Coherent backscattering enhancement under conditions of weak wave localization in disordered 3D and 2D systems

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An exact analytic solution is found for the problem of the weak localization of waves in semi-infinite disordered $3D$ and $2D$ systems with centers which scatter isotropically. Expressions derived for the angular distributions of the backscattered radiation are valid for an arbitrary angle of incidence of the photon flux on the surface of the medium and for an arbitrary relation between the elastic and inelastic cross sections. The angular distribution near the coherent-backscattering peak differs from the prediction of the diffusion theory. The intensity in the wings of the peak falls off monotonically with increasing angular deviation (ϑ) from the "exactly backward" direction, in a $1/\vartheta$ manner, in both the $3D$ and the $2D$ cases.

Interest in research on the coherent enhancement of the backscattering of waves, which yields important information on the fundamental properties of disordered media, has recently increased sharply.[1–10] As we know, the sharp peak in the angular distribution of backscattered waves in the "exactly backward" direction, which was predicted some time ago,[11–15] is a manifestation of a more general effect: a weak localization of waves in the course of multiple scattering by an ensemble of disordered centers.[16–19] The enhancement of the backscattering is of the same nature as the quantum-mechanical corrections to the electron kinetic coefficients in metals and semiconductors with impurities[16–19] and also in dense gases.[20] The weak-localization effect stems from an interference of waves which have undergone successive scattering by the same centers but which are propagating in opposite directions. The effect is a consequence of the symmetry of the scattering processes under time reversal.[19–21]

Since the first experimental observation of coherent backscattering enhancement,[22–24] there have been a number of studies carried out to observe weak localization of light in disordered media with scattering particles of submicron size.[1–4,10,25,26] Just recently, backscattering enhancement has been observed in $2D$ systems[8,27] and liquid crystals.[9]

The active experimental research has stimulated theoretical papers analyzing the coherent enhancement of backscattering from disordered media under weak-localization conditions.[3,5–7,28–32] The corresponding theory is based on the calculation of the intensity component from the so-called fan diagrams (or "most-crossed" or "cyclic" diagrams). The summation of these diagrams can be reduced to the problem of solving radiation transport equations, as was shown by Barabanenkov.[15]

Numerous papers have been devoted to calculations of the angular distribution of the scattering.[3,5–7,15,28–34] Even in the simplest case, of the scattering of scalar waves by a system of small-scale centers (with a size smaller than the wavelength), however, the calculations are only approximate.[1)] The calculations are based on either the diffusion approximation[3,5,15,21,29–32] or the incorporation of single and double scattering events.[28] The use of the diffusion approximation runs into the problem of boundary conditions,[7,35,36] as we know, and it assumes that the intensity is dominated by high-multiplicity collisions. Analysis shows that calculations based on the diffusion approximation cannot claim to give a correct description of the line shape of the backscattering peak, particularly in the wings of the line, or of the dependence of the enhancement factor for scattering exactly backward on the angle of incidence of the initial flux on the surface of the medium. It is thus necessary to resort to numerical methods in order to calculate the backscattering angular distribution. In Refs. 3 and 28, for example, a direct numerical integration of the transport equation was carried out for a $3D$ system of point scatterers. As we will show below, however, the problem of the weak localization of waves in a semi-infinite medium with centers which scatter isotropically can be solved analytically in both the $2D$ and $3D$ cases.

Below we given an exact analytic solution of the problem of weak localization of waves in semi-infinite disordered $3D$ and $2D$ systems with centers which scatter isotropically. We derive corresponding expressions for the angular distribution of the backscattered waves for arbitrary angles of incidence of the radiation flux on the surface of the medium and for an arbitrary relation between the cross sections for absorption and elastic scattering by an individual center. We calculate the enhancement factors for scattering exactly backward; in particular, we find the limiting values of this factor for $3D$ and $2D$ systems. We analyze the shape of the angular distribution near the backscattering peak. We show that the backscattering intensity in the wings of the angular distribution falls off with the angular deviation (ϑ) from the direction exactly backward, in accordance with $\propto 1/\vartheta$, in both the $2D$ and $3D$ cases. The components of the interference correction to the total albedo of the medium coming from various scattering orders are estimated. It is shown that in a $3D$ system the correction is determined primarily by double scattering, while in a $2D$ system the components of the correction from double scattering and from scattering processes of higher multiplicity are comparable in magnitude [their ratio is $\sim \ln(\lambda/l)$, where λ is the wavelength, and $l(\lambda \ll l)$ is the mean free path].

1. GENERAL RELATIONS

We consider the scattering of a plane wave incident on a system of randomly arranged small-scale centers (smaller than the wavelength). We assume that there is absolutely no correlation in the arrangement of the centers and that the scattering by each of them is isotropic.

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The problem of calculating the angular distribution and other characteristics of the scattered radiation reduces to one of finding the Green's function of the scattering problem and the mutual-coherence function (density matrix) of the wave field, both averaged over the positions of the centers:

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = \langle \psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\rangle. \qquad (1)$$

Under conditions such that the radiation wavelength λ is smaller by a large factor than l, the mean free path between successive collisions in the medium, the Green's function of the scattering problem, averaged over the positions of the centers, satisfies the wave equation[36–38]

$$\left(\frac{\partial^2}{\partial \mathbf{r}^2} + k_0^2\right)G(\mathbf{r},\mathbf{r}') - n\int d\mathbf{R}_a \int d\mathbf{r}''\langle \mathbf{r}|\hat{\mathscr{T}}_a|\mathbf{r}''\rangle G(\mathbf{r}'',\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad (2)$$

where $k_0 = 2\pi/\lambda$, n is the number of scattering centers per unit volume, and $\hat{\mathscr{T}}_a$ is the scattering matrix for the scattering by a center at the point $\mathbf{R}_a$. In the case of small-scale inhomogeneities, the matrix element $\langle \mathbf{r}|\hat{\mathscr{T}}_a|\mathbf{r}'\rangle$ can be written in the form

$$\langle \mathbf{r}|\hat{\mathscr{T}}_a|\mathbf{r}'\rangle = -4\pi f\delta(\mathbf{r}-\mathbf{R}_a)\delta(\mathbf{r}'-\mathbf{R}_a), \qquad (3)$$

where f is the scattering amplitude. The corresponding cross section for elastic scattering by the center is $\sigma_{\rm el} = 4\pi|f|^2$, the total interaction cross section is $\sigma_{\rm tot} = (4\pi/k_0)\,{\rm Im}\, f$, and the absorption cross section is $\sigma_a = \sigma_{\rm tot} - \sigma_{\rm el}$.

Using (3), we can put Eq. (2) in the following form[2)]:

$$[\partial^2/\partial\mathbf{r}^2 + k_0^2 + 4\pi n f\theta(\mathbf{r})]G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad (4)$$

where the function $\theta(\mathbf{r})$ is equal to unity inside the scattering medium and zero outside it. The solution of Eq. (2) [or (4)] describes the field distribution in a spherical wave which is propagating away from a radiation source at the point $\mathbf{r}'$. If the source is withdrawn to $z' = -\infty$ (this case corresponds to the incidence of a plane wave on the surface of the medium) the distribution of the average field is described by an equation of the same type as (4),

$$[\partial^2/\partial\mathbf{r}^2 + k_0^2 + 4\pi n f\theta(\mathbf{r})]\psi_0(\mathbf{r}) = 0, \qquad (5)$$

but with the boundary condition

$$\psi_{\rm inc}(\mathbf{r})|_{z=-\infty} = \exp(i\mathbf{k}_0\mathbf{r}). \qquad (6)$$

The mutual-coherence function of wave field (1) can be put in the form[38]

$$\rho(\mathbf{r}_1,\mathbf{r}_2) = \rho_0(\mathbf{r}_1,\mathbf{r}_2) + \int d\mathbf{R}_1\, d\mathbf{R}_2\, d\mathbf{r}_1'\, d\mathbf{r}_2' \times G(\mathbf{r}_1,\mathbf{R}_1)G^*(\mathbf{r}_2,\mathbf{R}_2)\Gamma(\mathbf{R}_1,\mathbf{r}_1';\mathbf{R}_2,\mathbf{r}_2')\rho_0(\mathbf{r}_1',\mathbf{r}_2'), \qquad (7)$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function of the scattering problem, which satisfies Eq. (4), and $\rho_0(\mathbf{r}_1,\mathbf{r}_2)$ is the mutual-coherence function of waves which have not undergone incoherent scattering in the medium. Far from the boundary of the medium, the function $\rho_0(\mathbf{r}_1,\mathbf{r}_2)$ describes a superposition of the incident wave and of a wave reflected coherently from the surface. In the case in which a plane wave is incident on the medium from $z = -\infty$, the function $\rho_0(\mathbf{r}_1,\mathbf{r}_2)$ can be written as the product

$$\rho_0(\mathbf{r}_1,\mathbf{r}_2) = \psi_0(\mathbf{r}_1)\psi_0^*(\mathbf{r}_2), \qquad (8)$$

where $\psi_0(\mathbf{r})$ satisfies Eq. (5) with boundary condition (6).

According to definition (7), the matrix $\hat{\Gamma}$ describes the evolution of the coherent field during multiple scattering in the medium. In terms of the ordinary diagram technique, $\hat{\Gamma}$ is determined by the sum of all coupled diagrams without the incoming and outgoing lines which correspond to the Green's functions.[38,39]

Under the condition $\lambda \ll l = (n\sigma_{\rm tot})^{-1}$, $\hat{\Gamma}$ is dominated by the sum $\hat{L}$ of ladder diagrams[38,39] (Fig. 1). The series of ladder diagrams correspond to the incoherent summation of waves during successive, independent scattering events. The summation of the series of ladder diagrams leads to an integral equation (Fig. 1). In the coordinate representation, this equation can be written

$$L(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2') = n\int d\mathbf{R}_a \langle \mathbf{r}_1|\hat{\mathscr{T}}_a|\mathbf{r}_1'\rangle\langle \mathbf{r}_2|\hat{\mathscr{T}}_a^+|\mathbf{r}_2'\rangle + n\int d\mathbf{R}_a \int d\mathbf{R}_1 d\mathbf{R}_1' d\mathbf{R}_2 d\mathbf{R}_2' \langle \mathbf{r}_1|\hat{\mathscr{T}}_a|\mathbf{R}_1\rangle \times \langle \mathbf{r}_2|\hat{\mathscr{T}}_a^+|\mathbf{R}_2\rangle G(\mathbf{R}_1,\mathbf{R}_1')G^*(\mathbf{R}_2,\mathbf{R}_2') \times L(\mathbf{R}_1',\mathbf{r}_1';\mathbf{R}_2',\mathbf{r}_2').$$

The function $L(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2')$ is the Green's function of an ordinary kinetic equation for the mutual-coherence function (density matrix), which describes multiple and coherent scattering of waves in the medium.

Substituting expression (3) for the matrix element $\langle \mathbf{r}|\hat{\mathscr{T}}_a|\mathbf{r}'\rangle$, into the equation which we have derived we find

$$L(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2') = (4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2')\delta(\mathbf{r}_1-\mathbf{r}_2) + (4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_2)\iint d\mathbf{R}_1'\, d\mathbf{R}_2' G(\mathbf{r}_1,\mathbf{R}_1')G^*(\mathbf{r}_2,\mathbf{R}_2') \times L(\mathbf{R}_1',\mathbf{r}_1';\mathbf{R}_2',\mathbf{r}_2'). \qquad (9)$$

It is not difficult to see that the solution of Eq. (9) can be written in the form

$$L(\mathbf{r}_1,\mathbf{r}_1',\mathbf{r}_2,\mathbf{r}_2') = (4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1'-\mathbf{r}_2')[\delta(\mathbf{r}_1-\mathbf{r}_1') + F(\mathbf{r}_1,\mathbf{r}_1')], \qquad (10)$$

where $F(\mathbf{r},\mathbf{r}')$ satisfies the equation

$$F(\mathbf{r},\mathbf{r}') = (4\pi)^2|f|^2 n|G(\mathbf{r},\mathbf{r}')|^2 + (4\pi)^2|f|^2 n\int d\mathbf{r}''|G(\mathbf{r},\mathbf{r}'')|^2 F(\mathbf{r}'',\mathbf{r}'). \qquad (11)$$

Equation (11) is the same as the well-known equation of transport theory which describes the spatial distribution of the energy density of incoherently scattered radiation (the density of particles or photons) from an isotropic point source of unit intensity.

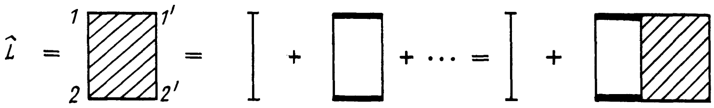

FIG. 1.

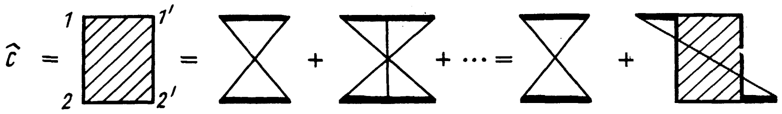

FIG. 2.

We know[15] that under conditions of weak localization, with $\lambda \ll l$, a calculation of the angular distribution of the waves scattered almost exactly backwards cannot be restricted to only the series of ladder diagrams in the expression for $\hat{\Gamma}$. The series of so-called fan diagrams[15,19] (Fig. 2) makes a contribution of the same order of magnitude as that of the ladder diagrams to the intensity of the radiation which is reflected into a narrow angular cone $\vartheta \lesssim \lambda / l$ around the backward direction. These fan diagrams are sometimes also called "most-crossed" or "cyclic" diagrams. They describe an interference of the waves which are scattered successively by the same inhomogeneities but which pass them in opposite directions.

The summation of the series of fan diagrams leads to an integral equation which is shown schematically in Fig. 2. In the coordinate representation, the integral equation for $C(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2')$ can be written as follows, where (3) has been taken into account:

$$\begin{aligned} C(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2') &= (4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_2')\delta(\mathbf{r}_2-\mathbf{r}_1')G(\mathbf{r}_1,\mathbf{r}_1')G^*(\mathbf{r}_2,\mathbf{r}_2') \\ &+(4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_2')\int d\mathbf{R}_1'\, d\mathbf{R}_2' G(\mathbf{r}_1,\mathbf{R}_1') \\ &\times G^*(\mathbf{R}_2',\mathbf{r}_2)C(\mathbf{R}_1',\mathbf{r}_1';\mathbf{r}_2,\mathbf{R}_2'). \end{aligned} \tag{12}$$

Comparing this equation with (9), we easily see that $C(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2')$ is the same as the integral term in (9) if we interchange $\mathbf{r}_2$ and $\mathbf{r}_2'$ in it. Consequently, the sum of fan diagrams $C(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2')$ can be expressed in terms of the function $F(\mathbf{r},\mathbf{r}')$ which we introduced above:

$$C(\mathbf{r}_1,\mathbf{r}_1';\mathbf{r}_2,\mathbf{r}_2')=(4\pi)^2|f|^2 n\delta(\mathbf{r}_1-\mathbf{r}_2')\delta(\mathbf{r}_2-\mathbf{r}_1')F(\mathbf{r}_1,\mathbf{r}_1'). \tag{13}$$

Now summing the components of the $\hat{\Gamma}$ matrix due to the ladder and fan diagrams, $\hat{\Gamma} = \hat{L} + \hat{C}$, we finally find the following expression for the mutual-coherence function of the waves under weak-localization conditions in accordance with (7):

$$\begin{aligned} \rho(\mathbf{r}_1,\mathbf{r}_2) &= \rho_0(\mathbf{r}_1,\mathbf{r}_2)+(4\pi)^2|f|^2 n \\ &\times \int d\mathbf{R}\, G(\mathbf{r}_1,\mathbf{R})G^*(\mathbf{r}_2,\mathbf{R})\rho_0(\mathbf{R},\mathbf{R}) \\ &+(4\pi)^2|f|^2 n\int d\mathbf{r}_1'\, d\mathbf{r}_2'\, d\mathbf{r}_3'\, d\mathbf{r}_4' G(\mathbf{r}_1,\mathbf{r}_1')G^*(\mathbf{r}_2,\mathbf{r}_2')F(\mathbf{r}_1',\mathbf{r}_2') \\ &\times [\delta(\mathbf{r}_1'-\mathbf{r}_3')\delta(\mathbf{r}_2'-\mathbf{r}_4')+\delta(\mathbf{r}_1'-\mathbf{r}_4')\delta(\mathbf{r}_2'-\mathbf{r}_3')]\rho_0(\mathbf{r}_3',\mathbf{r}_4'). \end{aligned} \tag{14}$$

The integration in (14) is carried out over the volume occupied by the scattering medium. The first term in (14) describes the unscattered radiation, the second the singly scattered radiation, and the third the multiple scattering.

The angular spectrum of the backscattered waves (the radiation flux crossing a unit area of the surface in the given direction) is determined, as we know, by the value of the Fourier transform of the mutual-coherence function in a plane an infinite distance from the boundary of the medium $(z_1 = z_2 = -\infty)$:

$$J(\mu,\mu_0,\varphi)=J(\mathbf{k},\mathbf{k}_0)=\frac{k_z^2}{(2\pi)^2}\frac{1}{\Sigma}\rho(\mathbf{k}_\parallel,z;\mathbf{k}_\parallel,z)\,|_{z\to-\infty}, \tag{15}$$

where $\mu_0 = \cos\theta_0$, $\mu = \cos\theta$, θ_0 and θ are the angles between the inward normal to the surface of the medium and the propagation directions of the incident and scattered waves, φ is the azimuthal scattering angle,

$$k_z=k_0|\cos\theta|,\quad \mathbf{k}_\parallel=(k_x,k_y)=(k_0\sin\theta\cos\varphi,\ k_0\sin\theta\sin\varphi)$$

are the components of the wave vector of the scatter field which are respectively perpendicular and parallel to the surface, and Σ is the area of the surface.

Expression (15) was derived for the $3D$ case. For scattering of waves by a $2D$ system, the relationship between the angular distribution and the mutual-coherence function is slightly different:

$$J(\mu,\mu_0)=\frac{k_z^2}{2\pi k_0 L}\rho(k_\parallel,z;k_\parallel,z)\,|_{z\to-\infty}, \tag{16}$$

where $k_\parallel = k_0\sin\theta$, and L is the length of the interface.

Using (8), (15) [or (16)], and the reciprocity theorem (Ref. 34, for example), according to which we have

$$\int d\boldsymbol{\rho}\exp(-i\mathbf{k}_\parallel\boldsymbol{\rho})G(\boldsymbol{\rho},z,\boldsymbol{\rho}',z')\,|_{z\to-\infty}=\frac{\exp(-ik_z z)}{2ik_z}\psi_0(\mathbf{r}',\mathbf{k}_1), \tag{17}$$

where $\psi_0(\mathbf{r},\mathbf{k}_1)$ is a solution of Eq. (5) with the boundary condition $\psi_{max}|_{z\to-\infty} = \exp(i\mathbf{k},\mathbf{r})$, $\mathbf{k}_1 = (-\mathbf{k}_\parallel, k_z)$, we find the following expression for the angular distribution of the backscattered waves:

$$\begin{aligned} J(\mathbf{k},\mathbf{k}_0)=\frac{n|f|^2}{\Sigma}\Big\{ &\int d\mathbf{r}\,|\psi_0(\mathbf{r},\mathbf{k}_1)|^2|\psi_0(\mathbf{r},\mathbf{k}_0)|^2 \\ &+\int d\mathbf{r}\,d\mathbf{r}'\,|\psi_0(\mathbf{r},\mathbf{k}_1)|^2F(\mathbf{r},\mathbf{r}')\,|\psi_0(\mathbf{r}',\mathbf{k}_0)|^2 \\ &+\int d\mathbf{r}\,d\mathbf{r}'\,\psi_0(\mathbf{r},\mathbf{k}_1)\psi_0^*(\mathbf{r},\mathbf{k}_0)F(\mathbf{r},\mathbf{r}')\psi_0^*(\mathbf{r}',\mathbf{k}_1)\psi_0(\mathbf{r}',\mathbf{k}_0)\Big\} \end{aligned} \tag{18}$$

(in the $2D$ case, the area Σ in (18) must be replaced by $Lk_0/2\pi$).

The first term in expression (18) is the single-scattering component, the second comes from ordinary incoherent multiple scattering, and the third is the result of an interference of multiply scattered waves. It follows immediately from (18) that in the case of scattering exactly backward, $\mathbf{k}_\parallel = -\mathbf{k}_{0\parallel}$, the multiple incoherent scattering and the wave interference make equal contributions to the angular distributions; i.e., the intensity of the waves scattered multiply (with a multiplicity of at least two) turns out to be twice that which would follow from classical transport theory.

Under conditions of weak localization, a calculation of the angular distribution of the scattered waves reduces thus to the problem of solving Eq. (11) for a function which describes the spatial distribution of the energy density of the incoherently scattered radiation from an isotropic point source of unit intensity. As we will show below, it is possible to derive an exact analytic solution for this equation in the case of a semi-infinite disordered system. The solution of Eq. (11) and the corresponding calculations of the angular distribution are the subjects of the following sections of this paper.

The entire discussion above applies equally well to $3D$ and $2D$ disordered systems. However, in calculating the angular distribution of the radiation below we will be obliged to take account of the specific features of the scattering of waves in the systems of different dimensionality, so we will treat these two cases separately.

2. ANGULAR DISTRIBUTION OF THE BACKSCATTERING UNDER CONDITIONS OF WEAK WAVE LOCALIZATION IN A DISORDERED $3D$ SYSTEM

Let us assume that a disordered system of small-radius scatterers fills the half-space $z>0$. The solution of Eq. (5) with boundary condition (6) is then

$$\psi_0(\mathbf{r},\mathbf{k}_0)=\psi_0(z)\exp(i\mathbf{k}_{0\|}\boldsymbol{\rho}), \tag{19}$$

$$\psi_0(z)=\begin{cases}\exp(ik_{0z}z)+\dfrac{k_{0z}-\varkappa_{0z}}{k_{0z}+\varkappa_{0z}}\exp(-ik_{0z}z), & z<0,\\[2mm] \dfrac{2k_{0z}}{k_{0z}+\varkappa_{0z}}\exp(i\varkappa_{0z}z), & z>0,\end{cases}$$

where $\varkappa_{0z}^2=k_{0z}^2+4\pi nf=k_0^2\mu_0^2+4\pi nf$. Solution (19) describes the coherent field in the system consisting of the vacuum plus the disordered medium, with refraction and specular reflection of waves at the interface being taken into account. The expression for the field $\psi_0(\mathbf{r},\mathbf{k}_1)$ differs from (19) in the replacement of $\mathbf{k}_{0\|},k_{0z},\varkappa_{0z},\mu_0$ by $-\mathbf{k}_\|,k_z,\varkappa_z,|\mu|$.

Now substituting the expressions for the average fields into general relation (18), we find the following simple representation of the angular distribution $J(\mathbf{k},\mathbf{k}_0)$:

$$J(\mathbf{k},\mathbf{k}_0)=\frac{n\sigma_{el}}{4\pi}\frac{4k_{0z}^2}{|k_{0z}+\varkappa_{0z}|^2}\frac{4k_z^2}{|k_z+\varkappa_z|^2}\Bigg[\frac{1}{2\,\mathrm{Im}(\varkappa_{0z}+\varkappa_z)}+F^{(3D)}(0,2\,\mathrm{Im}\,\varkappa_z,2\,\mathrm{Im}\,\varkappa_{0z})+F^{(3D)}(|\mathbf{k}_\|+\mathbf{k}_{0\|}|,i(\varkappa_{0z}^*-\varkappa_z),i(\varkappa_z-\varkappa_{0z}^*))\Bigg], \tag{20}$$

where

$$F^{(3D)}(q,p,p')=2\pi\int_0^\infty\rho\,d\rho J_0(q\rho)\int_0^\infty\int_0^\infty dz\,dz'\exp(-pz-p'z')F^{(3D)}(\rho,z,z'), \tag{21}$$

and $J_0(q\rho)$ is a Bessel function of the index zero. Expressions (20) and (21) incorporate the circumstance that we have $F(\mathbf{r},\mathbf{r}')\equiv F(|\boldsymbol{\rho}-\boldsymbol{\rho}'|,z,z')$ by virtue of the symmetry of Eq. (11).

Under the condition $n|f|\lambda^2\ll1$, the jump in the effective dielectric constant at the vacuum-medium interface is small,[3)] and the coherent-reflection and refraction effects are manifested only at small depths $z\lesssim(n|f|)^{-1/2}$ in a narrow interval of glancing angles of the wave propagation: $|\mu|,\mu_0\lesssim\mu_c$, where μ_c is the cosine of the angle of total external reflection ($\mathrm{Re}f<0$) or of total internal reflection ($\mathrm{Re}f>0$), given by $\mu_c\sim(n|f|\lambda^2)^{1/2}\ll1$ (Ref. 40). Accordingly, in calculating $J(\mathbf{k},\mathbf{k}_0)$ we can use the same approximations as in Ref. 40; i.e., we can assume that the refraction and the coherent reflection affect only the transmission of the incident waves and of the backscattered waves across the interface, while having no effect on the process of multiple scattering inside the medium. In accordance with this model, in calculating the angular distribution of the backscattering we must take the coherent interaction with the medium into account only in the average field $\psi_0(\mathbf{r},\mathbf{k})$, which describes the transmission of the incident waves and of the backscattered waves across the interface. These effects can be ignored in the calculation of $F(\mathbf{r},\mathbf{r}')$. The corrections for the interaction with the interface to the solution of the equation for the radiation energy density (integrated over angles) turn out to be small, on the order of the ratio of the angular dimensions of the region in which the coherent reflection and refraction are important to those of the entire region of scattering angles, $(n|f|\lambda^2)^{1/2}\ll1$ (Ref. 40). In calculating the angular distribution under conditions of weak localization, these effects must be neglected as small effects of higher order.

We substitute accordingly the Green's function of an infinite medium into Eq. (11) $(z,z'>0)$. Under the condition $n|f|\lambda^2\ll1$ this Green's function is

$$G(|\mathbf{r}-\mathbf{r}'|)=-\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\exp\left[i\left(k_0+\frac{2\pi nf}{k_0}\right)|\mathbf{r}-\mathbf{r}'|\right]. \tag{22}$$

As a result we find an integral equation with a difference kernel for $F^{(3D)}(\mathbf{r},\mathbf{r}')$:

$$F^{(3D)}(\mathbf{r},\mathbf{r}')=\frac{n|f|^2}{|\mathbf{r}-\mathbf{r}'|^2}\exp(-n\sigma_{tot}|\mathbf{r}-\mathbf{r}'|)+n|f|^2\int d\mathbf{r}''\frac{\exp(-n\sigma_{tot}|\mathbf{r}-\mathbf{r}''|)}{|\mathbf{r}-\mathbf{r}''|^2}F^{(3D)}(\mathbf{r}'',\mathbf{r}'). \tag{23}$$

Different methods were used to solve Eq. (23) in Refs. 3, 5, 15, and 28–30: The approximation of single scattering [i.e., only the first term on the right side of (23) is taken into account],[28] the diffusion approximation,[3,5,15,29,30] and direct numerical integration.[3,28] In contrast with Refs. 3, 5, 15, and 28–30, we will find an exact analytic solution of this equation.

Using $F^{(3D)}(\mathbf{r},\mathbf{r}')\equiv F^{(3D)}(|\boldsymbol{\rho}-\boldsymbol{\rho}'|,z,z')$ and a Bessel transformation in the variable $|\rho-\rho'|$, we can put equation (23) in the form

$$F^{(3D)}(q,z,z')=K_q(|z-z'|)+\int_0^\infty dz''\,K_q(|z-z''|)F^{(3D)}(q,z'',z'), \tag{24}$$

where

$$F^{(3D)}(q,z,z')=2\pi\int_0^\infty\rho\,d\rho J_0(q\rho)F^{(3D)}(\rho,z,z'), \tag{25}$$

$$K_q(|z-z'|)=2\pi n|f|^2\int_0^\infty\rho\,d\rho\frac{J_0(q\rho)\exp\{-n\sigma_{tot}[\rho^2+(z-z')^2]^{1/2}\}}{\rho^2+(z-z')^2}. \tag{26}$$

Equations of the type in (24) arise frequently in transport theory and have been studied quite comprehensively.[41]

It is not difficult to show that the sum of the derivatives of the function $F^{(3D)}(q,z,z')$ can be written as the product[41]

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z'}\right)F^{(3D)}(q,z,z')=\Phi_q(z)\Phi_q(z'), \quad (27)$$

where

$$\Phi_q(z)=F^{(3D)}(q,z,0)=F^{(3D)}(q,0,z).$$

As a result, we find the following expression for the function $F^{(3D)}(q,p,p')$, in terms of which angular distribution (20) is expressed:

$$F^{(3D)}(q,p,p')=\frac{1}{p+p'}[\Phi_q(p)\Phi_q(p')+\Phi_q(p)+\Phi_q(p')]. \quad (28)$$

The problem thus reduces to one of finding the function $\Phi_q(z)$ (more precisely, its Laplace transform), which is a function of only a single variable. Setting $z'=0$ in (24), we find the following equation for $\Phi_q(z)$:

$$\Phi_q(z)=K_q(|z|)+\int_0^\infty dz' K_q(|z-z'|)\Phi_q(z'). \quad (29)$$

Equation (29) is a linear integral equation on a half-line, with a symmetric difference kernel. It can be solved by the standard Wiener-Hopf method.[42]

We know[42] that the construction of a solution by the Wiener-Hopf method requires that the auxiliary function $\Lambda(q_z)$, which is related to the Fourier transform of the kernel of the integral equation by

$$\Lambda(q_z)=1-K_q(q_z),$$

where

$$K_q(q_z)=\int_{-\infty}^{\infty} dz K_q(|z|)\exp(-iq_z z),$$

be analytic and have a bounded number of zeros in a certain finite band running parallel to the real axis

$$-\tau_-<\operatorname{Im} q_z<\tau_+, \quad \tau_-,\ \tau_+>0.$$

In our case, in which the kernel is determined by integral (26), the function $\Lambda(q_z)$ takes the simple form

$$\Lambda(q_z)=1-\omega\frac{\operatorname{arctg}[(q_z^2+q^2)^{1/2}/n\sigma_{tot}]}{(q_z^2+q^2)^{1/2}/n\sigma_{tot}}, \quad (30)$$

where $\omega=4\pi|f|^2/\sigma_{\rm tot}=\sigma_{\rm el}/\sigma_{\rm tot}$ is the single-scattering albedo. In the complex q_z plane, $\Lambda(q_z)$ has two branch points $(q_z=\pm i[(n\sigma_{\rm tot})^2+q^2]^{1/2}$ and two zeros $(q_z=\pm i[(n\sigma_{\rm tot})^2\zeta^2(\omega)+q^2]^{1/2}$, where $\zeta(\omega)$ is the solution of the equation $\omega \operatorname{arth}\zeta=\zeta, \zeta(\omega)<1$, so the function $\Lambda(q_z)$ satisfies all of the requirements stated above in, for example, the band $-n\sigma_{\rm tot}<\operatorname{Im} q_z<n\sigma_{\rm tot}$. Now using the general relations of the Wiener-Hopf method, we find the following expression for the Laplace transform of the function $\Phi_q(z)$:

$$\Phi_q(p)=\exp\left[-\frac{p}{\pi}\int_0^\infty \ln\Lambda(q_z)\frac{dq_z}{q_z^2+p^2}\right]-1, \quad (31)$$

where $\Lambda(q_z)$ is given by (30).

Expression (31) solves the problem of calculating the function $F^{(3D)}(q,p,p')$ [and thus $F^{3D}(\mathbf{r},\mathbf{r}')$] and makes it possible to find an explicit analytic expression for the angular distribution of the backscattered waves. Substituting (31) into (28), and then substituting the result into (20), we finally find

$$J(\mathbf{k},\mathbf{k}_0)=\frac{\omega}{4\pi}\frac{4k_{0z}^2}{|k_{0z}+\varkappa_{0z}|^2}\frac{4k_z^2}{|k_z+\varkappa_z|^2}\frac{\operatorname{Re}(\varkappa_{0z}/k_0)\operatorname{Re}(\varkappa_z/k_0)}{\operatorname{Re}(\varkappa_{0z}/k_0)+\operatorname{Re}(\varkappa_z/k_0)} \times\left\{H\left(\operatorname{Re}\frac{\varkappa_{0z}}{k_0},\omega|0\right)H\left(\operatorname{Re}\frac{\varkappa_z}{k_0},\omega|0\right) +\left[\left|H\left(\frac{n\sigma_{tot}}{i(\varkappa_z^*-\varkappa_{0z})},\omega\left|\frac{|\mathbf{k}_\parallel+\mathbf{k}_{0\parallel}|}{n\sigma_{tot}}\right.\right)\right|^2-1\right]\right\}, \quad (32)$$

where

$$H(\mu,\omega|\nu)=\exp\left\{-\frac{\mu}{\pi}\int_0^\infty \ln\left[1-\omega\frac{\operatorname{arctg}(\xi^2+\nu^2)^{1/2}}{(\xi^2+\nu^2)^{1/2}}\right]\frac{d\xi}{1+\mu^2\xi^2}\right\}. \quad (33)$$

Expression (32) determines completely the angular distribution of the backscattering in the case in which a plane wave is incident on a disordered semi-infinite medium with centers which scatter isotropically. This result is valid for an arbitrary single-scattering albedo ($\omega=\sigma_{\rm el}/\sigma_{\rm tot}$) and for arbitrary angles of incidence of the primary flux on the surface of the medium. Angular distribution (32) describes both the effect of weak wave localization, which is manifested for directions which are nearly exactly backward, and the effects of coherent reflection and refraction of waves at the boundary of the medium, which are important at grazing angles of incidence and emission of the radiation $(|\mu|,\mu_0\sim\mu_c\sim(n|f|)^{1/2}\lambda\ll 1)$.

If the cosine of the angle of incidence of the wave on the surface is not very small, $\mu_0\gg\mu_c$, the effects of the refraction and coherent reflection can be ignored, and expression (32) simplifies:

$$J(\mathbf{k},\mathbf{k}_0)=\frac{\omega}{4\pi}\frac{|\mu|\mu_0}{|\mu|+\mu_0}\left\{H(|\mu|,\omega|0)H(\mu_0,\omega|0) +\left[\left|H\left(\tilde\mu,\omega\left|\frac{|\mathbf{k}_\parallel+\mathbf{k}_{0\parallel}|}{n\sigma_{tot}}\right.\right)\right|^2-1\right]\right\}, \quad (34)$$

where

$$\frac{1}{\tilde\mu}=\left(\frac{1}{\mu_0}+\frac{1}{|\mu|}\right)-\frac{ik_0}{n\sigma_{tot}}(\mu_0-|\mu|).$$

If the effects of the coherent interaction with the interface are ignored, expression (34) is the exact solution of the problem of calculating the angular distribution of the backscattering from a disordered medium with small-scale centers under conditions of weak wave localization ($\lambda\ll l$).

The function $H(\mu\omega|\nu)$ in (32) and (34) may be thought of as a generalization of the Chandrasekhar H-function.[35,41,43] With $\nu=0$, the function $H(\mu,\omega|\nu)$ determined by (33) becomes the ordinary Chandrasekhar function, which describes the angular distribution of incoherently scattered radiation.

In addition to the integral representation (33) for $H(\mu,\omega|\nu)$, we could use the method of Ref. 41 to derive the nonlinear equation

$$H(\mu,\omega|\nu)=1+\frac{\omega}{2}\mu H(\mu,\omega|\nu)\times\int_0^{(1+\nu^2)^{-1/2}} d\mu'\frac{H(\mu',\omega|\nu)}{(\mu+\mu')(1-\nu^2\mu'^2)^{1/2}}, \quad (35)$$

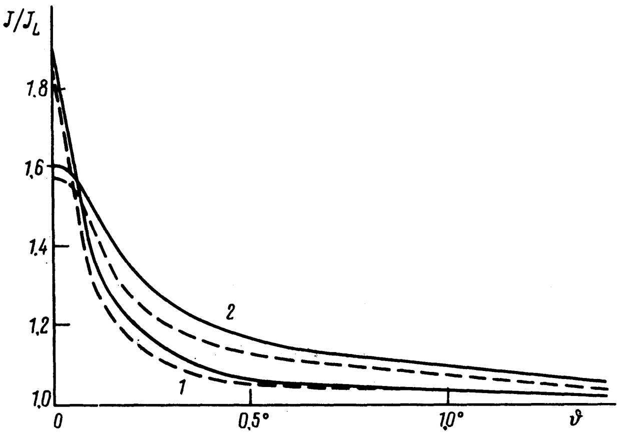

FIG. 3. Angular distribution of the backscattering of waves from $3D$ (solid lines) and $2D$ (dashed lines) disordered systems in the vicinity of the exactly backward direction. Normal incidence, $\mu_0 = 1$. The characteristics of the medium are $\lambda/2\pi l = 3\cdot 10^{-3}$ and (1) $\omega = 1$ or (2) $\omega = 0.8$. The curves have been normalized to the corresponding values of the incoherent intensity, J_L.

which is analogous to the familiar equation for the ordinary Chandrasekhar function and becomes the latter in the case $\nu = 0$. Nonlinear equation (35) turns out to be convenient for calculating $H(\mu,\omega|\nu)$ by an iterative method (an iterative method for the ordinary Chandrasekhar function is described in Refs. 35 and 43).

Let us analyze the features of the backscattering angular distribution. The first term in braces in (32) and (34) describes the angular distribution of the incoherent scattered waves and agrees with the existing results—the distribution found in Ref. 40—if we take the effects of the coherent interaction with the interface into account; it agrees with the classical result found by Chandrasekhar[41,43] if those effects are ignored.

The second term describes an interference of multiply scattered waves which have passed by the same inhomogeneities but in opposite directions. The presence of an additional negative term here reflects the absence of a contribution from single-scattering processes.

The interference term in $J(\mathbf{k},\mathbf{k}_0)$ leads to an coherent backscattering effect: the appearance in the angular distribution of the backscattered waves of a sharp peak, with a typical angular width $\Delta\vartheta \sim \lambda/l$, and with slowly decaying wings. The incoherent-scattering component of $J(\mathbf{k},\mathbf{k}_0)$, in contrast, is a very smooth function of the angle, and it can be treated as constant over the scales over which the interference term changes in (32) and (34). The angular distribution $J(\mathbf{k},\mathbf{k}_0)$ can thus be described as a plateau caused by incoherent scattering from which a sharp interference maximum rises in the direction exactly backward. Figure 3 shows the results of calculations of the angular distribution of the backscattered waves on the basis of (33) and (34).

3. BACKSCATTERING OF WAVES FROM A DISORDERED 2*D* SYSTEM

Just recently, a coherent enhancement of backscattering has also been observed in expreiments on the reflection of light and sound waves from disordered $2D$ media.[8,27] Two-dimensional systems were modeled in Refs. 8 and 27 by an ensemble of long filaments, parallel to each other with random distances between their axes.

In contrast with the $3D$ case discussed above, in the scattering of waves by a $2D$ system of disordered centers the interference effects in the multiple scattering become so strong that the condition $\lambda \ll l$ is no longer sufficient for the ocurrence of a regime of weak localization. There is the further requirement that the characteristic dimensions of the region with which the incident radiation interacts must be much smaller than the localization length[4)]: $l_D \ll L_{\text{loc}}$, where $l_D \sim l(1-\omega)^{-1/2}$ is the effective depth to which the incident radiation penetrates into the medium (the diffusion length[36,41]), and $L_{\text{loc}} \sim l_{\text{el}} \exp(l_{\text{el}}/\lambda)$ is the localization length in a disordered $2D$ system.[16,17,32] This condition imposes a definite, although far from severe, limitation on the absorption in the medium: $1-\omega \gg \exp(-l_{\text{el}}/\lambda)$.

In the $2D$ case the amplitude for scattering by an individual center, l, is conveniently chosen from the following representation of the singly scattered field:

$$\psi(\mathbf{r}) = \exp(i\mathbf{k}_0\mathbf{r}) + f(2\pi/k_0 r)^{1/2} \exp(ik_0 r + i\pi/4), \qquad (36)$$

where $k_0 = 2\pi/\lambda$ and $\mathbf{r} = (x,z)$ is a $2D$ radius vector. With this definition, the amplitude f becomes a dimensionless quantity. The total elastic cross section is related to the amplitude by $\sigma_{\text{el}} = (2\pi)^2|f|^2/k_0$ and has the dimensionality of a length. The optical theorem takes the same form as in the $3D$ case., The effective dielectric constant of the $2D$ medium is determined by the previous relation, $k_0^2\varepsilon = k_0^2 + 4\pi n f$, where n is now the number of scattering centers per unit area.

The calculation of the backscattering angular distribution in the $2D$ case is analogous to that above for a disordered $3D$ medium. We will accordingly focus on the distinctive features of the multiple scattering of waves in a disordered $2D$ system.

On the whole, the solution for the average field, (19), and expression (20) for the angular distribution of the backscattered waves are also valid in the $2D$ case. In contrast with a $3D$ medium, now there is no azimuthal dependence of the angular distribution, the common factor $1/4\pi$ in (20) must be replaced by $1/2\pi$, and $F^{(3D)}(q,p,p')$ must be replaced by $F^{(2D)}(q,p,p')$:

$$F^{(2D)}(q,p,p') = \int_{-\infty}^{\infty} \cos qx \, dx \int_0^{\infty}\int_0^{\infty} dz\, dz' \exp(-pz-p'z') F^{(2D)}(x,z,z'), \qquad (37)$$

where

$$q = k_0(\sin\theta + \sin\theta_0) = k_\parallel + k_{0\parallel},$$

and θ_0 and θ are the incidence and the scattering angles in the x, z plane ($0 \leqslant \theta_0$, $\theta \leqslant 2\pi$). In (37) we have taken into account the following relation, which holds by virtue of the symmetry of the problem:

$$F^{(2D)}(\mathbf{r},\mathbf{r}') \equiv F^{(2D)}(|x-x'|, z, z').$$

In Eq. (11), we must now replace (22) by the $2D$ Green's function, which describes the field distribution in a cylindrical wave which propagates without scattering from the point $\mathbf{r}'$ to the point $\mathbf{r}$:

$$G(|\mathbf{r}-\mathbf{r}'|) = -\tfrac{1}{4} i H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|), \qquad (38)$$

where $k = k_0 + 2\pi nf/k_0$ ($n|f| \ll k_0^2$), and $H_0^{(1)}(z)$ is a Hankel function of the first kind. After substituting (38) into (11), we find an integral equation with a difference kernel for $F^{(2D)}(\mathbf{r},\mathbf{r}')$. We find a solution of this equation by the same methods as were used above to solve Eq. (23). The only distinction is that in the first step we use a Fourier cosine transformation in the difference variable $x - x'$ instead of the Bessel transformation (25). All of the other results concerning the solution of the one-dimensional integral equation [(27), (28), (31)] remain in force for a $2D$ disordered medium.

The auxiliary function $\Lambda(q_z)$, in terms of which the basic function $\Phi_q^{(2D)}(p)$ and $F^{(2D)}(q,p,p')$ are expressed, takes the following form in the $2D$ case:

$$\Lambda(q_z)=1-(4\pi)^2\frac{n|f|^2}{2\pi|k|}\frac{\arcsin[(1-u)/2]}{(1-u^2)^{1/2}}, \tag{39}$$

where

$$u=[q_z^2+q^2+(n\sigma_{tot})^2]/2|k|^2-1.$$

Analysis shows that in the calculation of the angular distribution of the scattered waves the function (31) is dominated by the integration region $q_z \ll k_0$, in which the following approximation is valid for $\Lambda(q_z)$:

$$\Lambda(q_z)=1-\omega\,[(q_z^2+q^2)/(n\sigma_{tot})^2+1]^{-1/2}. \tag{40}$$

From the physical standpoint, the use of expression (40) [in place of (39)] in (31) means that the near-field effects, i.e., the contribution from the region $r \lesssim \lambda$ around the scattering center, are being ignored in density equation (11). This approach corresponds to the substitution into (11) of not the exact Green's function, (38), but its asymptotic expression in the far zone:

$$G(|\mathbf{r}-\mathbf{r}'|)\approx-(8\pi k|\mathbf{r}-\mathbf{r}'|)^{-1/2}\exp(ik|\mathbf{r}-\mathbf{r}'|+i\pi/4). \tag{41}$$

If we are not interested in the effects of the refraction and reflection of waves as the interface of the scattering medium is crossed, we can derive the following expression for the backscattering angular distribution $J(\mu,\mu_0)$:

$$J(\mu,\mu_0)=\frac{\omega}{2\pi}\frac{|\mu|\mu_0}{|\mu|+\mu_0}\Big\{h(\mu_0,\omega|0)h(|\mu|,\omega|0)+\Big[\Big|h\Big(\bar{\mu},\omega\,\Big|\frac{k_0}{n\sigma_{tot}}|\sin\theta+\sin\theta_0|\Big)\Big|^2-1\Big]\Big\}. \tag{42}$$

In contrast with (32) and (34), backscattering angular distribution (42) is expressed in terms of a $2D$ analog of the function $H(\mu,\omega|\nu)$: the function

$$h(\mu,\omega|\nu)=\exp\Big[-\frac{\mu}{\pi}\int_0^\infty\ln\Big(1-\frac{\omega}{(\xi^2+\nu^2+1)^{1/2}}\Big)\frac{d\xi}{1+\mu^2\xi^2}\Big]. \tag{43}$$

Like (32)–(34), expression (42) describes the angular distribution of the incoherently scattered radiation [the first term in braces in (42)] and the interference of multiply scattered waves. Figure 3 shows the results calculated on the angular distribution of the backscattered waves from (42) and (43).

The function $H(\mu,\omega|\nu)$ has the same properties as $H(\mu,\omega|\nu)$. In particular, using the methods of Ref. 41 we can derive the following nonlinear integral equation for $h(\mu,\omega|\nu)$:

$$h(\mu,\omega|\nu)=1+\frac{\omega}{\pi}\mu h(\mu,\omega|\nu)\times\int_0^{(1+\nu^2)^{-1/2}}d\mu'\frac{h(\mu',\omega|\nu)}{(\mu+\mu')[1-(1+\nu^2)\mu'^2]^{1/2}}. \tag{44}$$

It should also be noted that the function $h(\mu,\omega|0)$ is a $2D$ analog of the Chandrasekhar H-function, and the first term in braces in (42) is the angular distribution of the incoherently scattered radiation which was found by Chandrasekhar.[41,43] Using Eq. (44) with $\nu=0$, we can, as in Refs. 41 and 43, find the total reflection coefficient for the radiation scattered incoherently by a disordered $2D$ medium:

$$R(\mu_0)=\frac{1}{\mu_0}\int_{\pi/2}^{3\pi/2}d\theta J_L(\mu,\mu_0)=1-(1-\omega)^{1/2}h(\mu_0,\omega|0). \tag{45}$$

The generalization of (42)–(45) to take into account the coherent reflection and refraction at the interface is completely analogous to that of Ref. 40.

In contrast with $H(\mu,\omega|\nu)$ the function $h(\mu,\omega|\nu)$, has the additional universality property

$$h(\mu,\omega|\nu)=h\Big(\mu(\nu^2+1)^{1/2},\ \frac{\omega}{(\nu^2+1)^{1/2}}\Big|0\Big)=h\Big(\mu\omega,1\,\Big|\Big(\frac{\nu^2+1-\omega^2}{\omega^2}\Big)^{1/2}\Big), \tag{46}$$

which makes it possible to express a function of three variables, $h(\mu,\omega|\nu)$, in terms of a function of two variables—either the h-function, in the case of conservative scattering, or $h(\mu,\omega|\nu)$, which is a two-dimensional analog of the Chandrasekhar function which describes incoherent scattering of radiation.

4. DISCUSSION OF RESULTS

Universal expressions (32)–(34), (42), (43) derived above make it possible to calculate the angular distribution of the backscattering from disordered $3D$ and $2D$ systems with centers which scatter isotropically for an arbitrary angle of incidence of the primary flux on the surface of the medium and for an arbitrary relation between the cross sections for elastic scattering and absorption at an individual center. The case of most interest in the analysis of these expressions is the case of scattering exactly backward, when a sharp backscattering peak arises in the angular distribution because of the weak localization of the waves.

One of the basic characteristics of the weak-localization effect which can be actually measured in the reflection of waves from disordered media is the backscattering enhancement factor η, which is the ratio of the observed intensity to the intensity of incoherent scattering for the direction exactly backward.[3,4] From (18) we have

$$\eta=J(-\mathbf{k}_0,\mathbf{k}_0)/J_L(-\mathbf{k}_0,\mathbf{k}_0)=2-J_1(-\mathbf{k}_0,\mathbf{k}_0)/J_L(-\mathbf{k}_0,\mathbf{k}_0), \tag{47}$$

where $J_L(\mathbf{k},\mathbf{k}_0)$ is the angular distribution of the incoherently scattered radiation, and $J_1(\mathbf{k},\mathbf{k}_0)$ is the angular distribution of single scattering. In other words, the enhancement factor is determined exclusively by the relative size of the component of the overall intensity of incoherent scattering which is due to single scattering. Using expression (32), we find the following result for a disordered $3D$ medium:

$$\eta=2-H^{-2}(\mathrm{Re}(\varkappa_{0z}/k_0),\ \omega), \qquad (48)$$

where $H(\mu,\omega) = H(\mu,\omega|0$ is the Chandrasekhar function. In the $2D$ case, the expression for η differs from (48) only in that the H-function is replaced by $h(\mu,\omega|0)$.

It follows from (48) that η depends on the single-scattering albedo and on the angle at which the radiation flux is incident on the surface of the medium. For normal incidence, the enhancement factor reaches its extreme value in the case $\omega = 1$. For the reflection of waves from an disordered $3D$ system we have $\eta_{\max} \approx 1.882$, while that for a $2D$ system is $\eta_{\max} \approx 1.844$. With increasing absorption (with decreasing ω), the relative importance of single scattering increases, and the enhancement factor progressively decreases.

The behavior of η as a function of the angle of incidence is monotonic. At grazing angles of incidence the enhancement factor tends toward unity in accordance with

$$\eta\approx1+2\mathrm{Re}(\varkappa_{0z}/k_0)\alpha(\omega)+\ldots, \qquad (49)$$

where we would have

$$\alpha(\omega)=-\frac{1}{\pi}\int_0^\infty \ln\left(1-\omega\frac{\mathrm{arctg}\,\xi}{\xi}\right)d\xi$$

in the $3D$ case and

$$\alpha(\omega)=-\frac{1}{\pi}\int_0^\infty \ln\left(1-\frac{\omega}{(\xi^2+1)^{1/2}}\right)d\xi$$

in the $2D$ case.

In particular, we can draw conclusions from (49) about how the effects of the coherent interaction with the interface influence the enhancement factor. In a medium which is denser than vacuum ($\mathrm{Re}\, f > 0$), the glancing angle of the incident wave inside the medium is larger than that in vacuum. As a result, there is an increase in the effective multiplicity of the wave scattering in the medium and therefore in the enhancement factor η also.

Although result (49) was derived, strictly speaking, in the approximation that the jump in the effective dielectric constant at the vacuum-medium interface is small ($n|f| \ll k_0^2$), the qualitative conclusions which follow from this result regarding the influence of refraction and coherent reflection on the enhancement factor η remain valid in the general case in which the jump in ε at the interface is not small. This point should be kept in mind when comparing the theoretical results with experimental data. It is possible, as was pointed out in Ref. 5, that the scatter in the data from various experiments carried out to measure the backscattering enhancement factor is a consequence of specifically the coherent interaction with the interface.

The angular distribution of the radiation intensity near the backscattering peak contains information more comprehensive than the factor η about the interference of waves during multiple scattering in the medium.

Let us take a qualitative look at the behavior of the interference part $J_c(\mathbf{k},\mathbf{k}_0)$ of the complete angular distribution of the backscattering in the limiting cases of small ($\vartheta \ll \lambda/l$) and large ($\vartheta \gg \lambda/l$) angular deviations from the backward direction.

For directions which are approximately exactly backward ($\vartheta \ll \lambda/l$), $\nu \ll 1$), the behavior of $J_c(\mathbf{k},\mathbf{k}_0)$ can be characterized as follows: At a low absorption level ($1-\omega \ll 1$), in both the $3D$ and $2D$ cases, the angular distribution near the maximum is shaped primarily by high-multiplicity scattering processes and can be described by (see the Appendix)

$$J_c(\mathbf{k},\mathbf{k}_0)=\frac{\omega\mu_0}{4\pi(d-1)}\{A(\mu_0)-B(\mu_0)[d(1-\omega)+\nu^2]^{1/2}\}, \qquad (50)$$

where d is the dimensionality of the space. In the $3D$ case we would have

$$A(\mu_0)=H^2(\mu_0,\ 1|0)-1,\quad B(\mu_0)=2\mu_0H^2(\mu_0,\ 1|0). \qquad (51)$$

In the $2D$ case, the H-function in (51) would have to be replaced by $h(\mu_0,1/0)$.

It follows from this expression that the angular distribution of the intensity near the backscattering peak is determined exclusively by the factor $[d(1-\omega)+v^2]^{1/2}$. In particular, in the absence of absorption, $\omega = 1$, the peak would be triangular:

$$J_c=\frac{\mu_0}{4\pi(d-1)}[A(\mu_0)-B(\mu_0)|\nu|].$$

The result (50) and (51), which was derived on the basis of the exact solutions (34) and (42), agrees qualitatively with the equations of the diffusion theory.[3-5,29-32] On the other hand, in terms of the values of the coefficients $A(\mu_0)$ and $B(\mu_0)$ in (50) [and also in terms of the incoherent-scattering intensity in (34) and (42)], the results which follow from the exact solution disagree with those derived in the diffusion approximation [particularly in the case of oblique incidence ($\mu_0 < 1$)].

With increasing absorption, low-multiplicity scattering processes become progressively more important, and the backscattering peak becomes lower and progressively more rounded according to (34) and (42).[5)]

The wings in the angular distribution of $J_c(\mathbf{k},\mathbf{k}_0)$ near the backscattering peak are determined by the asymptotic behavior of the functions $H(\mu,\omega|\nu)$ and $h(\mu,\omega|\nu)$ at $\nu \gg 1$. Using the integral representations of these functions according to (33) and (43), we find, for $\nu \gg 1$,

$$H(\mu,\ \omega|\nu)\approx1+\pi\omega/4\nu,\qquad h(\mu,\ \omega|\nu)\approx1+\omega/2\nu. \qquad (52)$$

Substituting these expressions into (34) and (42), we find the following simple relation for the interference part of the angular distribution:

$$J_c(\mathbf{k},\mathbf{k}_0)\approx\frac{\omega^2}{2\beta}\frac{|\mu|\mu_0}{|\mu|+\mu_0}\frac{1}{\nu}, \qquad (53)$$

where $\beta = 4$ in the $3D$ case and $\beta = \pi$ in the $2D$ case. It follows in particular that for normal incidence ($\mu_0 = 1$) we would have

$$J_c(\mu,1)\approx\frac{\omega^2}{2\beta}\frac{|\mu|}{1+|\mu|}\frac{1}{(1-\mu^2)^{1/2}}\approx\frac{\omega^2}{4\beta}\frac{n\sigma_{tot}}{k_0}\frac{1}{\vartheta} \qquad (54)$$

($\vartheta \gg \lambda/l$). In other words, the intensity in the wings of the angular distribution near the backscattering peak falls off in inverse proportion to the angular deviation from the direction exactly backward, ϑ. This behavior ($J_c \propto \vartheta^{-1}$) has been observed experimentally in both 3D (Ref. 26) and 2D (Ref. 8) disordered media.

In the case of oblique incidence ($\mu_0 < 1$) the angular width of the backscattering peak in the 2D case and that in the azimuthal plane ($\varphi = \pi$) in the 3D case increase with decreasing μ_0, in accordance with $\Delta\vartheta \sim \lambda / l\mu_0$. The scatter in azimuthal angle in the 3D case for $|\mu| = \mu_0$ falls off in accordance with $\Delta\varphi \sim \lambda / l(1-\mu_0^2)^{1/2}$.

It is not difficult to show that the wings in the angular distribution far from the backscattering peak, $\vartheta \gg \lambda / \iota$, are determined exclusively by double scattering in both the 3D and 2D cases.

In addition to analyzing the backscattering angular distribution, it is interesting to determine how the weak wave localization contributes to the total albedo of the scattering medium and to determine the relative role played by the scattering processes of various multiplicities here.

In the 3D case, because of the slow decay of the intensity in the wings of the distribution, the area under the J_c curve is determined primarily by the region of large angular deviations from the backward direction, in which approximation (53), (54) is valid. It is thus a simple matter to derive an estimate of the interference component of the albedo of the disordered medium integrated over angles. In the case of normal incidence ($\mu_0 = 1$) we find

$$R_c = \frac{2\pi}{\mu_0} \int_{(\mu<0)} J_c \sin\theta \, d\theta \approx 0.071\omega^2(\lambda/l), \qquad (55)$$

In accordance with the discussion above, this result is determined by double scattering. The corrections to (55) for scattering of higher multiplicity, $k \geqslant 3$, are of the next higher order in the small quantity: $(\lambda/l)^2$.

In the 2D case the situation is quite different. The interference component of the overall albedo of the disordered medium is not determined exclusively by double scattering in this case. Scattering processes of high multiplicity, $k \geqslant 3$ (the region of relatively small angles near the peak, $\vartheta \lesssim \lambda / l$), make a contribution to

$$R_c = \int_{\pi/2}^{3\pi/2} J_c \, d\theta$$

which is on the order of λ / l, while double scattering (the wings of the spectrum) makes a contribution on the order of $(\lambda / l)\ln(\lambda / l)$. Consequently, and in contrast with the 3D case, the contribution of multiple scattering to the albedo in 2D systems is essentially the same as that of double scattering.

We note in conclusion that the results of the diffusion theory,[3-5,15,29-32] which is based on the initial assumption that the high-multiplicity scattering processes play a dominant role (so double scattering is completely ignored), are not valid for describing the wings of the peak in the backscattering of waves from a medium with small-scale inhomogeneities. In particular, the law $J_c \propto \vartheta^{-2}$ which has been derived in several places[3-5,15,29-31] is incorrect. For this reason, the use of the diffusion approximation in calculating integral quantities [the interference component of the total albedo, (55), etc.] may lead to erroneous results.

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APPENDIX

In an analysis of the angular distribution near the exactly backward direction ($\nu \ll 1$) under weak-absorption conditions ($1 - \omega \ll 1$), it is inconvenient to use representation (33), (43) directly. The reason is that the behavior of the functions $H(\mu,\omega|\nu)$ and $h(\mu,\omega|\nu)$ near the point $\nu = 0$, $\omega = 1$ is not analytic. In the limit $\nu \to 0$, $\omega \to 1$, the zeros of the function Λ in the logarithm in (31), (33), and (43) shift toward the real axis, with the result that a singularity appears in the integrand in (31), (33), and (43) at $\xi = 0$.

We will transform representation (33), (43) in order to single out the factor which is responsible for the nonanalytic behavior of $H(\mu,\omega|\nu)$ and $h(\mu,\omega|\nu)$ in the limit $\nu \to 0$, $\omega \to 1$. For this purpose we consider the function

$$Z(\mu,\omega|\nu) = \exp\left[-\frac{\mu}{\pi}\int_0^\infty \ln\Lambda(\xi,\omega|\nu)\frac{d\xi}{\mu^2\xi^2+1}\right], \qquad (A1)$$

which generalizes (33) and (43). We denote by $\xi_0 = \pm i\gamma$ the zeros of the function $\Lambda(\xi,\omega|\nu)$. We single out explicitly the singularity in $\ln\Lambda(\xi,\omega|\nu)$ associated with these zeros:

$$\ln\Lambda = \ln\frac{\xi^2+1}{\xi^2+\gamma^2}\Lambda + \ln\frac{\xi^2+\gamma^2}{\xi^2+1}. \qquad (A2)$$

An additional factor of $\xi^2 + 1$ has been introduced in (A2) in order to prevent the appearance of a singular point as $|\xi| \to \infty$. Substituting (A2) into (A1), and carrying out some straightforward calculations, we find

$$Z(\mu,\omega|\nu) = \frac{\mu+1}{\mu\gamma+1}\exp\left\{-\frac{\mu}{\pi}\int_0^\infty \ln\left[\frac{\xi^2+1}{\xi^2+\gamma^2}\Lambda(\xi,\omega|\nu)\right]\frac{d\xi}{\mu^2\xi^2+1}\right\}. \qquad (A3)$$

When we now use the expression for γ which corresponds to the case of weak absorption,

$$\gamma = [d(1-\omega)+\nu^2]^{1/2}, \qquad (A4)$$

where d is the dimensionality of the space, we easily see that the quantity in the argument of the exponential function in (A3) is an analytic function of ω and that the singularity of $Z(\mu,\omega|\nu)$ in the limit $\nu \to 0$, $\omega \to 1$ is a consequence of exclusively the coefficient of the exponential function. We can thus derive the following approximate expression for $Z(\mu,\omega|\nu)$ in the limit $\nu \ll 1$, $1 - \omega \ll 1$:

$$Z(\mu,\omega|\nu) = \frac{1}{\gamma\mu+1}Z(\mu,1|0). \qquad (A5)$$

If we set $\nu = 0$ in (A5), we find the known representation of the Chandrasekhar H-function in the case of weak absorption.[41]

[1)] There are some exceptional cases: Several exact results—on the angular distribution of the backscattering in the case of a pronounced gyrotropy and on the enhancement factor for scattering exactly backward in the absence of gyrotropy—have been derived with allowance for the polarization of the waves by Golubentsev.[33]

[2)] If the scattering centers are in a matrix with a dielectric constant ε, the quantity $4\pi nf$ in (4) must be replaced by $k_0^2(\varepsilon - 1) + 4\pi nf_1$, where f_1 is the scattering amplitude in the medium.

3) We should point out that representation (20) is generally valid not only in the case in which the jump in the dielectric constant at the interface is a consequence of the presence of scattering particles in the region $z>0$ but also in the more general situation of an arbitrary change in the effective dielectric constant at the interface, e.g., the case in which the scattering centers are in a medium with a dielectric constant ε (see also footnote 2)).

4) As in §2, we are considering a semi-infinite medium; i.e., we are assuming that the thickness of the scattering layer is large: $L \gg l_D$. In the opposite case $L \ll l_D$, a weak localization occurs if $L \ll L_{loc}$.

5) In the 2D case, this conclusion follows directly from the properties of the function $h(\mu,\omega|\nu)$ in (46): A decrease in ω is equivalent to an increase in the deviation from the backward direction and to a more oblique incidence of the primary radiation flux.

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