G}_{\mathbf{k}}{}^{A})^{s,s'}(t,t') = \sum_{\sigma,\sigma'} e_{\mathbf{k}}^{s,\sigma}(t)\, {}^{A}\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')\, e_{\mathbf{k}}^{s',\sigma'}(t')^*,$$
$${}^{A}\hat{\tilde{G}}_{\mathbf{k}} \equiv \hat{\tilde{G}}_{\mathbf{k}} \hat{\tilde{\Sigma}}_{\mathbf{k}}{}^{A} \hat{\tilde{G}}_{\mathbf{k}}{}^{+}. \quad (4.30)$$

The superscript A in ${}^{A}\hat{\tilde{G}}_{\mathbf{k}}$ is placed on the left to distinguish this operator from the generally different anti-Hermitian part $\hat{\tilde{G}}_{\mathbf{k}}^{A}$ of the operator $\hat{\tilde{G}}_{\mathbf{k}}$.

The modified functions have the same symmetry properties as the initial ones; in particular,

$$\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') = \tilde{G}_{-\mathbf{k}}^{-\sigma,-\sigma'}(t,t')^*,$$
$$\tilde{N}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') = \tilde{N}_{\mathbf{k}}^{\sigma',\sigma}(t',t)^* = \tilde{N}_{-\mathbf{k}}^{-\sigma',-\sigma}(t',t).$$

When the Green operator $\hat{\tilde{G}}_{\mathbf{k}}$ is subdivided into the contributions of the oscillations and the beats, $\hat{\tilde{G}}_{\mathbf{k}} = \hat{g}_{\mathbf{k}} + \delta\hat{\tilde{G}}_{\mathbf{k}}$, the pair correlator $\hat{\tilde{N}}_{\mathbf{k}}$ breaks up into four terms, i.e,

$$\hat{\tilde{N}}_{\mathbf{k}} = \hat{n}_{\mathbf{k}} + \hat{g}_{\mathbf{k}}\hat{\tilde{\Phi}}_{\mathbf{k}}\delta\hat{\tilde{G}}_{\mathbf{k}}{}^{+} + \delta\hat{\tilde{G}}_{\mathbf{k}}\hat{\tilde{\Phi}}_{\mathbf{k}}\hat{g}_{\mathbf{k}}{}^{+} + \delta\hat{\tilde{G}}_{\mathbf{k}}\hat{\tilde{\Phi}}_{\mathbf{k}}\delta\hat{\tilde{G}}_{\mathbf{k}}{}^{+}. \quad (4.31)$$

The first of them

$$\hat{n}_{\mathbf{k}} = \hat{g}_{\mathbf{k}}\hat{\tilde{\Phi}}_{\mathbf{k}}\hat{g}_{\mathbf{k}}{}^{+} \quad (4.32)$$

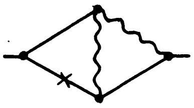
takes into account the mutual correlation of the natural oscillations, the second and third—the correlation between the natural oscillations and the beats, and the fourth—between the beats themselves. The matrix elements $n_{\mathbf{k}}^{\sigma,\sigma'}(t,t')$ of the operator $\hat{n}_{\mathbf{k}}$, which are diagonal in the indices σ and σ', are autocorrelation functions of the natural oscillations, while its off-diagonal elements ($\sigma' = -\sigma$) describe the usually called "anomalous" mutual correlation of the natural oscillations with equal and opposite wave vectors.

Similar equations for the operator ${}^{A}\hat{\tilde{G}}_{\mathbf{k}}$ are obtained by replacing the operator $\hat{\tilde{\Phi}}_{\mathbf{k}}$ in (4.31) and (4.32) by $\hat{\tilde{\Sigma}}_{\mathbf{k}}^{A}$:

$$ {}^{A}\hat{\tilde{G}}_{\mathbf{k}} = \hat{\eta}_{\mathbf{k}} + \hat{g}_{\mathbf{k}}\hat{\tilde{\Sigma}}_{\mathbf{k}}{}^{A}\delta\hat{\tilde{G}}_{\mathbf{k}}{}^{+} + \delta\hat{\tilde{G}}_{\mathbf{k}}\hat{\tilde{\Sigma}}_{\mathbf{k}}{}^{A}\hat{g}_{\mathbf{k}}{}^{+} + \delta\hat{\tilde{G}}_{\mathbf{k}}\hat{\tilde{\Sigma}}_{\mathbf{k}}{}^{A}\delta\hat{\tilde{G}}_{\mathbf{k}}{}^{+}, \tag{4.33} $$

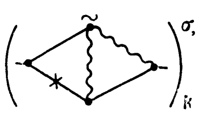
$$ \hat{\eta}_{\mathbf{k}} = \hat{g}_{\mathbf{k}}\hat{\tilde{\Sigma}}_{\mathbf{k}}{}^{A}\hat{g}_{\mathbf{k}}{}^{+}. \tag{4.34} $$

Assuming the equations for $\lambda^{\sigma}_{\mathbf{k}}(t)$, $e^{s,\sigma}_{\mathbf{k}}(t)$, and $c^{s,\sigma}_{\mathbf{k}}(t)$ in terms of $\hat{\tilde{\Sigma}}_{\mathbf{k}}$, to be known (see the Appendix), we can calculate the operators $\hat{\tilde{\Phi}}_{\mathbf{k}}$, $\hat{\tilde{\Sigma}}^{A}_{\mathbf{k}}$, and $\hat{\tilde{\Omega}}_{\mathbf{k}}$ directly by a diagram technique. For example, the term corresponding to the diagram



(4.35′)

in the expansion of the function $(\hat{\tilde{\Sigma}}^{A}_{\mathbf{k}})^{\sigma,\sigma'}(t,t')$ is given by



$$ = \sum_{\substack{\sigma_1,\ldots,\sigma_5\\ \sigma_1',\ldots,\sigma_5'}} \int dt_1\, dt_2 \prod_{i=1}^{5} \frac{d^3\mathbf{k}_i}{(2\pi)^3} \tilde{U}^{-\sigma|\sigma_1,\sigma_2}_{-\mathbf{k}|\mathbf{k}_1,\mathbf{k}_2}(t)\, \tilde{U}^{\sigma_3|-\sigma_2',\sigma_4}_{\mathbf{k}_3|-\mathbf{k}_2,\mathbf{k}_4}(t_2) $$
$$ \times \tilde{U}^{-\sigma_1'|-\sigma_4',\sigma_5}_{-\mathbf{k}_1|-\mathbf{k}_4,\mathbf{k}_5}(t_1) $$
$$ \times \tilde{U}^{-\sigma_3'|-\sigma_5',\sigma'}_{-\mathbf{k}_3|-\mathbf{k}_5,\mathbf{k}}(t')\cdot(2\pi)^9\,\delta^3(-\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,\delta^3(-\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) $$
$$ \times \delta^3(-\mathbf{k}_1-\mathbf{k}_4+\mathbf{k}_5)\, \tilde{G}^{\sigma_1,\sigma_1'}_{\mathbf{k}_1}(t,t_1)\, \tilde{G}^{-\sigma_3',-\sigma_3}_{-\mathbf{k}_3}(t',t_2) $$
$$ \times {}^{A}\tilde{G}^{\sigma_2,\sigma_2'}_{\mathbf{k}_2}(t,t_2)\, \tilde{N}^{\sigma_4,\sigma_4'}_{\mathbf{k}_4}(t_2,t_1)\, \tilde{N}^{\sigma_5,\sigma_5'}_{\mathbf{k}_5}(t_1,t'). \tag{4.35} $$

The modified vertex function $\tilde{U}^{\sigma_1|\sigma_2,\ldots,\sigma_n}_{\mathbf{k}_1|\mathbf{k}_2,\ldots,\mathbf{k}_n}(t)$ is connected with the initial one by the relation

$$ \tilde{U}^{\sigma_1|\sigma_2,\ldots,\sigma_n}_{\mathbf{k}_1|\mathbf{k}_2,\ldots,\mathbf{k}_n}(t) = \sum_{s_1,\ldots,s_n} c^{s_1,\sigma_1}_{\mathbf{k}_1}(t)\, e^{s_2,\sigma_2}_{\mathbf{k}_2}(t)\ldots e^{s_n,\sigma_n}_{\mathbf{k}_n}(t)\, U^{s_1,\ldots,s_n}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}. \tag{4.36} $$

If all the integrations over the wave vectors are carried out in (4.35) or in any other term of the expansion of the kernels of the operators $\hat{\tilde{\Phi}}_{\mathbf{k}}$, $\hat{\tilde{\Sigma}}^{A}_{\mathbf{k}}$, and $\hat{\tilde{\Omega}}_{\mathbf{k}}$, the resultant integrand is a function of all the instants of time ascribed to the vertices, and attenuates with a decrement of order $\Delta\omega$ as the difference of any pair of its arguments is increased. This time quasilocality of the expansions of the self-energy functions (which is obviously a stronger property than the assumption made above that $\Sigma^{s,s'}_{\mathbf{k}}(t,t')$ attenuates rapidly with increase of $t-t'$) is due to oscillations of the integrand. The factors

$$ \tilde{G}^{\sigma,\sigma'}_{\mathbf{k}}(t,t'),\quad \tilde{N}^{\sigma,\sigma'}_{\mathbf{k}}(t,t'),\quad {}^{A}\tilde{G}^{\sigma,\sigma'}_{\mathbf{k}}(t,t') $$

in this equation oscillate at $\tau - t' > 0$ as functions of τ with a frequency close to $\sigma\omega_{\sigma k}$. In the region $\tau < 0$, on the other hand, where the first function is equal to zero, the remaining ones oscillate with a frequency close to $-\sigma'\omega_{\sigma' \mathbf{k}}$. At a given "configuration" of its temporal arguments, the integrand oscillates over all their independent differences, the number of which is smaller by unity than the number of diagram vertices with fully defined frequencies that depend on the wave vectors. After elimination of all the delta-functions, the number of the remaining integrations over the wave-vector components still exceeds the number of vertices, so that the frequencies of the integrand oscillations can be chosen as new independent variables. Shifting the paths of integration over them in the complex plane by a distance of order $\Delta\omega$ away from the real axis,[5] it is easy to verify the quasilocality of the integration result.

The quasilocality property, i.e., the proximity of all the internal and external times in each term of the expansions of the self-energy functions, is preserved when Eqs. (4.31) and (4.33) are replaced in them for $\tilde{N}_{\mathbf{k}}$ and ${}^{A}\tilde{G}_{\mathbf{k}}$ and then $\hat{\tilde{\Phi}}_{\mathbf{k}}$ and $\hat{\tilde{\Sigma}}^{A}_{\mathbf{k}}$ are again expanded in the last three terms of these equations, and also in the off-diagonal matrix elements of the first terms replaced by expressions (4.32) and (4.34). It is impossible to proceed in this manner with the diagonal matrix elements of the operators $\hat{n}_{\mathbf{k}}$ and $\hat{\eta}_{\mathbf{k}}$, since the expressions corresponding to them are in principle nonlocal. It is helpful to simplify the nonlocal expressions by separating small corrections from the functions $n^{\sigma,\sigma}_{\mathbf{k}}(t,t')$ and $\eta^{\sigma,\sigma}_{\mathbf{k}}(t,t')$:

$$ \begin{aligned} n^{\sigma,\sigma}_{\mathbf{k}}(t,t') &= \tilde{n}_{\mathbf{k}}{}^{\sigma}(t,t') + \delta n_{\mathbf{k}}{}^{\sigma}(t,t'),\\ \eta^{\sigma,\sigma}_{\mathbf{k}}(t,t') &= \tilde{\eta}_{\mathbf{k}}{}^{\sigma}(t,t') + \delta\eta_{\mathbf{k}}{}^{\sigma}(t,t'). \end{aligned} \tag{4.37} $$

Here

$$ \tilde{n}_{\mathbf{k}}{}^{\sigma}(t+\tau/2,\ t-\tau/2) = n_{\mathbf{k}}{}^{\sigma}(t)\exp[-i\varphi_{\mathbf{k}}{}^{\sigma}(t,\ \tau)], \tag{4.38} $$

$$ \varphi_{\mathbf{k}}{}^{\sigma}(t,\tau) = \int_{t}^{t+\tau/2} d\tilde{t}\,\lambda_{\mathbf{k}}{}^{\sigma}(\tilde{t}) + \int_{t-\tau/2}^{t} d\tilde{t}\,\lambda_{\mathbf{k}}{}^{\sigma}(\tilde{t})^{*}, $$

$$ n_{\mathbf{k}}{}^{\sigma}(t) = \int_{-\infty}^{t} d\bar{t}\,\Phi_{\mathbf{k}}{}^{\sigma}(\bar{t})\exp\left[-2\int_{\bar{t}}^{t} d\tilde{t}\,\gamma_{\mathbf{k}}{}^{\sigma}(\tilde{t})\right], \tag{4.39} $$

$$ \Phi_{\mathbf{k}}{}^{\sigma}(\bar{t}) = \int_{-\infty}^{\infty} d\bar{\tau}\,\tilde{\Phi}^{\sigma,\sigma}_{\mathbf{k}}\left(\bar{t}+\frac{\bar{\tau}}{2},\bar{t}-\frac{\bar{\tau}}{2}\right)\exp[i\varphi_{\mathbf{k}}{}^{\sigma}(\bar{t},\bar{\tau})]. $$

It is convenient to represent the correction $\delta n^{\sigma}_{\mathbf{k}}(t,t')$ in the form

$$ \delta n_{\mathbf{k}}{}^{\sigma}(t,t') = -\int_{-\infty}^{t} d\bar{t} \int_{0}^{(t-\bar{t})/2} d\xi\,\tilde{\Phi}^{\sigma,\sigma}_{\mathbf{k}}(\bar{t}+\xi,t'+\xi) $$
$$ \times\exp\left\{-i\int_{\bar{t}}^{t} d\tilde{t}\,\lambda_{\mathbf{k}}{}^{\sigma}(\tilde{t}) + i\int_{0}^{\xi} d\tilde{\xi}\,[\lambda_{\mathbf{k}}{}^{\sigma}(\bar{t}+\tilde{\xi}) - \lambda_{\mathbf{k}}{}^{\sigma}(t'+\tilde{\xi})^{*}]\right\} $$
$$ -\int_{-\infty}^{t'} d\bar{t} \int_{0}^{(t'-\bar{t})/2} d\xi\,\Phi^{\sigma,\sigma}_{\mathbf{k}}(t+\xi,\bar{t}+\xi)\exp\left\{i\int_{\bar{t}}^{t'} d\tilde{t}\,\lambda_{\mathbf{k}}{}^{\sigma}(\tilde{t})^{*}\right. $$
$$ \left. + i\int_{0}^{\xi} d\tilde{\xi}\,[\lambda_{\mathbf{k}}{}^{\sigma}(t+\tilde{\xi}) - \lambda_{\mathbf{k}}{}^{\sigma}(\bar{t}+\tilde{\xi})]\right\}. \tag{4.40} $$

Similar equations for $\tilde{\eta}^{\sigma}_{\mathbf{k}}(t,t')$ and $\delta\eta^{\sigma}_{\mathbf{k}}(t,t')$ are obtained from (4.38)–(4.40) by replacing $\hat{\tilde{\Phi}}_{\mathbf{k}}$ by $\hat{\tilde{\Sigma}}^{A}_{\mathbf{k}}$. The function, obtained by making this replacement of $\Phi^{\sigma}_{\mathbf{k}}(t)$, is designated by $A\Sigma^{\sigma}_{\mathbf{k}}(t)$.

In the region $|t-t'| \lesssim (\Delta\omega)^{-1}$ essential for the calculation of the self-energy functions the corrections $\delta n^{\sigma}_{\mathbf{k}}(t,t')$ and $\delta\eta^{\sigma}_{\mathbf{k}}(t,t')$ are small compared with $\tilde{n}^{\sigma}_{\mathbf{k}}(t,t')$ and $\tilde{\eta}^{\sigma}_{\mathbf{k}}(t,t')$ and can be excluded, just as the correlators of the natural oscillations with beats, from the expansions of the self-energy functions, which turn out as a result to be expressed quasilocally in time in terms of the functions $n^{\sigma}_{\mathbf{k}}(t)$ and $\eta^{\sigma}_{\mathbf{k}}(t)$. The latter satisfy the simple evolution equations

$$ \begin{aligned} \frac{\partial n_{\mathbf{k}}{}^{\sigma}}{\partial t} &= \Phi_{\mathbf{k}}{}^{\sigma}(t) - 2\gamma_{\mathbf{k}}{}^{\sigma}(t)\,n_{\mathbf{k}}{}^{\sigma}(t),\\ \frac{\partial \eta_{\mathbf{k}}{}^{\sigma}}{\partial t} &= {}^{A}\Sigma_{\mathbf{k}}{}^{\sigma}(t) - 2\gamma_{\mathbf{k}}{}^{\sigma}(t)\,\eta_{\mathbf{k}}{}^{\sigma}(t). \end{aligned} \tag{4.41} $$

Under the condition

$$ \gamma_{\mathbf{k}}{}^{\sigma}(t) = -\sigma\,{}^{A}\Sigma_{\mathbf{k}}{}^{\sigma}(t) \tag{4.42} $$

[which decreases the previously allowed leeway in the definitions of the quantities $e_{\mathbf{k}}^{s,\sigma}(t)$, $c_{\mathbf{k}}^{s,\sigma}(t)$, and $\lambda_{\mathbf{k}}^{\sigma}(t)$], the function $\eta_{\mathbf{k}}^{\sigma}(t)$ is actually constant:

$$\eta_{\mathbf{k}}^{\sigma}(t)=-\sigma/2 \tag{4.43}$$

and (4.41) is transformed into the closed equation

$$\partial n_{\mathbf{k}}/\partial t=\Phi_{\mathbf{k}}^{+}(t)+2{}^{A}\Sigma_{\mathbf{k}}^{+}(t)n_{\mathbf{k}}(t)\equiv F_{\mathbf{k}}(t). \tag{4.44}$$

for the function

$$n_{\mathbf{k}}^{\sigma}(t)=n_{-\mathbf{k}}^{-\sigma}(t)\equiv n_{\sigma\mathbf{k}}(t) \tag{4.45}$$

The possibility of expressing, locally with respect to time, the "collision integral" $F_{\mathbf{k}}(t)$ in terms of the "quasiparticle distribution function" $n_{\mathbf{k}'}(t)$ is obvious from the derivation of (4.44).

5. STRUCTURE OF THE COLLISION INTEGRAL

We represent the collision integral $F_{\mathbf{k}}(t)$ of all the diagrams (3.2) containing n principal lines, and corresponding to the diagrams for $\tilde{\Sigma}_{\mathbf{k}}^{A}$, in the form

$$\begin{aligned}
{}^{(n)}F_{\mathbf{k}}(t) = &\frac{1}{n!}\int_{-\infty}^{\infty} d\tau \exp[i\varphi_{\mathbf{k}}^{+}(t,\tau)] \sum_{\substack{\sigma_1,\ldots,\sigma_n\\ \sigma_1',\ldots,\sigma_n'}} \int \prod_{i=1}^{n} dt_i\, d\tau_i\, d^3\mathbf{k}_i \\
&\times \sum_{\alpha,\beta} {}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t+\frac{\tau}{2}, t_1+\frac{\tau_1}{2},\ldots,t_n+\frac{\tau_n}{2}\right) \\
&\times {}^{\beta}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1',\ldots,\sigma_n'}\left(t-\frac{\tau}{2}, t_1-\frac{\tau_1}{2},\ldots,t_n-\frac{\tau_n}{2}\right)^{*} \\
&\times \delta^3(-\mathbf{k}+\mathbf{k}_1+\ldots+\mathbf{k}_n) \\
&\times\Big\{ N_{\mathbf{k}_1}^{\sigma_1,\sigma_1'}\left(t_1+\frac{\tau_1}{2}, t_1-\frac{\tau_1}{2}\right)\ldots N_{\mathbf{k}_n}^{\sigma_n,\sigma_n'}\left(t_n+\frac{\tau_n}{2}, t_n-\frac{\tau_n}{2}\right) \\
&+ 2n_{\mathbf{k}}(t)\Big[{}^{A}G_{\mathbf{k}_1}^{\sigma_1,\sigma_1'}\left(t_1+\frac{\tau_1}{2}, t_1-\frac{\tau_1}{2}\right) \\
&\times N_{\mathbf{k}_2}^{\sigma_2,\sigma_2'}\left(t_2+\frac{\tau_2}{2}, t_2-\frac{\tau_2}{2}\right)\ldots \\
&\times N_{\mathbf{k}_n}^{\sigma_n,\sigma_n'}\left(t_n+\frac{\tau_n}{2}, t_n-\frac{\tau_n}{2}\right)+\ldots \\
&+ N_{\mathbf{k}_1}^{\sigma_1,\sigma_1'}\left(t_1+\frac{\tau_1}{2}, t_1-\frac{\tau_1}{2}\right)\ldots \\
&\times N_{\mathbf{k}_{n-1}}^{\sigma_{n-1},\sigma_{n-1}'}\left(t_{n-1}+\frac{\tau_{n-1}}{2}, t_{n-1}-\frac{\tau_{n-1}}{2}\right) \\
&\times {}^{A}G_{\mathbf{k}_n}^{\sigma_n,\sigma_n'}\left(t_n+\frac{\tau_n}{2}, t_n-\frac{\tau_n}{2}\right)\Big]\Big\}.
\end{aligned} \tag{5.1}$$

The coefficients ${}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t,t_1,\ldots,t_n)$ [which have the meaning of renormalized $(n+1)$-wave vertex functions] differ from zero for $t>t_1,\ldots t_n$ and depend smoothly on t at fixed differences $t-t_i$ $(i=1,\ldots,n)$. The superscript α numbers all possible diagrams for the renormalized quantities, i.e., all the possible halves to the left of the principal section and containing each n principal lines of the diagrams (3.2). The summation in (5.1) is over those pairs of values of α and β at which the diagram made up by "gluing together" the halves α and β does not contain weakly bound (i.e., separated by cutting two lines) fragments. The integration in (5.1) is carried out in fact over the region

$$t-t_1,\ldots, t-t_n, |\tau|, |\tau_1|,\ldots, |\tau_n|\lesssim(\Delta\omega)^{-1},$$

since the integral over the wave vector attenuates outside this region, with a decrement of order $\Delta\omega$.

Eliminating from the collision integral, with the aid of (4.31) and (4.33), the anomalous correlators, the correlators of the beats with the natural oscillations and with one another, and also the analogous components of the operators ${}^{A}\hat{G}_{\mathbf{k}}$, we can reduce $F_{\mathbf{k}}(t)$ to a sum of the terms obtained from (5.1), supplemented by a renormalization of the vertices and by replacement of the functions

$$N_{\mathbf{k}}^{\sigma_i,\sigma_i'}(t_i+\tau_i/2, t_i-\tau_i/2),\quad {}^{A}G_{\mathbf{k}_i}^{\sigma_i,\sigma_i'}(t_i+\tau_i/2, t_i-\tau_i/2)$$

respectively by the functions

$$n_{\mathbf{k}}^{\sigma_i,\sigma_i}(t_i+\tau_i/2, t_i-\tau_i/2)\,\delta_{\sigma_i\sigma_i'},$$
$$\eta_{\mathbf{k}}^{\sigma_i,\sigma_i}(t_i+\tau_i/2, t_i-\tau_i/2)\,\delta_{\sigma_i\sigma_i'}.$$

The next step consists of eliminating from the collision integral the corrections $\delta n_{\mathbf{k}_i}^{\sigma_i}(t_i+\tau_i/2, t_i-\tau_i/2)$ and $\delta\eta_{\mathbf{k}}^{\sigma_i}(t_i+\tau_i/2, t_i-\tau_i/2)$ with the aid of (4.40) and of the similar equation for $\delta\eta_{\mathbf{k}}^{\sigma_i}(t_i+\tau_i/2, t_i-\tau/2)$. Expanding in these equations in powers of ξ near the point $\xi=0$ and explicitly integrating with respect to ξ, we can preserve the previous form of the collision terms also in this stage (accurate to one more redefinition of the terms and to one more renormalization of the vertices):

$$\begin{aligned}
{}^{(n)}F_{\mathbf{k}}(t)=&\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n}\int d\tau\prod_{i=1}^{n}dt_i\,d\tau_i\,d^3\mathbf{k}_i\delta^3 \\
&\times(-\mathbf{k}+\mathbf{k}_1+\ldots+\mathbf{k}_n) \\
&\times\sum_{\alpha,\beta}{}^{\alpha}\bar{V}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t+\frac{\tau}{2}, t_1+\frac{\tau_1}{2},\ldots,t_n+\frac{\tau_n}{2}\right) \\
&\times{}^{\beta}\bar{V}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t-\frac{\tau}{2}, t_1-\frac{\tau_1}{2},\ldots,t_n-\frac{\tau_n}{2}\right)^{*} \\
&\times\{n_{\mathbf{k}_1}^{\sigma_1}(t_1)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t_n)+2n_{\mathbf{k}}(t) \\
&\times[\eta_{\mathbf{k}_1}^{\sigma_1}(t_1)n_{\mathbf{k}_2}^{\sigma_2}(t_2)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t_n)+\ldots \\
&+n_{\mathbf{k}_1}^{\sigma_1}(t_1)\ldots n_{\mathbf{k}_{n-1}}^{\sigma_{n-1}}(t_{n-1})\eta_{\mathbf{k}_n}^{\sigma_n}(t_n)]\} \\
&\times\exp\left[i\varphi_{\mathbf{k}}^{+}(t,\tau)-i\sum_{i=1}^{n}\varphi_{\mathbf{k}_i}^{\sigma_i}(t_i,\tau_i)\right].
\end{aligned} \tag{5.2}$$

In what follows it is convenient to rewrite the argument of the exponential in (5.2) in a somewhat different form

$$\begin{aligned}
&\varphi_{\mathbf{k}}^{+}(t,\tau)-\sum_{i=1}^{n}\varphi_{\mathbf{k}_i}^{\sigma_i}(t_i,\tau_i)=\varphi_{\mathbf{k}}^{+}(t,\tau)-\sum_{i=1}^{n}\varphi_{\mathbf{k}_i}^{\sigma_i}(t,\tau) \\
&+\sum_{i=1}^{n}\Big[\int_{t_i+\tau_i/2}^{t+\tau/2}d\tilde{t}\,\lambda_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})-\int_{t_i-\tau_i/2}^{t-\tau/2}d\tilde{t}\,\lambda_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})^{*}+2i\int_{t_i}^{t}d\tilde{t}\,\gamma_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\Big].
\end{aligned} \tag{5.3}$$

The combinations resulting from the substitution of (5.3) in (5.2)

$$n_{\mathbf{k}_i}^{\sigma_i}(t_i)\exp\left[-2\int_{t_i}^{t}d\tilde{t}\gamma_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\right],\quad \eta_{\mathbf{k}_i}^{\sigma_i}(t_i)\exp\left[-2\int_{t_i}^{t}d\tilde{t}\gamma_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\right]$$

must be expanded, following (4.33), in powers of the difference t_i-t:

$$\begin{pmatrix} n_{\mathbf{k}_i}^{\sigma_i}(t_i) \\ \eta_{\mathbf{k}_i}^{\sigma_i}(t_i)\end{pmatrix} \exp\left[-2\int_{t_i}^{t} d\tilde{t}\,\gamma_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\right] = \begin{pmatrix} n_{\mathbf{k}_i}^{\sigma_i}(t) \\ \eta_{\mathbf{k}_i}^{\sigma_i}(t)\end{pmatrix}$$

$$+\sum_{n=1}^{\infty}\frac{(t_i-t)^n}{n!}\left[\frac{\partial}{\partial t}+2\gamma_{\mathbf{k}_i}^{\sigma_i}(t)\right]^{n-1}\begin{pmatrix} \Phi_{\mathbf{k}_i}^{\sigma_i}(t) \\ {}^{A}\Sigma_{\mathbf{k}_i}^{\sigma_i}(t)\end{pmatrix}. \tag{5.4}$$

With the aid of (5.3), (5.4), and the identity

$$t_i-t={}^1/_2\,[t_i+\tau_i/2-(t+\tau/2)+t_i-\tau_i/2-(t-\tau/2)]$$

we can, without losing the structural features of the collision integral, change over in it from the quantities $n_{\mathbf{k}_i}^{\sigma_i}(t_i)$ and $\eta_{\mathbf{k}_i}^{\sigma_i}(t_i)$ to $n_{\mathbf{k}_i}^{\sigma_i}(t)$ and $\eta_{\mathbf{k}_i}^{\sigma_i}(t)$. The collision terms and the vertex functions are redefined here in succession, while the vertex functions acquire a small increment that has a smooth dependence on the external time t:

$$F_{\mathbf{k}}(t)=\sum_n {}^{(n)}F_{\mathbf{k}}(t), \tag{5.5}$$

$${}^{(n)}F_{\mathbf{k}}(t)$$

$$=\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n}\int d\tau\prod_{i=1}^{n} dt_i\,d\tau_i\,d^3\mathbf{k}_i\,\delta^3(-\mathbf{k}+\mathbf{k}_1+\ldots+\mathbf{k}_n)$$

$$\times\sum_{\alpha,\beta}{}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t;t+\frac{\tau}{2},t_1+\frac{\tau_1}{2},\ldots,t_n+\frac{\tau_n}{2}\right)$$

$$\times{}^{\beta}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t;t-\frac{\tau}{2},t_1-\frac{\tau_1}{2},\ldots,t_n-\frac{\tau_n}{2}\right)^{*}$$

$$\times\{n_{\mathbf{k}_1}^{\sigma_1}(t)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t)$$

$$+\,2n_{\mathbf{k}}(t)\,[\eta_{\mathbf{k}_1}^{\sigma_1}(t)\,n_{\mathbf{k}_2}^{\sigma_2}(t)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t)$$

$$+\ldots+n_{\mathbf{k}_1}^{\sigma_1}(t)\ldots n_{\mathbf{k}_{n-1}}^{\sigma_{n-1}}(t)\,\eta_{\mathbf{k}_n}^{\sigma_n}(t)]\}$$

$$\times\exp\left\{i\left[\varphi_{\mathbf{k}}^{+}(t,\tau)-\sum_{i=1}^{n}\varphi_{\mathbf{k}_i}^{\sigma_i}(t,\tau)+\sum_{i=1}^{n}\int_{t_i+\tau_i/2}^{t+\tau/2}d\tilde{t}\,\lambda_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\right.\right.$$

$$\left.\left.-\sum_{i=1}^{n}\int_{t_i-\tau_i/2}^{t-\tau/2}d\tilde{t}\,\lambda_{\mathbf{k}_i}^{\sigma_i}(t)^{*}\right]\right\}. \tag{5.6}$$

The form of the collision term ${}^{(n)}\tilde{F}_{\mathbf{k}}(t)$ is noticeably simplified by integrating formally in (5.6) with respect to the internal times and introducing the symbols

$${}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)\equiv\int\prod_{i=1}^{n}dt_i\,{}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t,t_1,\ldots,t_n)$$

$$\times\exp\left[i\sum_{i=1}^{n}\int_{t_i}^{t}d\tilde{t}\,\lambda_{\mathbf{k}_i}^{\sigma_i}(\tilde{t})\right] \tag{5.7}$$

(which become quantitative by integration over the wave vectors). We get ultimately

$${}^{(n)}F_{\mathbf{k}}(t)=\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n}\int d\tau\prod_{i=1}^{n}d^3\mathbf{k}_i\,\delta^3(-\mathbf{k}+\mathbf{k}_1+\ldots+\mathbf{k}_n)$$

$$\times\sum_{\alpha,\beta}{}^{\alpha}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t;t+\frac{\tau}{2}\right){}^{\beta}V_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}\left(t;t-\frac{\tau}{2}\right)$$

$$\times\{n_{\mathbf{k}_1}^{\sigma_1}(t)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t)+2n_{\mathbf{k}}(t)\,[\eta_{\mathbf{k}_1}^{\sigma_1}(t)\,n_{\mathbf{k}_2}^{\sigma_2}(t)\ldots n_{\mathbf{k}_n}^{\sigma_n}(t)$$

$$+\ldots+n_{\mathbf{k}_1}^{\sigma_1}(t)$$

$$\ldots n_{\mathbf{k}_{n-1}}^{\sigma_{n-1}}(t)\,\eta_{\mathbf{k}_n}^{\sigma_n}(t)]\}\exp\left\{i\left[\varphi_{\mathbf{k}}^{+}(t,\tau)-\sum_{i=1}^{n}\varphi_{\mathbf{k}_i}^{\sigma_i}(t,\tau)\right]\right\}. \tag{5.8}$$

It is necessary next to expand with respect to τ with the aid of the equation

$$\varphi_{\mathbf{k}}^{\sigma}(t,\tau)=\omega_{\mathbf{k}}^{\sigma}(t)\,\tau+\sum_{m=1}^{\infty}\frac{\tau^{2m}}{2^{2m}(2m)!}\left[-2i\frac{\partial^{2m-1}\gamma_{\mathbf{k}}^{\sigma}}{\partial t^{2m-1}}\right.$$

$$\left.+\frac{\tau}{2m+1}\frac{\partial^{2m}\tilde{\omega}_{\mathbf{k}}^{\sigma}}{\partial t^{2m}}\right],$$

bear in mind that the functions $\eta_{\mathbf{k}_i}^{\sigma_i}(t)$ are in fact constants equal to $-\sigma_i/2$, and replace the integration variables $\mathbf{k}_i$ by $\sigma_i\mathbf{k}_i$. The result is

$${}^{(n)}\tilde{F}_{\mathbf{k}}(t)=\sum_{p=0}^{\infty}{}^{(p,n)}\tilde{F}_{\mathbf{k}}(t);$$

$$\tag{5.9}$$

$${}^{(p,n)}\tilde{F}_{\mathbf{k}}(t)=\frac{\partial^p}{\partial\omega^p}\sum_{\sigma_1,\ldots,\sigma_n}\int\prod_{i=1}^{n}d^3\mathbf{k}_i\,\delta^3(-\mathbf{k}+\sigma_1\mathbf{k}_1+\ldots\sigma_n\mathbf{k}_n)$$

$$\times\,\delta(-\omega+\sigma_1\tilde{\omega}_{\mathbf{k}_1}(t)+\ldots+\sigma_n\tilde{\omega}_{\mathbf{k}_n}(t))\,{}^{(p)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t)$$

$$\times\{n_{\mathbf{k}_1}(t)\ldots n_{\mathbf{k}_n}(t)$$

$$-\,n_{\mathbf{k}}(t)\,[\sigma_1 n_{\mathbf{k}_2}(t)\ldots n_{\mathbf{k}_n}(t)$$

$$+\ldots+n_{\mathbf{k}_1}(t)\ldots n_{\mathbf{k}_{n-1}}(t)\,\sigma_n]\}\big|_{\omega=\tilde{\omega}_{\mathbf{k}}(t)}$$

Here

$${}^{(0)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t)$$

$$=\frac{2\pi}{n!}\sum_{\alpha,\beta}{}^{\alpha}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t;t)\,{}^{\beta}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t;t)^{*},$$

$${}^{(1)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t)=\frac{2\pi}{n!}\,\mathrm{Im}\sum_{\alpha,\beta}\left[\frac{\partial}{\partial t}\,{}^{\alpha}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)\right]$$

$$\times\,{}^{\beta}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)^{*}\,\big|_{\tilde{t}=t},$$

$$\tag{5.10}$$

$${}^{(2)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t)=\frac{2\pi}{n!}\,\mathrm{Re}\sum_{\alpha,\beta}\frac{1}{4}\left[\frac{\partial}{\partial t}\,{}^{\alpha}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)\right.$$

$$\times\,\frac{\partial}{\partial t}\,{}^{\beta}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)^{*}$$

$$\left.-\,{}^{\alpha}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)\,\frac{\partial^2}{\partial t^2}\,{}^{\beta}V_{\sigma_1\mathbf{k}_1,\ldots,\sigma_n\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(\tilde{t};t)^{*}\right]\Bigg|_{\tilde{t}=t}$$

$$+\frac{1}{4}\left[\frac{\partial\gamma_{\mathbf{k}}}{\partial t}-\sum_{i=1}^{n}\frac{\partial\gamma_{\mathbf{k}_i}(t)}{\partial t}\right]{}^{(0)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t),\ldots$$

In the stationary case there remain in the expansion of the collision integral only the terms ${}^{(0,n)}\tilde{F}_{\mathbf{k}}(t)$, which have formally the standard structure (1.3). It is clear from the derivation of (5.9), however, that the "probabilities" ${}^{(0)}W_{\mathbf{k}_1,\ldots,\mathbf{k}_n}^{\sigma_1,\ldots,\sigma_n}(t)$ of all but the lowest nonlinear processes contain symbols that acquire a quantitative meaning only after integration over the wave vectors. This is the price paid for the formal preservation of the standard structure and a trace left by the divergences eliminated with the aid of the diagram technique.

It is easy to show that the increments ${}^{(p,n)}\tilde{F}_{\mathbf{k}}(t)\,(p=1,2,\ldots)$ introduced into the collision term ${}^{(0,n)}\tilde{F}_{\mathbf{k}}(t)$ by the nonstationarity contain, compared with this term, an additional smallness $(\gamma/\Delta\omega)^m(T\Delta\omega)^{-m}$ for $p=2m$ or $(\gamma/\Delta\omega)^m(T\Delta\omega)^{-m-1}$ for $p=2m+1\,(m=1,2,\ldots)$. This rule does not extend to the increment ${}^{(1,n)}\tilde{F}_{\mathbf{k}}(t)$, which is of the same order as

$$\frac{\gamma}{\Delta\omega}(T\Delta\omega)^{-1\,(0,n)}\tilde{F}_{\mathbf{k}}(t)$$

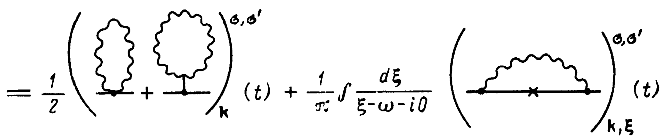
[and not $(T\Delta\omega)^{-1(0,n)}\tilde{F}_{\mathbf{k}}(t)$], since the part of the coefficient ${}^{\alpha}\tilde{V}^{\sigma_1,\ldots,\sigma_n}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}$ that is small in the parameter $\gamma/\Delta\omega$ varies with time.

6. CUBIC COLLISION TERM FOR WAVES WITH A DECAY DISPERSION LAW

As seen from (5.9), (5.10), and the estimates deduced from them that the terms ${}^{(0,2)}\tilde{F}_{\mathbf{k}}(t)$ and ${}^{(0,3)}\tilde{F}_{\mathbf{k}}(t)$ suffice to calculate the collision integral to third order in the wave energy. It is necessary to take into account in ${}^{(0,2)}\tilde{F}_{\mathbf{k}}(t)$ the probability of the processes and the renormalized frequency, accurate to corrections linear in the wave energy, and in ${}^{(0,3)}\tilde{F}_{\mathbf{k}}(t)$ only in the zeroth approximation. The normalized frequency is given with sufficient accuracy by the equations

$$\tilde{\omega}_{\mathbf{k}}(t)=\omega_{\mathbf{k}}+\mathrm{Re}\,\tilde{\Sigma}^{+,+}_{\mathbf{k},\omega_{\mathbf{k}}}(t), \tag{6.1}$$

$$\tilde{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$$



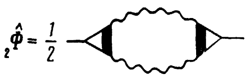
$$=\frac{1}{2}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}n_{\mathbf{k}_1}(t)\sum_{\sigma_1}\Big(U^{-,\sigma,\sigma_1,-\sigma_1}_{-\mathbf{k},\mathbf{k},\sigma_1\mathbf{k}_1,-\sigma_1\mathbf{k}_1}$$

$$-\frac{1}{\omega_0}\sum_{\sigma_2}U^{-\sigma,\sigma,\sigma_2}_{-\mathbf{k},\mathbf{k},0}\,U^{-\sigma_2,\sigma_1,-\sigma_1}_{0,\sigma_1\mathbf{k}_1,-\sigma_1\mathbf{k}_1}-\sum_{\sigma_1,\sigma_2}\int\frac{d^3\mathbf{k}_1\,d^3\mathbf{k}_2}{(2\pi)^3}$$

$$\times\delta^3(-\mathbf{k}+\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2)\frac{\sigma_2 n_{\mathbf{k}_2}(t)}{\sigma_1\omega_{\mathbf{k}_1}+\sigma_2\omega_{\mathbf{k}_2}-\omega-i0}\,|U^{-\sigma,\sigma_1,\sigma_2}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2}|^2. \tag{6.2}$$

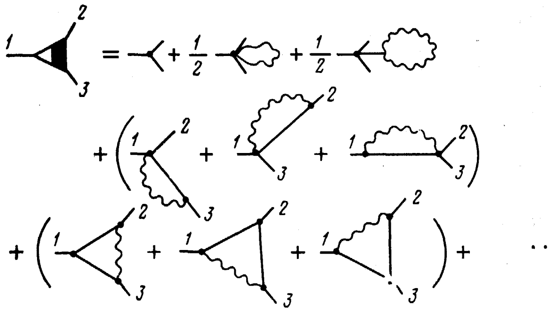
The unperturbed frequency ω_0 of natural oscillations having infinite wavelength is assumed here and hereafter to differ from zero. (For $\omega_0=0$ and $U^{\sigma,\sigma_1,\sigma_2}_{-\mathbf{k},\mathbf{k},0}\neq 0$ the long-wave eigenmodes must be described dynamically, for in this case the long-wave turbulence is generally speaking not weak no matter how low its energy.)

To find the probabilities of the three- and four-wave processes, ${}^{(0)}W^{\sigma_1,\sigma_2}_{\mathbf{k}_1,\mathbf{k}_2}(t)$ and ${}^{(0)}W^{\sigma_1,\sigma_2,\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ respectively, it suffices to track the vertex renormalizations used, when deriving (5.9) and (5.10), to express the function $\Phi^+_{\mathbf{k}}(t)$ in terms of the quasiparticle distribution $n_{\mathbf{k}}(t)$. No weakly bound fragments are produced when arbitrary halves of the diagrams (3.3), which have two principal lines, are joined together. The sum of such diagrams can therefore be represented in the form



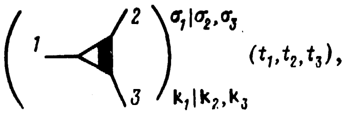
(6.3)

The diagrams entering in the renormalized three-wave vertex



(6.4)

are numbered by the different values of the exponent α (or β) in Eq. (5.1) with $n=2$. For $n=2$ the summation in (5.1) was carried out over all values of α and β without exception. The analytic equation corresponding to (6.4) is determined by using the usual rules for reading the diagrams. When operating in terms of the modified functions, it is necessary to set the left-hand side of (6.4) in correspondence with a modified renormalized vertex function



(t_1,t_2,t_3), (6.4′)

and the first diagram in the right-hand side of (6.4) to the function

$$\tilde{U}^{\sigma_1|\sigma_2,\sigma_3}_{\mathbf{k}_1|\mathbf{k}_2,\mathbf{k}_3}(t_1)\,\delta(t_2-t_1)\,\delta(t_3-t_1).$$

According to (4.36) and (A20), $\tilde{U}^{\sigma_1|\sigma_2,\sigma_3}_{\mathbf{k}_1|\mathbf{k}_2,\mathbf{k}_3}(t)$ is given, with accuracy sufficient for the solved problem, by the equation

$$\tilde{U}^{\sigma_1|\sigma_2,\sigma_3}_{\mathbf{k}_1|\mathbf{k}_2,\mathbf{k}_3}(t)=U^{\sigma_1,\sigma_2,\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\Big[1+\frac{1}{2}\sigma_1\frac{\partial}{\partial\omega_1}\tilde{\Sigma}^{\sigma_1,\sigma_1}_{\mathbf{k}_1,\omega_1}(t)^*+\frac{1}{2}\sigma_2\frac{\partial}{\partial\omega_2}\tilde{\Sigma}^{\sigma_2,\sigma_2}_{\mathbf{k}_2,\omega_2}(t)+\frac{1}{2}\sigma_3\frac{\partial}{\partial\omega^3}\tilde{\Sigma}^{\sigma_3,\sigma_3}_{\mathbf{k}_3,\omega_3}(t)\Big]$$
$$-\frac{\tilde{\Sigma}^{\sigma_1,-\sigma_1}_{\mathbf{k}_1,\omega_1}(t)^*\,U^{-\sigma_1,\sigma_2,\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}}{\omega_{\mathbf{k}_1}+\omega_{-\mathbf{k}_1}}-\frac{U^{\sigma_1,-\sigma_2,\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\tilde{\Sigma}^{-\sigma_2,\sigma_2}_{\mathbf{k}_2,\omega_2}(t)}{\omega_{\mathbf{k}_2}+\omega_{-\mathbf{k}_2}}-\frac{U^{\sigma_1,\sigma_2,-\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\tilde{\Sigma}^{-\sigma_3,\sigma_3}_{\mathbf{k}_3,\omega_3}(t)}{\omega_{\mathbf{k}_3}+\omega_{-\mathbf{k}_3}}\Bigg|_{\omega_i=\sigma_i\omega_{\sigma_i\mathbf{k}_i},\ i=1,2,3}, \tag{6.5}$$

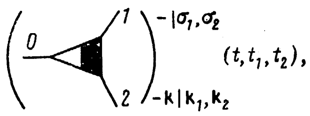
where $\tilde{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega}$ satisfies (6.2) as before. In the remaining terms of the right-hand side of (6.4), the modified vertex functions can be regarded as equal to the initial ones.

No additional renormalizations of the three-wave vertex are needed on going from expansion of the collision integral (5.1) to the expansion (5.6). The probability ${}^{(0)}W^{\sigma_1,\sigma_2}_{\mathbf{k}_1,\mathbf{k}_2}(t)$ of the three-wave process can therefore be written in the form

$${}^{(0)}W^{\sigma_1,\sigma_2}_{\mathbf{k}_1,\mathbf{k}_2}(t)=1/2\cdot 2\pi\,|\tilde{V}^{\sigma_1,\sigma_2}_{\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2}(t)|^2, \tag{6.6}$$

where $\tilde{V}^{\sigma_1,\sigma_2}_{\mathbf{k}_1,\mathbf{k}_2}(t)$ is the "probability amplitude" defined in accordance with (5.7)

$$\tilde{V}^{\sigma_1,\sigma_2}_{\mathbf{k}_1,\mathbf{k}_2}(t)=\int dt_1\,dt_2\exp\Big[i\int_{t_1}^{t}d\tilde{t}\,\lambda^{\sigma_1}_{\mathbf{k}_1}(\tilde{t})+i\int_{t_2}^{t}d\tilde{t}\,\lambda^{\sigma_2}_{\mathbf{k}_2}(\tilde{t})\Big]$$

$\times$ 

(t,t_1,t_2), (6.7)

where $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$.

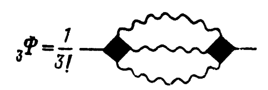
Accurate to corrections linear in the turbulence energy inclusive we have

$$V^{\sigma_1,\sigma_2}_{\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2}(t)=U^{-|\sigma_1,\sigma_2}_{-\mathbf{k}|\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2}(t)$$
$$+\frac{1}{2}\sum_{\sigma_3}\int\frac{d^3\mathbf{k}_3}{(2\pi)^3}n_{\mathbf{k}_3}(t)\Big(U^{-,\sigma_1,\sigma_2,\sigma_3,-\sigma_3}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3,-\sigma_3\mathbf{k}_3}$$
$$-\frac{1}{\omega_0}\sum_{\sigma_4}U^{-,\sigma_1,\sigma_2,\sigma_4}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,0}U^{-\sigma_4,\sigma_3,-\sigma_3}_{0,\sigma_3\mathbf{k}_3,-\sigma_3\mathbf{k}_3}\Big)$$
$$+\hat{S}\sum_{\sigma_3,\sigma_4}\int\frac{d^2\mathbf{k}_3\,d^3\mathbf{k}_4}{(2\pi)^3}\delta^3(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3-\sigma_4\mathbf{k}_4)\cdot$$
$$\times U^{-,\sigma_1,\sigma_3,\sigma_4}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma_3\mathbf{k}_3,\sigma_4\mathbf{k}_4}\frac{U^{\sigma_2,-\sigma_3,-\sigma_4}_{\sigma_2\mathbf{k}_2,-\sigma_3\mathbf{k}_3,-\sigma_4\mathbf{k}_4}\sigma_3 n_{\mathbf{k}_3}(t)}{\sigma_2\omega_{\mathbf{k}_2}-\sigma_3\omega_{\mathbf{k}_3}-\sigma_4\omega_{\mathbf{k}_4}+i0}$$
$$+\hat{S}\sum_{\sigma_3,\sigma_4,\sigma_5}\int\frac{d^3\mathbf{k}_3\,d^3\mathbf{k}_4\,d^3\mathbf{k}_5}{(2\pi)^3}\delta^3(\sigma_1\mathbf{k}_1+\sigma_3\mathbf{k}_3-\sigma_4\mathbf{k}_4)$$
$$\times\delta^3(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3-\sigma_5\mathbf{k}_5)U^{-,\sigma_4,\sigma_5}_{-\mathbf{k},\sigma_4\mathbf{k}_4,\sigma_5\mathbf{k}_5}U^{-\sigma_4,\sigma_1,\sigma_3}_{-\sigma_4\mathbf{k}_4,\sigma_1\mathbf{k}_1,\sigma_3\mathbf{k}_3}$$
$$\times U^{-\sigma_5,\sigma_2,-\sigma_3}_{-\sigma_5\mathbf{k}_5,\sigma_2\mathbf{k}_2,-\sigma_3\mathbf{k}_3}n_{\mathbf{k}_3}(t)\frac{\sigma_4}{\sigma_1\omega_{\mathbf{k}_1}+\sigma_3\omega_{\mathbf{k}_3}-\sigma_4\omega_{\mathbf{k}_4}+i0}$$
$$\times\frac{\sigma_5}{\sigma_2\omega_{\mathbf{k}_2}-\sigma_3\omega_{\mathbf{k}_3}-\sigma_5\omega_{\mathbf{k}_5}+i0}\Bigg|_{\mathbf{k}=\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2}. \quad (6.8)$$

Here $\hat{S}$ the operator of summation over cyclic permutations of three pairs of independent variables:

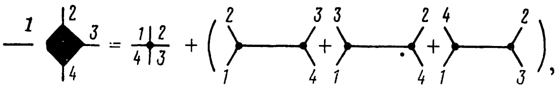
$$\begin{pmatrix}-\\-\mathbf{k}\end{pmatrix},\ \begin{pmatrix}\sigma_1\\\sigma_1\mathbf{k}_1\end{pmatrix},\ \begin{pmatrix}\sigma_2\\\sigma_2\mathbf{k}_2\end{pmatrix}.$$

The sum of diagrams (3.2) having three principal lines cannot be represented in a form similar to (6.3)



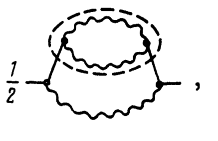
(6.9)

by introducing the renormalized four-wave vertex



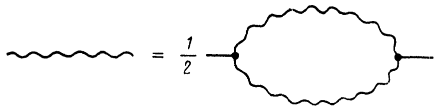
(6.10)

inasmuch as substitution of (6.10) in (6.9) adds to the diagrams listed in (3.2) an extra diagram



(6.11)

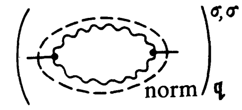
which contains a weakly bound fragment. This diagram appears in a natural manner after the first iteration of the equation



(6.12)

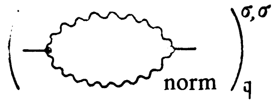
A similar iteration procedure is realized on going from (5.1) to (5.6), but not completely, and only to the extent that it does not lead to violation of the temporal quasilocality of the expressions. The additional four-wave-vertex renormalization carried out for this transition can be reconstructed by a suitable reading of the diagram (6.11). A specific feature of this reading, symbolized by the dashed line enclosing the weakly coupled fragment of the diagram in (6.11), consists of removing from the expression obtained in the usual manner the term that is substantially nonlocal in time and has already been taken into account in the three-wave collision term. Such a term is contained only in the expression for the normal correlator integrated with the aid of (4.32).

After removing the term that is essentially nonlocal in time, the contribution of the normal correlation of the natural oscillations and expression corresponding to the weakly coupled fragment of diagram (6.11) turns out to be



$$=-\Phi^{\sigma,\sigma}_{\mathbf{q}}\frac{\partial}{\partial\omega}\frac{\mathscr{P}}{\omega-\sigma\omega_\sigma\mathbf{k}}. \quad (6.13)$$

A Fourier transform with respect to time is taken, $\mathbf{q}=(\mathbf{k},\omega)$ is a four-component vector, and the symbol $\mathscr{P}$ indicates the impending integration by parts and evaluation of the integral in the sense of principal value. Substitution of the unperturbed Green function in (6.11) and the usual reading of the fragment instead of (6.13) would lead to the expression



$$=\frac{\Phi^{\sigma,\sigma}_{\mathbf{q}}}{(\omega-\sigma\omega_{\sigma\mathbf{k}})^2} \quad (6.14)$$

the integral of which with respect to frequency diverges. By agreeing to eliminate such divergences with the aid of the symbolic equality

$$\frac{1}{\xi^2}=-\frac{\partial}{\partial\xi}\frac{\mathscr{P}}{\xi}, \quad (6.15)$$

we can formally preserve the usual method of reading the diagram (6.11), add meaning to relation (6.9), and arbitrarily represent the four-wave-process probability in the form

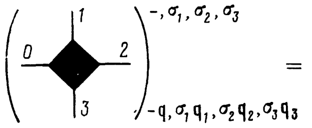
$$^{(0)}W^{\sigma_1,\sigma_2,\sigma_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}=\frac{1}{3!}2\pi\,|\tilde{V}^{\sigma_1,-\sigma_2,\sigma_3}_{\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3}|^2, \quad (6.16)$$

where the "probability amplitude" $\tilde{V}^{\sigma_1,\sigma_2,\sigma_3}_{\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3}$ coincides with the value of the renormalized vertex function (6.10) on the so-called energy surface

$$\mathbf{q}=\sum_{i=1}^{3}\sigma_i\mathbf{q}_i\qquad(\omega_i=\omega_{\mathbf{k}_i}),$$

on which we have

$$\tilde{V}^{\sigma_1,\sigma_2,\sigma_3}_{\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3}=$$



$$=U^{-,\sigma_1,\sigma_2,\sigma_3}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3}$$
$$+\hat{S}\left\{\sum_{\sigma'}\frac{\sigma' U^{-,\sigma_1,\sigma'}_{-\mathbf{k},\sigma_1\mathbf{k}_1,\sigma'\mathbf{k}'}}{\sigma_3\omega_{\mathbf{k}_3}+\sigma_2\omega_{\mathbf{k}_2}-\sigma'\omega_{\mathbf{k}'}+i0}U^{-\sigma',\sigma_2,\sigma_3}_{-\sigma'\mathbf{k}',\sigma_2\mathbf{k}_2,\sigma_3\mathbf{k}_3}\right\}, \quad (6.17)$$

$$\mathbf{k}=\sum_{i=1}^{3}\sigma_i\mathbf{k}_i,\qquad \sigma'\mathbf{k}'=\sum_{i=2}^{3}\sigma_i\mathbf{k}_i$$

(the operator $\hat{S}$ denotes summation over the three cyclic permutations of the subscripts 1, 2, and 3). It is easily seen that the singularities that need be determined in accordance with (6.15) are introduced in (6.16) by the squared moduli of all the terms contained in (6.17) except the first, and are connected with three-wave resonances. In the absence of the latter, i.e., in the case of a non-decay wave-dispersion law, Eq. (6.16) has not merely a symbolic but also a literal meaning and coincides with the universally known one. At the borderline of the decay and non-decay cases (i.e., where a new wave-interaction channel "cuts through") the kinetic equation, as noted above, cannot be used no matter how low the turbulence energy.

7. CONCLUSION

The main result of the present paper is the expansion of the collision integral (5.9) supplemented by equations for all its constituent quantities. The fact that the structure of this expansion is predetermined facilitates greatly, if the particle-distribution function is suitably chosen, the calculation of the higher collision terms, since the problem reduces to finding renormalized vertex functions and natural frequency. This makes it possible to use all the advantages of the lucid "semi-quantum" approach and at the same time be rid of its flaws. In the case of stationary turbulence it is even possible to retain the collision-term structure usually proposed on the basis of the quantum analogy. To be sure, the probabilities of all but the lowest nonlinear processes lose in this case their quantitative meaning, owing to the introduction of symbols that are not integrable in the usual sense. These symbols are introduced exclusively to obtain collision terms with the desired structure and are "decoded" in accordance with strictly prescribed rules that make it possible to return to expressions containing no divergences whatever and arising in natural fashion in a correct calculation. The divergences artificially produced to standardize the collision integral are connected with contributions of resonances of lower order to the higher collision terms.

In the case of nonstationary turbulence, the desired structure of the collision integral cannot be conserved even formally. A qualitative explanation of this fact is the following. The natural oscillations of the turbulent medium, i.e., the quasiparticles, are formed not only by the medium but also by the turbulence. The turbulence-induced normalization of a natural oscillation is altered within the same characteristic time T as the turbulence spectrum. In the essentially nonstationary case, when $T \sim \gamma^{-1}$, the quasiparticles have time to become noticeably restructured during their lifetime γ^{-1}. Comparing this with the intuitive quasiparticle concept, on which the "semiquantum" approach is based, one can regard as surprising not so much the presence of terms ${}^{(p,n)}\widetilde{F}_{\mathbf{k}}(t)$ with $p \geqslant 1$ in the collision integral (5.3), as their structural similarity to the standard collision terms ${}^{(0,n)}\widetilde{F}_{\mathbf{k}}(t)$. Nothing like this would occur if the quasiparticle distribution function were not chosen carefully enough; moreover, a small but substantially nonlocal (in time) deviation from the choice made would lead to a similar collision-integral nonlocality that is removable only by expansion in the parameter $(\gamma^{T})^{-1}$. There is fertile soil here for "refuting" the weak-turbulence theory. This problem, as is clear now, is technical and can be completely avoided by an adequate description of the turbulence.

Fundamental rather than technical difficulties are observed near the thresholds for the onset of new wave-interaction channels, where the temporal quasilocality of the expansions of the self-energy functions is violated. In terms of Fourier transforms with respect to time, violation of quasilocality means an abrupt (with a variation scale of order γ) dependence of the self-energy functions $\widetilde{\Phi}^{\sigma,\sigma'}_{\mathbf{k},\omega}$ and $\widetilde{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega}$ on the frequency ω at threshold values of the wave vector $\mathbf{k}$, for example for $\mathbf{k} = 2\mathbf{k}_0$ and the widely used dispersion law $\omega_{\mathbf{k}} = \omega_0(1 + k^2/2k_0^2)$. For such $\mathbf{k}$ the kinetic equation (5.9) turns out unsuitable no matter how low the turbulence energy. The corresponding waves require a dynamic or a quasidynamic description (see Ref. 15).

The authors thank V. E. Zakharov and V. S. L'vov for a stimulating interest in the work and for helpful discussions.

APPENDIX

The functions $\lambda^{\sigma}_{\mathbf{k}}(t)$, $e^{s,\sigma}_{\mathbf{k}}(t)$, and $c^{s,\sigma}_{\mathbf{k}}(t)$ can be expressed, quasilocally in time, in terms of $(\widetilde{\Sigma}^{A}_{\mathbf{k}})^{\sigma,\sigma'}(t,t')$ or, even more conveniently, in terms of the Fourier transform

$$(\hat{\Sigma}^{A})^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) = \int_{-\infty}^{\infty} d\tau (\hat{\Sigma}_{\mathbf{k}}{}^{A})^{\sigma,\sigma'}\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{i\omega\tau}. \quad \text{(A1)}$$

By virtue of the Kramers–Kroning relation that is similar to (3.11) it is possible to regard as known, together with $(\widetilde{\Sigma}^{A})^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$, also the function $\widetilde{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$. The imaginary part of $\lambda^{\sigma}_{\mathbf{k}}(t)$, which is connected by Eq. (4.42) with $A\Sigma^{\sigma}_{\mathbf{k}}(t)$, is simply expressed by

$$(\hat{\Sigma}^{A})^{\sigma,\sigma}_{\mathbf{k},\omega}(t) = \operatorname{Im} \tilde{\Sigma}^{\sigma,\sigma}_{\mathbf{k},\omega}(t)$$

and the real part of $\lambda^{\sigma}_{\mathbf{k}}(t)$ by

$$\gamma_{\mathbf{k}}{}^{\sigma}(t) = \left\{ -\sigma \operatorname{Im} \tilde{\Sigma}^{\sigma,\sigma}_{\mathbf{k},\omega}(t) - \frac{1}{4} \frac{\partial^2 \operatorname{Im} \tilde{\Sigma}^{\sigma,\sigma}_{\mathbf{k},\omega}(t)}{\partial\omega^2} \left[\frac{\partial}{\partial t} + \frac{\partial\omega_{\mathbf{k}}{}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega}\right] \times \operatorname{Im} \tilde{\Sigma}^{\sigma,\sigma}_{\mathbf{k},\omega}(t) + O\left(\frac{\gamma^2}{T^2(\Delta\omega)^3}\right)\right\}\Bigg|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}. \quad \text{(A2)}$$

From (4.4) follows the relation

$$\sum_s c_{\mathbf{k}}^{s,\sigma'}(t)\left[-is\frac{\partial}{\partial t} + s\lambda_{\mathbf{k}}{}^{\sigma}(t) - \omega_{s\mathbf{k}}\right] c_{\mathbf{k}}^{s,\sigma}(t)^{*} = \Sigma_{\mathbf{k}}^{\sigma,\sigma'}(t), \quad \text{(A3)}$$

where

$$\Sigma_{\mathbf{k}}^{\sigma,\sigma'}(t) \equiv \int_{-\infty}^{\infty} dt_1\, \Sigma_{\mathbf{k}}^{\sigma,\sigma'}(t_1,t) \exp\left[i\int_t^{t_1} d\tilde{t}\, \lambda_{\mathbf{k}}{}^{\sigma}(\tilde{t})\right] = \left\{1 - \frac{i}{2}\left[\frac{\partial}{\partial t} + \frac{\partial\lambda_{\mathbf{k}}{}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega}\right]\frac{\partial}{\partial\omega} + O\left(\frac{1}{T^2(\Delta\omega)^2}\right)\right\} \Sigma_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)\Big|_{\omega=\lambda^{\sigma}_{\mathbf{k}}(t)}.$$

The anti-Hermitian part of (A3)

$$\left[i\frac{\partial}{\partial t} + \lambda_{\mathbf{k}}{}^{\sigma'}(t)^{*} - \lambda_{\mathbf{k}}{}^{\sigma}(t)\right]\sum_s s c_{\mathbf{k}}^{s,\sigma}(t)^{*} c_{\mathbf{k}}^{s,\sigma'}(t) = \Sigma_{\mathbf{k}}^{\sigma',\sigma}(t)^{*} - \Sigma_{\mathbf{k}}^{\sigma,\sigma'}(t) \quad \text{(A4)}$$

can be integrated with respect to time, without loss of the temporal quasilocality, and represented in the form

$$\sum_s sc_{\mathbf{k}}^{s,\sigma}(t)^* c_{\mathbf{k}}^{s,\sigma'}(t) = \sigma\delta_{\sigma,\sigma'} + P_{\mathbf{k}}^{\sigma,\sigma'}(t). \tag{A5}$$

For the off-diagonal components of $P_{\mathbf{k}}^{\sigma,\sigma'}(t)$ we obtain directly from (A4)

$$\begin{aligned} P_{\mathbf{k}}^{\sigma,-\sigma}(t) &= [\lambda_{\mathbf{k}}^{\sigma}(t) + \lambda_{-\mathbf{k}}^{\sigma}(t)]^{-1}\Big[\tilde{\Sigma}_{\mathbf{k}}^{\sigma,-\sigma}(t) - \tilde{\Sigma}_{-\mathbf{k}}^{\sigma,-\sigma}(t) + i\frac{\partial}{\partial t}P_{\mathbf{k}}^{\sigma,-\sigma}(t)\Big] \\ &= \frac{1}{\omega+\omega'}\Big\{1 + i\Big[\frac{\partial}{\partial t} + \frac{\partial\lambda_{\mathbf{k}}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega} + \frac{\partial\lambda_{-\mathbf{k}}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega'}\Big] \\ &\quad\times\Big[-\frac{1}{2}\Big(\frac{\partial}{\partial\omega} + \frac{\partial}{\partial\omega'}\Big) + \frac{1}{\omega+\omega'}\Big][\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,-\sigma}(t) - \tilde{\Sigma}_{-\mathbf{k},\omega}^{\sigma,-\sigma}(t)] \\ &\quad + O\Big(\frac{1}{T^2(\Delta\omega)^2}\Big)\Big\}\Big|_{\omega=\lambda_{\mathbf{k}}^{\sigma}(t),\ \omega'=\lambda_{-\mathbf{k}}^{\sigma}(t)}. \end{aligned} \tag{A6}$$

At first glance, the missing possibility of conserving the temporal quasilocality when integrating the diagonal components of (A4) is connected with the identity (4.43), allowance for which leads to the following expression for $P_{\mathbf{k}}^{\sigma,\sigma}(t)$:

$$\begin{aligned} P_{\mathbf{k}}^{\sigma,\sigma}(t) &= -2\,\mathrm{Re}\int_0^\infty d\tau\int_0^\tau d\xi\,(\tilde{\Sigma}^A)_{\mathbf{k}}^{\sigma,\sigma}\Big(t+\frac{\xi+\tau}{2}, t+\frac{\xi-\tau}{2}\Big) \\ &\quad\times\exp\Big[i\varphi_{\mathbf{k}}^{\sigma}\Big(t+\frac{\xi}{2},\tau\Big) + \int_0^\xi d\tilde{\xi}\gamma_{\mathbf{k}}^{\sigma}\Big(t+\frac{\tilde{\xi}}{2}\Big)\Big] \\ &= -\mathrm{Im}\Big\{\Big[1 + \frac{1}{2}\frac{\partial^2}{\partial t\,\partial\gamma} + \frac{1}{4}\frac{\partial\gamma_{\mathbf{k}}^{\sigma}}{\partial t}\Big(\frac{\partial^2}{\partial\gamma^2} - \frac{\partial^2}{\partial\omega^2}\Big) + \frac{1}{2}\frac{\partial\tilde{\omega}_{\mathbf{k}}^{\sigma}}{\partial t}\frac{\partial^2}{\partial\gamma\,\partial\omega} \\ &\quad + O\Big(\frac{1}{T^2(\Delta\omega)^2}\Big)\Big]\frac{\tilde{\Sigma}_{\mathbf{k},\omega-i\gamma}^{\sigma,\sigma}(t) - \tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)}{\gamma}\Big\}\Big|_{\omega=\omega_{\mathbf{k}}^{\sigma}(t),\ \gamma=\gamma_{\mathbf{k}}^{\sigma}(t)}. \end{aligned} \tag{A7}$$

It is convenient to replace (A3) and (A5) by the single equation

$$\begin{aligned} c_{\mathbf{k}}^{s,\sigma}(t)^* c_{\mathbf{k}}^{s,\sigma'}(t) &\equiv A_{\mathbf{k}}^{s,\sigma,\sigma'}(t) = (\omega_{\mathbf{k}} + \omega_{-\mathbf{k}})^{-1}\Big\{[\lambda_{\mathbf{k}}^{\sigma}(t) + s\omega_{-s\mathbf{k}}] \\ &\quad\times[\sigma\delta_{\sigma,\sigma'} + P_{\mathbf{k}}^{\sigma,\sigma'}(t)] - \tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t) \\ &\quad - i\sum_{s'} s' c_{\mathbf{k}}^{s',\sigma'}(t)\frac{\partial}{\partial t}c_{\mathbf{k}}^{s',\sigma}(t)^*\Big\}. \end{aligned} \tag{A8}$$

The quantities $A_{\mathbf{k}}^{s,\sigma,\sigma'}(t)$ introduced here obviously have the properties

$$A_{\mathbf{k}}^{s,\sigma,\sigma'} = (A_{\mathbf{k}}^{s,\sigma',\sigma})^*, \quad A_{\mathbf{k}}^{s,\sigma,\sigma}A_{\mathbf{k}}^{s,\sigma',\sigma'} = |A_{\mathbf{k}}^{s,\sigma,\sigma'}|^2. \tag{A9}$$

The second property is in essence an equation for the real part of the function $\lambda_{\mathbf{k}}^{\sigma}(t)$:

$$\begin{aligned} \tilde{\omega}_{\mathbf{k}}^{\sigma} &= \sigma\omega_{\sigma\mathbf{k}} + [\sigma + P_{\mathbf{k}}^{\sigma,\sigma}(t)]^{-1}\Big\{\mathrm{Re}\,\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma}(t) + (\omega_{\mathbf{k}} \\ &\quad + \omega_{-\mathbf{k}})\frac{|A_{\mathbf{k}}^{-\sigma,\sigma,-\sigma}(t)|^2}{A_{\mathbf{k}}^{-\sigma,-\sigma,-\sigma}(t)} \\ &\quad + \mathrm{Im}\sum_s sc_{\mathbf{k}}^{s,\sigma}(t)^*\frac{\partial}{\partial t}c_{\mathbf{k}}^{s,\sigma}(t)\Big\}. \end{aligned} \tag{A10}$$

In the zeroth approximation in the parameter $(T\Delta\omega)^{-1}$ we obtain from (A8) and (A10) closed equations for $A_{\mathbf{k}}^{s,\sigma,\sigma'}$ and $\tilde{\omega}_{\mathbf{k}}^{\sigma}(t)$. In the calculations that follow it is necessary to recall the identity (4.22) rewritten in the form

$$\sigma\Big[\sum_s sc_{\mathbf{k}}^{s,\sigma}(t)^* e_{\mathbf{k}}^{s,\sigma'}(t) - S_{\mathbf{k}}^{\sigma,\sigma'}(t)\Big] = \delta_{\sigma,\sigma'}, \tag{A11}$$

where

$$\begin{aligned} S_{\mathbf{k}}^{\sigma,\sigma'}(t) &\equiv i\int_t^\infty dt_1\int_{-\infty}^t dt_2\,\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t_1,t_2) \\ &\quad\times\exp\Big[i\int_t^{t_1} d\tilde{t}\lambda_{\mathbf{k}}^{\sigma}(\tilde{t}) + i\int_{t_2}^{t} d\tilde{t}\lambda_{\mathbf{k}}^{\sigma'}(\tilde{t})\Big], \\ \tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') &= \sum_{s,s'} c_{\mathbf{k}}^{s,\sigma}(t)^*\Sigma_{\mathbf{k}}^{s,s'}(t,t')e_{\mathbf{k}}^{s',\sigma'}(t'). \end{aligned} \tag{A12}$$

Introducing the matrix of the transition from the functions $e_{\mathbf{k}}^{s,\sigma}(t)$ to the functions $c_{\mathbf{k}}^{s,\sigma}(t)$, defined as

$$e_{\mathbf{k}}^{s,\sigma}(t) = \sum_{\sigma_1} c_{\mathbf{k}}^{s,\sigma_1}(t)\,b_{\mathbf{k}}^{\sigma_1,\sigma}(t), \tag{A13}$$

we can rewrite (A12) in the form

$$\begin{aligned} \tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t') &= \sum_{\sigma_1}\tilde{\tilde{\Sigma}}_{\mathbf{k}}^{\sigma,\sigma_1}(t,t')\,b_{\mathbf{k}}^{\sigma_1,\sigma'}(t'), \\ S_{\mathbf{k}}^{\sigma,\sigma'}(t) &= \sum_{\sigma_1}\sum_{n=0}^{\infty}{}^{(n)}S_{\mathbf{k}}^{\sigma,\sigma_1,\sigma'}(t)\frac{\partial^n}{\partial t^n}b_{\mathbf{k}}^{\sigma_1,\sigma'}(t). \end{aligned} \tag{A14}$$

The coefficients ${}^{(n)}S_{\mathbf{k}}^{\sigma,\sigma_1,\sigma'}(t)$ depend only on the functions $\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)$ and $\lambda_{\mathbf{k}}^{\sigma}(t)$:

$$\begin{aligned} {}^{(n)}S_{\mathbf{k}}^{\sigma,\sigma_1,\sigma'}(t) &\equiv \frac{i}{n!}\int_t^\infty dt_1\int_{-\infty}^t dt_2\,\tilde{\tilde{\Sigma}}_{\mathbf{k}}^{\sigma,\sigma_1}(t_1,t_2)(t_2-t)^n \\ &\quad\times\exp\Big[i\int_t^{t_1} d\tilde{t}\lambda_{\mathbf{k}}^{\sigma}(\tilde{t}) + i\int_{t_2}^{t} d\tilde{t}\lambda_{\mathbf{k}}^{\sigma'}(\tilde{t})\Big] \\ &= \frac{i^n}{n!}\Big(\frac{\partial}{\partial\omega'}\Big)^n\Big\{1 - \frac{i}{2}\Big[\frac{\partial}{\partial t} + \frac{\partial\lambda_{\mathbf{k}}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega}\Big]\frac{\partial}{\partial\omega} \\ &\quad + \frac{i}{2}\Big[\frac{\partial}{\partial t} + \frac{\partial\lambda_{\mathbf{k}}^{\sigma'}}{\partial t}\frac{\partial}{\partial\omega'}\Big]\frac{\partial}{\partial\omega'} + O\Big(\frac{1}{T^2(\Delta\omega)^2}\Big)\Big\}\tilde{S}_{\mathbf{k},\omega,\omega'}^{\sigma,\sigma_1}(t), \\ &\omega = \lambda_{\mathbf{k}}^{\sigma}(t),\quad \omega' = \lambda_{\mathbf{k}}^{\sigma'}(t), \\ S_{\mathbf{k},\omega,\omega'}^{\sigma,\sigma'}(t) &= \frac{\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t) - \tilde{\Sigma}_{\mathbf{k},\omega'}^{\sigma,\sigma'}(t)}{\omega-\omega'}. \end{aligned} \tag{A15}$$

Given $\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)$ and $\lambda_{\mathbf{k}}^{\sigma}(t)$, (A11) is an equation for the function $b_{\mathbf{k}}^{\sigma,\sigma'}(t)$:

$$\sum_{\sigma_1}\Big\{\delta_{\sigma,\sigma_1} + \sigma\Big[P_{\mathbf{k}}^{\sigma,\sigma_1}(t) - \sum_{n=0}^{\infty}{}^{(n)}S_{\mathbf{k}}^{\sigma,\sigma_1,\sigma'}(t)\frac{\partial^n}{\partial t^n}\Big]\Big\}b_{\mathbf{k}}^{\sigma_1,\sigma'}(t) = \delta_{\sigma,\sigma'}. \tag{A16}$$

In the zeroth approximation in the parameter $(T\Delta\omega)^{-1}$, the solution of this equation is given by

$$\begin{aligned} b_{\mathbf{k}}^{\sigma_1,\sigma} &= [\delta_{\sigma_1,\sigma} - \sigma_1(P_{\mathbf{k}}^{-\sigma,-\sigma_1} - S_{\mathbf{k},\lambda_-,\lambda}^{-\sigma,-\sigma_1})]/Q_{\mathbf{k}}^{\sigma}, \\ Q_{\mathbf{k}}^{\sigma} &= [1 + \sigma(P_{\mathbf{k}}^{\sigma,\sigma} - S_{\mathbf{k},\lambda,\lambda}^{\sigma,\sigma})][1 - \sigma(P_{\mathbf{k}}^{-\sigma,-\sigma} - S_{\mathbf{k},\lambda_-,\lambda}^{-\sigma,-\sigma})] \\ &\quad + (P_{\mathbf{k}}^{\sigma,-\sigma} - S_{\mathbf{k},\lambda,\lambda}^{\sigma,-\sigma})(P_{\mathbf{k}}^{-\sigma,\sigma} - S_{\mathbf{k},\lambda_-,\lambda}^{-\sigma,\sigma}), \\ &\lambda \equiv \lambda_{\mathbf{k}}^{\sigma},\quad \lambda_- \equiv \lambda_{\mathbf{k}}^{-\sigma}. \end{aligned} \tag{A17}$$

It is convenient to specify the connection between the phases of the components $e_{\mathbf{k}}^{s,\sigma}(t)$ and $c_{\mathbf{k}}^{s,\sigma}(t)$, a connection needed for an unambiguous determination of the functions $e_{\mathbf{k}}^{\sigma,\sigma}(t)$ and $c_{\mathbf{k}}^{\sigma,\sigma}(t)$, by the condition

$$e_{\mathbf{k}}^{\sigma,\sigma}(t)/|e_{\mathbf{k}}^{\sigma,\sigma}(t)|=c_{\mathbf{k}}^{\sigma,\sigma}(t)^*/|c_{\mathbf{k}}^{\sigma,\sigma}(t)|. \quad \text{(A18)}$$

We obtain then for $e_{\mathbf{k}}^{s,\sigma}(t)$ and $c_{\mathbf{k}}^{s,\sigma}(t)$ the equations

$$c_{\mathbf{k}}^{\sigma,\sigma}(t)^* =|c_{\mathbf{k}}^{\sigma,\sigma}(t)|\left[\frac{b_{\mathbf{k}}^{\sigma,\sigma}(t)+A_{\mathbf{k}}^{\sigma,\sigma,-\sigma}(t)\,b_{\mathbf{k}}^{-\sigma,\sigma}(t)/|c_{\mathbf{k}}^{\sigma,\sigma}(t)|^2}{|b_{\mathbf{k}}^{\sigma,\sigma}(t)+A_{\mathbf{k}}^{\sigma,\sigma,-\sigma}(t)\,b_{\mathbf{k}}^{-\sigma,\sigma}(t)/|c_{\mathbf{k}}^{\sigma,\sigma}(t)|^2|}\right]^{1/2},$$

$$|c_{\mathbf{k}}^{\sigma,\sigma}(t)|^2\equiv A_{\mathbf{k}}^{\sigma,\sigma,\sigma}(t)=1+\sigma P_{\mathbf{k}}^{\sigma,\sigma}(t)+\frac{|A_{\mathbf{k}}^{-\sigma,\sigma,-\sigma}(t)|^2}{A_{\mathbf{k}}^{-\sigma,-\sigma,-\sigma}(t)},$$

$$c_{\mathbf{k}}^{-\sigma,\sigma}(t)^*=A_{\mathbf{k}}^{-\sigma,\sigma,-\sigma}(t)/c_{\mathbf{k}}^{\sigma,-\sigma}(t); \quad \text{(A19)}$$

$$e_{\mathbf{k}}^{\sigma,\sigma}(t)=c_{\mathbf{k}}^{\sigma,\sigma}(t)\,b_{\mathbf{k}}^{\sigma,\sigma}(t)+c_{\mathbf{k}}^{\sigma,-\sigma}(t)\,b_{\mathbf{k}}^{-\sigma,\sigma}(t),$$
$$e_{\mathbf{k}}^{-\sigma,\sigma}(t)=c_{\mathbf{k}}^{-\sigma,-\sigma}(t)\,b_{\mathbf{k}}^{-\sigma,\sigma}(t)+c_{\mathbf{k}}^{-\sigma,\sigma}(t)\,b_{\mathbf{k}}^{\sigma,\sigma}(t).$$

In conjunction with (A8), (A10), and (A16) they are easily solved by successive approximations in the parameter $(T\Delta\omega)^{-1}$. The zeroth-approximation equations are

$$^{(0)}\omega_{\mathbf{k}}{}^{\sigma}=\sigma\omega_{0\mathbf{k}}+\sigma\operatorname{Re}\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)-\frac{1}{2}\frac{\partial}{\partial\omega}|\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)|^2 +\frac{\sigma|\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,-\sigma}(t)|^2}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}\bigg|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^3}{(\Delta\omega)^2}\right);$$

$$^{(0)}c_{\mathbf{k}}^{\sigma,\sigma}(t)^*=1+\frac{1}{2}\sigma\frac{\partial}{\partial\omega}\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)\big|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^2}{(\Delta\omega)^2}\right);$$

$$^{(0)}e_{\mathbf{k}}^{\sigma,\sigma}(t)={}^{(0)}c_{\mathbf{k}}^{\sigma,\sigma}(t)^*+O\left(\frac{\gamma^2}{(\Delta\omega)^2}\right); \quad \text{(A20)}$$

$$^{(0)}c_{\mathbf{k}}^{-\sigma,\sigma}(t)^*=-\frac{\tilde{\Sigma}_{\mathbf{k},\tilde{\omega}_{\mathbf{k}}{}^{\sigma}}^{\sigma,-\sigma}(t)}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}+O\left(\frac{\gamma^2}{\omega_{\mathbf{k}}\Delta\omega}\right);$$

$$^{(0)}e_{\mathbf{k}}^{-\sigma,\sigma}(t)=-\frac{\tilde{\Sigma}_{\mathbf{k},\tilde{\omega}_{\mathbf{k}}{}^{\sigma}}^{-\sigma,\sigma}(t)}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}+O\left(\frac{\gamma^2}{\omega_{\mathbf{k}}\Delta\omega}\right).$$

The increments linear in the parameter $(T\Delta\omega)^{-1}$ turn out to be:

$$^{(1)}\lambda_{\mathbf{k}}{}^{\sigma}(t)=O\left(\frac{\gamma^2}{T(\Delta\omega)^2}\right),$$

$$^{(1)}c_{\mathbf{k}}^{\sigma,\sigma}(t)^*=\frac{\sigma}{8}\frac{\partial^3}{\partial t\partial\omega^2}\operatorname{Im}\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)\bigg|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^2}{T(\Delta\omega)^3}\right),$$

$$^{(1)}c_{\mathbf{k}}^{-\sigma,\sigma}(t)^*=\frac{i}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\partial\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,-\sigma}(t)}{\partial\omega}-\sigma\frac{\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,-\sigma}(t)}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}\right]\bigg|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^2}{T(\Delta\omega)^3}\right),$$

$$^{(1)}e_{\mathbf{k}}^{\sigma,\sigma}(t)=-\frac{\sigma}{8}\frac{\partial^3}{\partial t\partial\omega^2}\operatorname{Im}\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma}(t)\big|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^2}{T(\Delta\omega)^3}\right),$$

$$^{(1)}e_{\mathbf{k}}^{-\sigma,\sigma}(t)=\frac{i}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}\frac{\partial}{\partial t}\left[-\frac{1}{2}\frac{\partial\tilde{\Sigma}_{\mathbf{k},\omega}^{-\sigma,\sigma}(t)}{\partial\omega}+\sigma\frac{\tilde{\Sigma}_{\mathbf{k},\omega}^{-\sigma,\sigma}(t)}{\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}}\right]\bigg|_{\omega=\tilde{\omega}_{\mathbf{k}}{}^{\sigma}(t)}+O\left(\frac{\gamma^2}{T(\Delta\omega)^3}\right). \quad \text{(A21)}$$

Knowing $\lambda_{\mathbf{k}}^{\sigma}(t)$, $e_{\mathbf{k}}^{s,\sigma}$, and $c_{\mathbf{k}}^{s,\sigma}(t)$, we can calculate the collision integral directly in terms of the modified functions. The expression, needed for such a calculation, of the modified beat Green function $\delta\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')$ in terms of $\hat{\Sigma}_{\mathbf{k}}$ is obtained from the known result of the action exerted on $\delta\hat{\tilde{G}}_{\mathbf{k}}$ by the operator $\hat{\tilde{L}}_{\mathbf{k}}$ which is the inverse of $\hat{\tilde{G}}_{\mathbf{k}}$. The matrix elements of the operator $\hat{\tilde{L}}_{\mathbf{k}}$ are connected with the matrix elements of the operator $\hat{L}_{\mathbf{k}}$ by the transformation

$$\tilde{L}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')=\sum_{s,s'}c_{\mathbf{k}}^{s,\sigma}(t)^*L_{\mathbf{k}}^{s,s'}(t,t')\,e_{\mathbf{k}}^{s',\sigma'}(t') \quad \text{(A22)}$$

and can be written in the form

$$\tilde{L}_{\mathbf{k}}^{\sigma,\sigma'}(t,t')=\left\{[\sigma\delta_{\sigma,\sigma'}+S_{\mathbf{k}}^{\sigma,\sigma'}(t)]\left[i\frac{\partial}{\partial t}-\lambda_{\mathbf{k}}{}^{\sigma'}(t)\right]+\bar{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t)\right\}\delta(t-t')-\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t'), \quad \text{(A23)}$$

$$\bar{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t)\equiv\int dt_1\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma_1}(t,t_1)\exp\left[i\int_{t_1}^{t}d\tilde{t}\,\lambda_{\mathbf{k}}{}^{\sigma'}(\tilde{t})\right] =\left[1+\frac{i}{2}\left(\frac{\partial}{\partial t}+\frac{\partial\lambda_{\mathbf{k}}{}^{\sigma}}{\partial t}\frac{\partial}{\partial\omega}\right)\frac{\partial}{\partial\omega}+O\left(\frac{1}{(T\Delta\omega)^2}\right)\right]\bar{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)\bigg|_{\omega=\lambda_{\mathbf{k}}{}^{\sigma'}(t)}.$$

The Fourier transform $\bar{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)$ of the function $\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t+\tau/2,t-\tau/2)$ with respect to τ is connected with $\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)$ by the relation

$$\bar{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)=\sum_{\sigma_1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\frac{\partial^n\tilde{\Sigma}_{\mathbf{k}\omega}^{\sigma,\sigma_1}(t)}{\partial\omega^n}\frac{\partial^n b_{\mathbf{k}}^{\sigma_1,\sigma'}(t)}{\partial t^n}.$$

With the above definition of the natural-oscillation Green's function that satisfies the equation

$$\hat{\tilde{L}}_{\mathbf{k}}\delta\hat{\tilde{G}}_{\mathbf{k}}=\hat{1}-\hat{\tilde{L}}_{\mathbf{k}}\hat{g}_{\mathbf{k}} \quad \text{(A24)}$$

the function $\delta G_{\mathbf{k}}^{\sigma,\sigma'}(t,t')$ attenuates with a decrement of order $\Delta\omega$ as the difference $t-t'$ is increased. For the Fourier transform of $\delta\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t-\tau)$ with respect to τ:

$$\delta\tilde{G}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)\equiv\int_0^{\infty}d\tau\,\delta\tilde{G}_{\mathbf{k}}^{\sigma,\sigma'}(t,t-\tau)\,e^{i\omega\tau} \quad \text{(A25)}$$

which depends smoothly on t and ω for all real values of ω, we obtain from (A24) the equation

$$\sum_{\sigma_1}\left\{[\sigma\delta_{\sigma,\sigma_1}+S_{\mathbf{k}}^{\sigma,\sigma_1}(t)]\left[i\frac{\partial}{\partial t}+\omega-\lambda_{\mathbf{k}}^{\sigma_1}(t)\right]+\bar{\Sigma}_{\mathbf{k}}^{\sigma,\sigma_1}(t)-\sum_{n=0}^{\infty}\frac{i^n}{n!}\frac{\partial^n{}_-\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma_1}}{\partial\omega^n}\frac{\partial^n}{\partial t^n}\right\}\delta\tilde{G}_{\mathbf{k},\omega}^{\sigma_1,\sigma'}(t) =\sigma'[S_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)-S_{\mathbf{k}}^{\sigma,\sigma'}(t)]. \quad \text{(A26)}$$

We have introduced here the notation

$${}_-\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)\equiv\int_0^{\infty}d\tau\,\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t-\tau)\,e^{i\omega\tau} =\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\frac{\partial^{2n}}{\partial\omega^n\partial t^n}\tilde{\Sigma}_{\mathbf{k},\omega}^{\sigma,\sigma'}(t),$$

$$S_{\mathbf{k},\omega}^{\sigma,\sigma'}(t)\equiv i\int_0^{\infty}d\tau_1\int_0^{\tau_1}d\tau_2\tilde{\Sigma}_{\mathbf{k}}^{\sigma,\sigma'}(t,t-\tau_1)\times\exp\left[i\omega\tau_2+i\int_{\tau_2}^{\tau_1}d\tilde{\tau}\lambda_{\mathbf{k}}{}^{\sigma'}(t-\tilde{\tau})\right]$$

$$= \left\{ 1 + \frac{i}{2} \left(\frac{\partial}{\partial \omega} + \frac{\partial}{\partial \omega'} \right) \left(\frac{\partial}{\partial t} + \frac{\partial \lambda_{\mathbf{k}}^{\sigma'}}{\partial t} \frac{\partial}{\partial \omega'} \right) + \frac{i}{2} \frac{\partial \lambda_{\mathbf{k}}^{\sigma'}}{\partial t} \frac{\partial^2}{\partial \omega \partial \omega'} + O\left(\frac{1}{(T\Delta\omega)^2} \right) \right\} \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\omega'}(t) \Big|_{\omega'=\lambda_{\mathbf{k}}^{\sigma'}(t)}, \qquad (A27)$$

$$\bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\omega'}(t) \equiv \frac{\bar{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) - \bar{\Sigma}^{\sigma,\sigma'}_{\mathbf{k},\omega'}(t)}{\omega - \omega'}.$$

In the zeroth approximation in the parameter $(T\Delta\omega)^{-1}$ Eq. (A26) reduces to

$$\sum_{\sigma_1} \tilde{L}^{\sigma,\sigma_1}_{\mathbf{k},\omega}(t)\, {}^{(0)}\delta G^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) = \sigma'(\omega - \lambda_{\mathbf{k}}^{\sigma}(t))\, T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t). \qquad (A28)$$

The functions $\tilde{L}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$ and $T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$ in this equation are expressed in terms of $\bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\omega'}(t)$ by the relations

$$\tilde{L}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) = \sigma(\omega - \lambda_{\mathbf{k}}^{\sigma}(t))\,\delta_{\sigma,\sigma'} - (\omega - \lambda_{\mathbf{k}}^{\sigma}(t))(\omega - \lambda_{\mathbf{k}}^{\sigma'}(t))\, T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t),$$

$$T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) \equiv \frac{\bar{S}^{\sigma,\sigma'}_{\mathbf{k},\lambda,\omega}(t) - \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\lambda,\lambda_1}(t)}{\omega - \lambda_{\mathbf{k}}^{\sigma'}(t)} = \frac{\bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\lambda}(t) - \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\lambda_1}(t)}{\lambda_{\mathbf{k}}^{\sigma} - \lambda_{\mathbf{k}}^{\sigma'}}, \qquad (A29)$$

in the subscripts of which $\lambda \equiv \lambda_{\mathbf{k}}^{\sigma}$ and $\lambda_1 \equiv \lambda_{\mathbf{k}}^{\sigma'}$. This form is convenient for inversion of $\hat{\tilde{L}}_{\mathbf{k},\omega}(t)$ and for the calculation of the modified beat Green function:

$${}^{(0)}\delta G^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) = [\sigma\sigma' T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) + \sigma(\omega - \lambda_{\mathbf{k}}^{-\sigma}(t))\,\delta_{\sigma,\sigma'} \det \| T_{\mathbf{k},\omega}(t) \|]/Q_{\mathbf{k},\omega}(t),$$

$$Q_{\mathbf{k},\omega}(t) = 1 + (\omega - \lambda_{\mathbf{k}}^{-}(t))\, T^{-,-}_{\mathbf{k},\omega}(t) - (\omega - \lambda_{\mathbf{k}}^{+}(t))\, T^{+,+}_{\mathbf{k},\omega}(t) - (\omega - \lambda_{\mathbf{k}}^{-}(t))(\omega - \lambda_{\mathbf{k}}^{+}(t)) \det \| T_{\mathbf{k},\omega}(t) \|,$$

$$\det \| T_{\mathbf{k},\omega}(t) \| = T^{+,+}_{\mathbf{k},\omega}(t)\, T^{-,-}_{\mathbf{k},\omega}(t) - T^{+,-}_{\mathbf{k},\omega}(t)\, T^{-,+}_{\mathbf{k},\omega}(t). \qquad (A30)$$

As seen from (A30), the beat Green function is not only smoothly frequency-dependent on the real ω axis, but is also uniformly small there in the parameter $\gamma/(\Delta\omega)$:

$$| {}^{(0)}\delta \tilde{G}^{\sigma,\sigma}_{\mathbf{k},\omega} | \lesssim \frac{1}{\Delta\omega} \frac{\gamma}{\Delta\omega}, \qquad | {}^{(0)}\delta \tilde{G}^{\sigma,-\sigma}_{\mathbf{k},\omega} | \lesssim \frac{1}{\omega_{\mathbf{k}} + \omega_{-\mathbf{k}}} \frac{\gamma}{\Delta\omega}.$$

The corrections that distinguish $\delta \tilde{G}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$ from ${}^{(0)}\delta \tilde{G}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t)$ can be obtained by iterations. In particular, the correction linear in the parameter $(T\Delta\omega)^{-1}$ is:

$${}^{(1)}\delta \tilde{G}^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) = i\sigma\sigma' \frac{\partial}{\partial t} \left\{ [\omega - \lambda_{\mathbf{k}}^{\sigma}(t)]^{-1} \left[\frac{1}{2} \left(\frac{\partial}{\partial \omega} + \frac{\partial}{\partial \lambda'} \right) \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\omega,\lambda'}(t) - \frac{1}{2} \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \lambda'} \right) \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\lambda,\lambda'}(t) + \frac{\partial}{\partial \lambda} \bar{S}^{\sigma,\sigma'}_{\mathbf{k},\lambda,\lambda'}(t) - T^{\sigma,\sigma'}_{\mathbf{k},\omega}(t) \right] \right\} \Big|_{\lambda=\lambda_{\mathbf{k}}^{\sigma}(t),\, \lambda'=\lambda_{\mathbf{k}}^{\sigma'}(t)} + O\left(\frac{\gamma\, {}^{(0)}\delta G^{\sigma,\sigma'}_{\mathbf{k},\omega}}{T(\Delta\omega)^2} \right). \qquad (A31)$$

As it should, it is a smooth function of the frequency on the real ω axis.

[1)]All energies are presumed to be positive and no waves can be generated by an unperturbed medium.

[2)]The problem of correctly calculating the collision integral is acute also for wave-particle interaction. This very problem has raised the prolonged discussion concerning the validity of the so-called quasilinear equations that appear when the first nonvanishing term is retained in the expansion of the quasiparticle–particle collision integral (see, e.g., Ref. 7 and the citations therein).

[3)]When account is taken in the equation for $\hat{N}$ of a small term proportional to the correlator of the random force *f*, a similar correction appears in (3.14) and is due to the weak damping of the natural modes of the linear problem, i.e., to the presence of a small anti-Hermitian part of the operator ${}^{0}\hat{G}^{-1}$.

[4)]Such a transformation of the field variables is generally speaking not canonical [see Eq. (A20)].

[5)]This can be done in view of the smooth dependence of the pre-exponential factors on the oscillation frequencies, provided only that the Jacobian of the transition to the new variables has no singularities. The latter cannot be eliminated near the thresholds of appearance of new interaction channels. The kinetic equation given below is therefore, strictly speaking, inapplicable to oscillations with near-threshold wave vectors.

[1]B. B. Kadomtsev, Plasma Turbulence, in: *Reviews of Plasma Physics*, M. A. Leontovich, ed., Vol. 4, Plenum (1968).

[2]A. A. Galeev and R. Z. Sagdeev, *ibid.*, Vol. 7 (1979).

[3]V. E. Zakharov, Izv. Vyssh. Ucheb. Zaved. Radiofizika **17**, 431 (1974).

[4]V. M. Malkin, Fiz. Plazmy **8**, 357 (1982) [Sov. J. Plasma Phys. **8**, 202 (1982)]. Preprint No. 158, Nucl. Phys. Inst., Novosibirsk, 1980.

[5]V. M. Malkin, Zh. Eksp. Teor. Fiz. **83**, 88 (1982) [Sov. Phys. JETP **56**, 48 (1982)].

[6]J. A. Krommes, in: *Basic Plasma Physics*, North Holland, 1985, Vol. II.

[7]G. Laval and D. Pesme, Phys. Fluids **26**, 52 (1982).

[8]V. E. Zakharov and V. S. L'vov, Izv. Vyssh. Ucheb. Zaved. Radiofizika **18**, 147 (1975).

[9]H. W. Wyld, Ann. Phys. (N.Y.) **14**, 143 (1961).

[10]A. A. Vedenov, in: *Reviews of Plasma Physics*, Plenum (1967), Vol. 3.

[11]B. I. Sturman, Usp. Fix. Nauk **144**, 497 (1984) [Sov. Phys. Usp. **27**, 881 (1984)].

[12]V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, *ibid.* **114**, 609 (1974) [**17**, 896 (1975)].

[13]V. S. L'vov, Zh. Eksp. Teor. Fiz. **69**, 2079 (1975) [Sov. Phys. JETP **42**, 1057 (1975)]. Preprint No. 190, Inst. of Automation and Electrometry, USSR Acad. Sci., Novosibirsk, 1982.

[14]L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon, 1969. § 125.

[15]V. S. L'vov and A. M. Rubenchik, Zh. Eksp. Teor. Fiz. **72**, 127 (1977) [Sov. Phys. JETP **45**, 67 (1977)].

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