Symmetry transformations of 2D binary electrically conducting system

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The local and integral electrical characteristics of a piecewise-homogeneous binary medium with a checkerboard structure are studied. This system is studied as an exactly solvable model of plane, heterogeneous, isotropic systems with equal concentrations of components. A physical interpretation is offered for the symmetry transformations of such systems. The form of both the known symmetry relations and the generalizing expressions for these relations derived in the present paper is shown to be determined primarily by the conditions for the flow of an external current in the system.

1. INTRODUCTION

In a study of the effective conductivity and average electrical characteristics of binary thin films with equal concentrations of randomly arranged inhomogeneities, Dykhne[1] found that such systems have an exact analytic description if their components are under geometrically equivalent conditions. The method for solving the problem turned out to be comparatively simple because the 2D equations for the steady-state electric field for media with the specified properties allow certain linear symmetry transformations. These transformations were interpreted as reciprocity relations for the two systems, which can be obtained from each other by interchanging the resistivities of the corresponding cells[2]: $\rho_1 \rightleftarrows \rho_2$.

Falling in the category of these heterogeneous systems, as was pointed out by Dykhne,[1] is a piecewise-homogeneous medium with a regular, periodic arrangement of components in a regular checkerboard structure. The black and white squares of the checkerboard are identified with different values of the resistivity. This medium is of interest because it has a clearly defined structural symmetry, which substantially simplifies the study of its local and macroscopic properties. It thus becomes possible to point out a fairly simple and completely transparent method for finding symmetry transformations of field quantities and to show how they depend on the orientation of the external current vector in the system. The physical meaning of the local symmetries of the field is determined in the process. These local symmetries are actually linear relations between the vector current densities (or the vector electric fields) at complex-conjugate points in adjacent cells.

Since the inhomogeneous medium is isotropic on the average, its effective parameters do not depend on the direction in which the external current flows in the system. This cannot be said of the average characteristics of the distributions of the currents and fields with respect to components, in particular, the energy dissipation and dispersion of the fields. For example, Joule energy is evolved at equal densities in the black and white cells, regardless of their resistivities, and the system as a whole is heated uniformly. This result is correct for a current distribution such that the external current in the medium is directed along any one of the diagonals of the squares.

These and certain other properties of the system can be established in two independent ways: through the direct application of local symmetry transformations and their average analogs; or through the use of the results of a solution of the boundary-value problem on the current distribution. The two approaches complement each other. The second approach makes it possible to lay a rigorous foundation for the existence of generalized symmetry transformations.

The inhomogeneous medium in which we are interested will be discussed below in terms of its electrical properties. However, a completely analogous approach could be taken to calculate the thermal, diffusive, magnetic, and other physical fields in such a medium. The formulation of the boundary-value problem for these other fields is mathematically equivalent.

2. BASIC EQUATIONS

It is thus assumed that the continuous, piecewise-homogeneous binary medium consists of square cells in which the resistivity ρ takes on different values in a doubly periodic alternation: $\rho = \{\rho_1, \rho_2\}$.

The steady-state current flow in this conducting medium is described by the equations

$$\mathrm{rot}\,\mathbf{E}=0, \quad \mathrm{div}\,\mathbf{j}=0, \quad \mathbf{E}=\rho\mathbf{j}, \tag{2.1}$$

where $\mathbf{E} = \{\mathbf{E}_1, \mathbf{E}_2\}$ and $\mathbf{j} = \{\mathbf{j}_1, \mathbf{j}_2\}$ are the piecewise-uniform vector electric field and vector current density, respectively (the subscripts 1 and 2 specify that the corresponding quantities pertain to cells with the resistivity values ρ_1 and ρ_2).

Consider a 2D electric field in the xy plane, which coincides with the medium. In this case one can use a complex representation of the field and current vectors:

$$E(z)=E_x-iE_y, \quad j(z)=j_x-ij_y \quad (z=x+iy). \tag{2.2}$$

According to the original equations, (2.1), these quantities are analytic functions within the cells (at the boundaries of the cells, their analyticity is disrupted, since the relations $\mathrm{curl}\,\mathbf{j}\neq 0$ and $\mathrm{div}\,\mathbf{E}\neq 0$ hold the contour lines).

For complex vectors (2.2), Ohm's law remains the same in form:

$$E(z)=\rho j(z). \tag{2.3}$$

Going over to the plane of a complex variable simplifies the calculations, and in the solution of the boundary-value problem it accordingly becomes possible to apply the effective methods of the theory of analytic functions. It should be kept in mind here that while the complex vectors $E(z)$ and

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$j(z)$ are equal to the actual vectors **E** and **j** in magnitude they do not have the same direction. They specify the field pattern which is the mirror image of the physical field and which transforms into the latter if the operation of complex conjugation is carried out on the complex vectors (2.2).

3. BOUNDARY CONDITIONS

If there is an ohmic contact at a boundary between the unlike electrically conducting media, the normal components of the current density vector and the tangential components of the electric field vector are continuous: $j_n^+ = j_n^-$, $E_\tau^+ = E_\tau^-$. By virtue of the symmetry, it is sufficient to calculate the field in two adjacent cells, S_1 and S_2 (Fig. 1). The field pattern is reproduced in a double periodic fashion in the other squares.

For the piecewise-analytic vector $j(z) = \{j_1(z), j_2(z)\}$ in cell S_1, the boundary conditions can be written in the form

$$\begin{aligned} &\mathrm{Re}\{n(t)j_1(t)\} = \mathrm{Re}\{n(t)j_2(t)\}, \\ &\qquad\qquad\qquad\qquad t\in L, \\ &\mathrm{Im}\{n(t)\rho_1 j_1(t)\} = \mathrm{Im}\{n(t)\rho_2 j_2(t)\}, \end{aligned} \tag{3.1}$$

where $n(t)$ is the outward unit normal to L, which is the contour bounding region S_1.

The boundary conditions (3.1) will be used in a substantial way below in the derivation of the symmetry transformations of this system, so the representation of these conditions will be discussed in somewhat more detail.

In expanded form, conditions (3.1) become

$$\begin{aligned} &j_1(t)-\overline{j_1(t)} = j_2(t)-\overline{j_2(t)}, \\ &\qquad\qquad\qquad\qquad t\in L_g; \\ &\rho_1[j_1(t)+\overline{j_1(t)}] = \rho_2[j_2(t)+\overline{j_2(t)}], \\ &j_1(t)+\overline{j_1(t)} = j_2(t)+\overline{j_2(t)}, \\ &\qquad\qquad\qquad\qquad t\in L_v, \\ &\rho_1[j_1(t)-\overline{j_1(t)}] = \rho_2[j_2(t)-\overline{j_2(t)}], \end{aligned} \tag{3.2}$$

where the superior bar means complex conjugation, and L_g and L_v are respectively the horizontal and vertical parts of the contour $L = L_g \cup L_v$.

After one of the vectors, e.g., $-j_1(t)$, is eliminated from each pair of equalities in (3.2), and after the relative resistivity

$$\Delta = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}, \quad -1 \leqslant \Delta \leqslant 1, \tag{3.3}$$

is introduced, the boundary conditions become

$$\begin{aligned} &(1+\Delta)j_1(t) = j_2(t) - \Delta\overline{j_2(t)}, \quad t\in L_g; \\ &(1+\Delta)j_1(t) = j_2(t) + \Delta\overline{j_2(t)}, \quad t\in L_v. \end{aligned} \tag{3.4}$$

The local boundary equalities must be supplemented with the integral relations

$$\begin{aligned} &J = J_x - iJ_y, \\ &J_x = \int_0^h j_{x1}(0,y)\,dy, \quad J_y = \int_0^h j_{y1}(x,0)\,dx, \end{aligned} \tag{3.5}$$

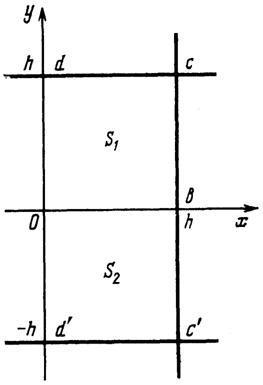

FIG. 1.

which specify the vector external current $J = J_x - iJ_y$ on a step of one cell.

Equations (3.4) have been written for contour L, which bounds region S_1. Boundary conditions on boundary line L' of cell S_2 (Fig. 1) are written in a corresponding way:

$$\begin{aligned} &(1-\Delta)j_2(t) = j_1(t) + \Delta\overline{j_1(t)}, \quad t\in L_g'; \\ &(1-\Delta)j_2(t) = j_1(t) - \Delta\overline{j_1(t)}, \quad t\in L_v'. \end{aligned} \tag{3.6}$$

Here L'_g and L'_v are respectively the horizontal and vertical parts of contour L'.

4. SYMMETRY TRANSFORMATION

The primary distinguishing feature of a heterogeneous medium with a checkerboard structure is the presence of a variety of symmetries, which reflect its structure and color image. The symmetry is obviously also reflected in the formation of physical field in the system. To determine the symmetry, it is necessary to try to find the relation between the current distributions in two adjacent cells with different resistivities, ρ_1 and ρ_2 (Fig. 1).

The vector current densities $j_1(z)$ and $j_2(z)$ at complex-conjugate points with respect to the z axis can be compared by means of the linear transformation

$$j_1(z) = T\,\overline{j_2(\bar{z})}, \tag{4.1}$$

where T is, generally speaking, some complex function of the variable z which is to be determined.

The transformation (4.1) means that the vector $j_2(z)$ is mapped in a mirror fashion out of region S_2 to the point in region S_1 at which the vector $j_1(z)$ is taken stretched out in modulus by an amount $|T|$, and rotated through an angle arg T, being brought into coincidence with the vector $j_1(z)$.

The transformation which is the inverse of transformation (4.1) is written

$$j_2(z) = \overline{T^{-1}}\,\overline{j_1(\bar{z})}. \tag{4.2}$$

Remarkably, under certain conditions, which are determined exclusively by the nature of the flow of the external current in the system, the value of T remains constant for all points in each cell.

To find an explicit expression for T it is necessary to make use of the boundary conditions (3.4) and (3.6). When transformations (4.1) and (4.2) are used, these conditions transform into each other, thereby reflecting the fact that the field patterns in adjacent cells are linearly similar.

For example, the boundary conditions on the horizontal lines of square S_1 [the first equality in system (3.4)] are transformed by (4.1) and (4.2) to the form

$$(1+\Delta)\overline{T j_2(t)}=\overline{T^{-1} j_1(t)}-\Delta T^{-1} j_1(t),\ t\in L_g',$$

or, equivalently,

$$(1+\Delta)T\overline{T} j_2(t)=j_1(t)-\Delta T\overline{T}^{-1}\overline{j_1(t)},\ t\in L_g'. \tag{4.3}$$

The last equality is the same as the corresponding condition in (3.6) if

$$(1+\Delta)T\overline{T}=(1-\Delta),\ T\overline{T}_1^{-1}=1.$$

It follows that

$$T=\pm i\left(\frac{1-\Delta}{1+\Delta}\right)^{1/2} \tag{4.4}$$

or

$$T=\pm i(\rho_2/\rho_1)^{1/2}. \tag{4.4'}$$

The same expression for T could be derived from the transformations of the boundary conditions on the vertical lines.

The relationship between the current density vectors in adjacent cells, (4.1), is thus as follows according to (4.4) and (4.4′):

$$j_1(z)=\pm i\left(\frac{1-\Delta}{1+\Delta}\right)^{1/2}\overline{j_2(\overline{z})}=\pm i\left(\frac{\rho_2}{\rho_1}\right)^{1/2}\overline{j_2(\overline{z})}. \tag{4.5}$$

The sign in front of the square root in (4.4) and (4.5) is important and absolutely must be taken into account in the calculation of the electric field. As will be shown below, the choice of sign is determined by the direction in which the average (or external) current flows in the system.

The transformation (4.5) was derived in vector form by Dykhne, who employed considerations different from those used here. He did not take account of the presence of two signs on the square root.[1] His approach is legitimate if the calculations are restricted to the effective properties of the inhomogeneous medium.

In addition to this symmetry, of the reflection of the field pattern in each cell, there is one more symmetry, which can be expressed by relations of the following type:

$$j_1(z)=P\overline{j_1(\overline{z}^*)},\ j_2(z)=Q\overline{j_2(\overline{z}^*)}\ \ (z^*=\pm iz), \tag{4.6}$$

where the complex numbers P and Q are determined with the help of the boundary conditions. The transformations (4.6) demonstrate a reflection symmetry with respect to the diagonals of the square cells. They send the boundary conditions on the horizontal lines into conditions on the vertical lines, and vice versa. For example, substitution of (4.6) into the first equality in (3.4) puts the latter in the form

$$(1+\Delta)\overline{P j_1(t)}=\overline{Q j_2}(t)-\overline{\Delta} Q j_2(t),\ t\in L_v,$$

which can be written

$$(1+\Delta)j_1(t)=\overline{P}^{-1}\overline{Q} j_2(t)-\Delta\overline{P^{-1}}\overline{Q j_2(t)},\ t\in L_v.$$

This equality is the same as the second condition in (3.4) if

$$\overline{P^{-1}}\overline{Q}=1,\ \overline{P^{-1}}Q=-1, \tag{4.7}$$

i.e., if the numbers P and Q have the values

$$P=\pm i,\ P=Q. \tag{4.8}$$

Here, as in the earlier equations, the particular sign is determined by the choice of the direction in which the external current flows in the system.

A similar result is found if the boundary conditions on the vertical lines are sent by rotation transformations (4.6) into conditions on horizontal lines.

An inhomogeneous system with a checkerboard structure thus allows reflection symmetry transformations of the following type:

$$j_k(z)=\pm i\overline{j_k(\overline{z}^*)}\quad (k=1,\ 2). \tag{4.9}$$

The symmetries which we have identified show that linearly similar field patterns are formed in adjacent cells. A current distribution of this sort is established, however, only for a completely definite direction of the external current flow in the system, a will be shown below.

5. AVERAGING OF THE ELECTRIC FIELD

By virtue of the periodicity of the structure, the field averaged over the entire inhomogeneous medium can be found by simply averaging the local field in two adjacent cells:

$$\langle E\rangle=\frac{1}{S}\int_s E\,ds,\quad \langle j\rangle=\frac{1}{S}\int_s j\,ds, \tag{5.1}$$

where S is the sum of the areas of cells S_1 and S_2: $S=S_1\cup S_2$ (Fig. 1). The field averaged in this fashion consists of the sum of fields of two cells, differing in resistivity:

$$\langle E\rangle={}^1/_2(\langle E_1\rangle+\langle E_2\rangle),\ \langle j\rangle={}^1/_2(\langle j_1\rangle+\langle j_2\rangle). \tag{5.2}$$

The average field in the cells is uniform; by virtue of (5.2) the average field in the system is then seen to be uniform also. Accordingly, with the piecewise-homogeneous medium is associated a medium with some constant effective resistivity $\rho_{\rm eff}$:

$$\langle E\rangle=\rho_{eff}\langle j\rangle. \tag{5.3}$$

The resistivity $\rho_{\rm eff}$ is found by means of average symmetry transformations. The application of averaging operation (5.1) to expressions (4.5) and (4.9) within a single cell, e.g., S_1, leads to the relations

$$\langle j_1\rangle=\pm i(\rho_2/\rho_1)^{1/2}\langle\overline{j}_2\rangle,\ \langle j_k\rangle=\pm i\langle\overline{j}_k\rangle,\ k=1,\ 2. \tag{5.4}$$

Hence

$$\langle j_1\rangle=(\rho_2/\rho_1)^{1/2}\langle j_2\rangle, \tag{5.5}$$

and also, by virtue of Ohm's law,

$$\langle E_1\rangle=(\rho_1/\rho_2)^{1/2}\langle E_2\rangle. \tag{5.6}$$

Averaging in cell S_2 again leads to expressions (5.5) and (5.6).

From (5.2), (5.3), (5.5), and (5.6) one finds

$$\rho_{eff}=(\rho_1\rho_2)^{1/2}. \tag{5.7}$$

The effective conductivity of the system is determined by the expression

$$\sigma_{eff}=(\sigma_1\sigma_2)^{1/2}, \tag{5.8}$$

which is of the same form as the preceding expression, from which it was found through an inversion of the resistivities

($\sigma = 1/\rho$). These results were derived by Dykhne in Ref. 1 by a different method, which also made use of the symmetry properties of the system.

Relations (5.4) can be used to find the meaning of the different signs on the square root in symmetry transformations (4.5) and (4.9). Taking the positive value of the root, for example, one finds

$$\langle j_{x1}\rangle = -(\rho_2/\rho_1)^{1/2}\langle j_{y2}\rangle,\ \langle j_{y1}\rangle = -(\rho_2/\rho_1)^{1/2}\langle j_{x2}\rangle,$$
$$\langle j_{xk}\rangle = -\langle j_{yk}\rangle\ (k=1,\ 2) \tag{5.9}$$

We thus see that the average current (which has the same direction and the same magnitude as the external current, as we will see below) flows along diagonals in one direction in all of the square cells.

If one takes the negative root, from (5.4) one obtains the equalities

$$\langle j_{x1}\rangle = (\rho_2/\rho_1)^{1/2}\langle j_{y2}\rangle,\ \langle j_{y1}\rangle = (\rho_2/\rho_1)^{1/2}\langle j_{x2}\rangle, \tag{5.10}$$
$$\langle j_{xk}\rangle = \langle j_{yk}\rangle\ (k=1,\ 2).$$

In this case the average current in the system is directed along the other diagonal of the squares.

The existence of two signs in the symmetry transformations does not affect the calculations of the effective resistivity, since the inhomogeneous medium is isotropic on the whole, and its (electrical) conduction properties do not depend on the direction in which the average current flows. It is important, however, to take the signs into account in these transformations from the standpoint of the completeness of the solution of the boundary-value problem of the current distribution in the system and its average characteristics.

Using (5.2)–(5.6) and the average Ohm's law in the cells, relations between the electric fields and the currents, averaged over the individual cells and over the system as a whole, are easily found:

$$\langle E_{1,2}\rangle = \frac{2\rho_{1,2}^{1/2}}{\rho_1^{1/2}+\rho_2^{1/2}}\langle E\rangle,\qquad \langle j_{1,2}\rangle = \frac{2\rho_{2,1}^{1/2}}{\rho_1^{1/2}+\rho_2^{1/2}}\langle j\rangle. \tag{5.11}$$

A measure of the dispersion of the field can also be determined:

$$D = \frac{\langle E\bar{E}\rangle - \langle E\rangle\langle\bar{E}\rangle}{\langle E\rangle\langle\bar{E}\rangle} = \frac{\langle j\bar{j}\rangle - \langle j\rangle\langle\bar{j}\rangle}{\langle j\rangle\langle\bar{j}\rangle} = \left(\left(\frac{\rho_1}{\rho_2}\right)^{1/4} - \left(\frac{\rho_2}{\rho_1}\right)^{1/4}\right)^2. \tag{5.12}$$

The local and average symmetry transformations in (4.5) and (5.4) make it possible to prove that the values of the Joule dissipation are equal at complex-conjugate points in adjacent cells:

$$\rho_1 j_1(z)\overline{j_1(z)} = \rho_2 j_2(\bar{z})\overline{j_2(\bar{z})}. \tag{5.13}$$

It can also be shown that, on the whole over the cells we have

$$\rho_1\langle j_1\rangle\langle\bar{j}_1\rangle = \rho_2\langle j_2\rangle\langle\bar{j}_2\rangle. \tag{5.14}$$

Relations (5.11)–(5.14) were derived by Dykhne.[1] We wish to stress that in the system under consideration here these relations hold only if the external current is flowing along a diagonal of the squares, i.e., only under the reflection symmetry transformation (4.5). For all other directions of the external current in the system, neither the symmetry relation (4.5) nor relations (5.11)–(5.14) hold.

As a characteristic of the overall properties of the system, the symmetry transformation (4.5) allows one to distinguish the quantity

$$\mathrm{In} = (1-\Delta^2)^{1/2}, \tag{5.15}$$

which is, for arbitrary values of the resistivities of the components, always equal to the ratio of the effective parameters of their average values:

$$\mathrm{In} = \frac{\rho_{eff}}{\langle\rho\rangle} = \frac{\sigma_{eff}}{\langle\sigma\rangle} = \frac{\langle j\rangle\langle\bar{j}\rangle}{\langle j\bar{j}\rangle} = \frac{\langle E\rangle\langle\bar{E}\rangle}{\langle E\bar{E}\rangle}.$$

This result involves, along with quantities already introduced, the use of

$$\langle\rho\rangle = {}^1/_2(\rho_1+\rho_2),\ \langle\sigma\rangle = {}^1/_2(\sigma_1+\sigma_2).$$

In this sense, In is an invariant of the system. This quantity has the range

$$0 \leqslant \mathrm{In} \leqslant 1.$$

The value $\mathrm{In} = 0$ corresponds to the case in which the resistivity of one of the components of the inhomogeneous medium is finite, while that of the other is zero (or tends toward infinity); i.e., a medium of this type is an insulator (or an ideal conductor). Here one has $|\Delta| = 1$. If $\mathrm{In} = 1$, then the medium is homogeneous: $\rho_1 = \rho_2$ and $\Delta = 0$.

6. STUDY OF THE BOUNDARY-VALUE PROBLEM

The reflection symmetry transformation (4.5) is valid only if the vector external current is directed along a diagonal of the squares. This transformation is therefore a particular transformation. It is natural to ask whether it is possible to derive a general expression for this transformation which would be valid for any direction of the external current in the system. Apparently the simplest way to find the answer is to work from a complete analytic solution of the original problem.

The standard approach to the solution of the Markushevich boundary-value problem with boundary conditions as in (3.4) and (3.5) requires some rather lengthy calculations.[3] It becomes a substantially simpler matter to solve this problem by making immediate use of the symmetry transformation (4.5). Boundary relations (3.4) then transform into the boundary conditions of the ordinary two-element Riemann problem, and its solution can be constructed by known methods. It is necessary to recall that (4.5) contains two signs. Considering each of them separately simply leads to particular solutions; the general solution of the problem is found as the sum of these particular solutions.

For example, in the case in which the positive sign is taken in the transformation (4.5),the boundary conditions (3.4) become

$$j_1(t) = (-\Delta + i(1-\Delta^2)^{1/2})\overline{j_1(t)},\ t\in L_g;$$
$$j_1(t) = (\Delta + i(1-\Delta^2)^{1/2})\overline{j_1(t)},\ t\in L_v. \tag{6.1}$$

A direct check shows that a solution of this problem is given by the expression

$$j_1(z) = C_1\exp\left\{\frac{i\pi}{2}\left(\frac{1}{2}+\gamma\right)\right\}\left(\frac{\mathrm{cn}\,u}{\mathrm{sn}\,u\,\mathrm{dn}\,u}\right)^{2\gamma},\qquad u = \frac{K}{h}z, \tag{6.2}$$

where C_1 is a real constant; sn, cn, and dn are the elliptic functions; K is the complete elliptic integral of the first kind (for a square, $K = 1.854$); h is the side of a square; and the parameter γ is defined by

$$\gamma=\frac{1}{\pi}\operatorname{arctg}\left(\frac{\Delta}{(1-\Delta^2)^{1/2}}\right)=\frac{1}{\pi}\operatorname{arctg}\left(\frac{\rho_1-\rho_2}{2(\rho_1\rho_2)^{1/2}}\right),$$

$$0\leqslant|\gamma|\leqslant\frac{1}{2}. \qquad (6.3)$$

The current distribution in the adjacent cell is found with the help of (4.5):

$$j_2(z)=C_1\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\exp\left\{\frac{i\pi}{2}\left(\frac{1}{2}-\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{2\gamma}. \qquad (6.4)$$

In these expressions, the constant C_1 is found from the integral conditions (3.5), in which the vector J is treated as given. Calculations lead to

$$J_x=J_y=C_1\left(\frac{\rho_1}{\rho_1+\rho_2}\right)^{1/2}I. \qquad (6.5)$$

Here I is a definite integral, given by

$$I=\frac{h}{2K}\frac{\pi^{3/2}}{\cos(\pi\gamma)}\left\{\Gamma\left(\frac{3}{4}+\frac{\gamma}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\gamma}{2}\right)\right\}^{-1},$$

where Γ is the gamma function.

In place of the vector external current J in this problem one could specify the potential drops over an interval equal to the cell size:

$$U_x=\int_0^h E_{x1}(x)\,dx=\rho_1\int_0^h j_{x1}(x)\,dx,$$

$$U_y=\int_0^h E_{y1}(y)\,dy=\rho_1\int_0^h j_{y1}(y)\,dy.$$

Calculations reproducing the derivation of (6.5) lead to

$$U_x=U_y=C_1\rho_1\left(\frac{\rho_2}{\rho_1+\rho_2}\right)^{1/2}I. \qquad (6.6)$$

The external current and potential drops are related by the integral Ohm's law:

$$U_x=\rho_{eff}J_x,\quad U_y=\rho_{eff}J_y. \qquad (6.7)$$

From (6.5), (6.6), and (6.7) the known result $\rho_{\rm eff} = (\rho_1\rho_2)^{1/2}$ follows.

The effective resistivity of this inhomogeneous medium can thus in principle be determined in two ways: first, by means of average currents and fields, with the help of the symmetry transformations which have been established; second, from the integral Ohm's law, in which the external currents and potential drops are calculated from the results of the solution of the boundary-value problem.

The solution found for the boundary-value problem shows that the current distribution in the present heterogeneous system has the following features. The flow of current from one square cell and to another is accompanied by a concentration of the current near two corners which lie on the same diagonal of each square. Directly at this points, the current density vector increases without bound. Near the two other corners of the square the current density vector is bounded and right at the corners it vanishes. The average current in the system, which is also the external current flowing in the medium, is directed along the corresponding diagonal of the squares. The currents averaged over the black and white cells are not the same in magnitude.

A second particular solution of the boundary-value problem is found under the assumption that the current distribution in the system has the same symmetry, given by the transformation (4.5), in which the negative sign is taken.

In this case, the boundary conditions (3.4) become

$$j_1(t)=(-\Delta-i(1-\Delta^2)^{1/2})\overline{j_1(t)},\quad t\in L_g;$$
$$j_1(t)=(\Delta-i(1-\Delta^2)^{1/2})\overline{j_1(t)},\quad t\in L_v. \qquad (6.8)$$

These results differ from the corresponding conditions in (6.1) only in the coefficients of the vector $j_1(t)$: The moduli of the complex coefficients are the same in the two cases, while the arguments are opposite in sign. This difference is reflected in the form of the solution of the Riemann boundary-value problem, which is now

$$j_1(z)=C_2\exp\left\{-\frac{i\pi}{2}\left(\frac{1}{2}+\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{-2\gamma},$$
$$j_2(z)=C_2\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\exp\left\{\frac{i\pi}{2}\left(-\frac{1}{2}+\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{-2\gamma}. \qquad (6.9)$$

Here C_2 is a real constant, determined by the integral conditions of the problem, (3.5), and the notation is otherwise the same as in (6.2) and (6.4).

Although the particular solutions are given by superficially similar expressions, they actually describe different field patterns. For example, while the first particular solution is characterized by a concentration of the current, with a theoretically unbounded growth of the current at a certain pair of corners of the squares, lying on a diagonal, the second solution vanishes at these corners and increases without bound at the other pair of corners. In other works, the singular and nonsingular points of the two solutions trade places.

In both the first and second cases, the external current in the cells of the system is directed along diagonals of the squares—different diagonals for the two solutions.

The sum of the particular solutions found is also a solution of the boundary-value problem:

$$j(z)=j_1(z)=C_1\exp\left\{\frac{i\pi}{2}\left(\frac{1}{2}+\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{2\gamma}$$
$$+C_2\exp\left\{-\frac{i\pi}{2}\left(\frac{1}{2}+\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{-2\gamma},\quad z\in S_1;$$

$$j(z)=j_1(z)=\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\left\{C_1\exp\left\{\frac{i\pi}{2}\left(\frac{1}{2}-\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{2\gamma}\right.$$
$$\left.+C_2\exp\left\{\frac{i\pi}{2}\left(-\frac{1}{2}+\gamma\right)\right\}\left(\frac{\operatorname{cn}u}{\operatorname{sn}u\operatorname{dn}u}\right)^{-2\gamma}\right\},\quad z\in S_2$$
$$(u=Kz/h). \qquad (6.10)$$

As is well known, under general assumptions a linear combination of particular solutions with arbitrary real coefficients is a general solution if the boundary conditions are satisfied. According to (6.10), with any prespecified vector external current $J = J_x - iJ_y$ flowing in the system we can associate a sum of particular solutions for each of which the current is directed along a diagonal, and the given current J is found upon summation.

The complete solution of the problem has some features which distinguish it from the particular solutions. While the latter are characterized by an unbounded growth of the current density at one pair of corners of the square cells and by zero values of the current density and the other pair, in the complete solution the vector current density has integrable singularities at all corners of the squares. This result means that in general the flow of a current out of a cell into the adjacent cells is accompanied by a concentration of the current near the corners of the square cells. The unbounded increase in the current at the corners of the squares results from the particular way in which the cells make contact at the corner points and the tendency toward a predominant current flow through the cells with the smaller resistivity.

The constants C_1 and C_2, which are determined by (3.5), can be described by the expressions

$$C_1=\frac{J_x-J_y}{I(2(1+\Delta))^{1/2}},\quad C_2=\frac{J_x+J_y}{I(2(1+\Delta))^{1/2}}. \qquad (6.11)$$

We thus see that when the components of the external current are equal ($J_x = \pm J_y$), i.e., when this current is flowing along one of the diagonals of the square cells, one of the two constants is zero, and the complete solution becomes the corresponding particular solution.

7. GENERALIZED SYMMETRY TRANSFORMATION

A general solution of the boundary-value problem for the spatial distribution of the current in the system is invariant under linear symmetry transformations similar to those characteristic of the particular solutions. It is not difficult to show that for a general solution the vector current densities in adjacent cells are related by the quadratic relation

$$j_1^2(z)+\frac{\rho_2}{\rho_1}\overline{j_2^2(\bar{z})}=4C_1C_2=\frac{J_x^2-J_y^2}{I^2(1+\Delta)}, \qquad (7.1)$$

which can be written in the equivalent form

$$[j_1(z)+i(\rho_2/\rho_1)^{1/2}\overline{j_2(\bar{z})}][j_1(z)-i(\rho_2/\rho_1)^{1/2}\overline{j_2(\bar{z})}]=4C_1C_2. \qquad (7.2)$$

A relationship of this nature between the currents makes it impossible to extend to the general case the conclusion, which was drawn above for the particular solutions, that the Joule energy dissipation is the same in all the cells, as can be confirmed by direct calculation.

Furthermore, relations (5.11) and (5.12) between the average currents and fields cannot be extended to the general solution of the problem. The derivation of those relations leaned heavily on linear symmetry transformations (4.5). For a general solution, the latter would have to be replaced by quadratic relations (7.1) and (7.2), which become linear when their right sides are zero. This situation is possible under the condition $C_1 = 0$ or $C_2 = 0$, i.e., under the equalities $J_x = \pm J_y$, which correspond to the particular solutions of the problem, as stated above.

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