Properties of photocount statistics for short observation times and short optical pulses

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(Submitted 1 March 1988)
Zh. Eksp. Teor. Fiz. **96**, 528–542 (August 1989)

Glauber's theory of photocounts leads to violation of the most important principle in physics—causality. This breakdown is especially notable for short observation times and short (femtosecond) optical pulses. We analyze the origins of this violation of causality, and present an alternative, corrected theory of photocounts. An n-photon state is used to demonstrate the differences between the two theories.

1. INTRODUCTION

Glauber's widely known theory of photodetection (see Refs. 1–3, for example) enables one to calculate photon counting statistics for different states of the radiation field, and is in good agreement with experimental results.[3,4] Specifically, it yields a Poisson distribution for photocounts from a coherent state of the electromagnetic field. In those cases to which the semiclassical formalism of Mandel[5] applies, fluctuations in the number of photocounts must be at least as great as for a Poisson distribution. For states of the field that are highly nonclassical, however (for example, states with a definite number of photons[6] or squeezed states[7]), fluctuations can be smaller (sub-Poissonian statistics). In principle, this situation cannot be accounted for by Mandel's semiclassical formula. Interest in such states has recently experienced an upsurge: for instance, the coordinates of an oscillator in a squeezed state can be measured much more accurately than they could be otherwise.[8]

In a previous note,[9] we pointed out that Glauber's theory of photodetection has a fundamental problem—it violates causality: in other words, there is a nonzero probability of recording the arrival of a photon before it has reached the detector. Bykov[10] subsequently noted that there is an analogous virtual breakdown of causality involved in the transmission of an excitation between atoms. Normally, when the duration of a light pulse (signal) is much greater than the period of oscillation, this virtual advance, while somewhat puzzling, is nevertheless of little practical import, since it only amounts to approximately one period. But for short (femtosecond) pulses,[11] this defect of the theory also becomes significant in practical terms.

In this paper, we analyze the premises underlying Glauber's theory that lead to the violation of causality, formulate an emended theory free of this drawback, and examine the resulting changes in the photocount probability distribution. It turns out, as might be expected *a priori*, that substantial changes become manifest only when the signals involved are sufficiently brief. These changes, however, can be quite significant. For example, photocount fluctuations that are predicted to be sub-Poissonian by the Glauber theory can be larger than for a Poisson distribution. This must be borne in mind when one studies the states of highly nonclassical fields.

In the Glauber theory of photodetection, the electromagnetic field is described by a set of quantum correlation functions of the form

$$G_{i_1\ldots i_n j_1\ldots j_n}(x_1,\ldots,x_n) = \mathrm{Sp}[\hat\rho \hat E^{(+)}_{i_1}(x_1)\ldots\hat E^{(+)}_{i_n}(x_n)\hat E^{(-)}_{j_1}(x_1)\ldots\hat E^{(-)}_{j_n}(x_n)]. \quad (1)$$

Here $x = (\mathbf{r},t)$, $\hat p$ is the operator describing the photon statistics (density matrix), and the field $\hat E_i^{(-)}(x)$ is the positive-frequency part of the transverse electric field operator:

$$\hat E_i^{(-)}(x) = \frac{i(\hbar c)^{1/2}}{2\pi}\int d^3k\, k^{1/2} e_{i\alpha}(\mathbf{k}) \exp[i(\mathbf{kr}-ckt)]\hat a_\alpha(k). \quad (2)$$

In Eq. (2), $k = |\mathbf{k}|, e_{i\alpha}(\mathbf{k})$ is the ith component of the polarization vector $\mathbf{e}_\alpha(\mathbf{k})$, $\alpha = 1,2$, which satisfies the relations $\mathbf{k}\mathbf{e}_\alpha(\mathbf{k}) = 0, e_{i\alpha}e_{j\alpha} = \delta_{ij} - k_ik_j/k^2$; $\hat a_\alpha(\mathbf{k})$ is the annihilation operator for a photon with polarization $\mathbf{e}_\alpha$ and wave vector $\mathbf{k}$:

$$[\hat a_\alpha(\mathbf{k}),\ \hat a_\beta^+(\mathbf{k}')] = \delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}'). \quad (3)$$

The negative-frequency part of the electric field $\hat E_j^{(+)}(x)$ is defined by $\hat E_i^{(+)}(x) = (\hat E_i^{(-)}(x))^+$, and the operator yielding the transverse part of the electric field is

$$\hat E_i(x) = \hat E_i^{(+)}(x) + \hat E_i^{(-)}(x). \quad (4)$$

According to the Glauber theory, if a group of isolated atoms, with dipole moments lined up in the direction of the unit vector $\mathbf{n}$, are located at the points $\mathbf{r}_1\cdots\mathbf{r}_N$, the joint probability of detecting at photon at $\mathbf{r}_1$ in the time interval $(t_1, t_1 + dt_1)$,...and one at $\mathbf{r}_N$ in the interval $(t_N, t_N + dt_N)$ is

$$dW_N = \nu^N n_{i_1}\ldots n_{i_N} n_{j_1}\ldots n_{j_N} G_{i_1\ldots i_N j_1\ldots j_N}(x_1,\ldots,x_N)\,dt_1\ldots dt_N. \quad (5)$$

In this formula, ν is the quantum efficiency of the sensor, which depends on how large a dipole moment the atom has. Specifically, the probability of detecting a single photon at the point $\mathbf{r}$ in some infinitesimal time interval is

$$dW_1 = \nu n_i n_j G_{ij}(x)\,dt. \quad (6)$$

In deriving Eq. (5), Glauber[1] carried out his calculations to the first nonvanishing order of perturbation theory. To find the probability of detecting n photons in a finite time T, we proceed as follows. Let the field be in a coherent state $|z\rangle$ for which

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$$\hat{a}_\lambda(\mathbf{k})|z\rangle = z_\lambda(\mathbf{k})|z\rangle,\ \hat{\rho}=\hat{\rho}_z=|z\rangle\langle z|. \quad (7)$$

According to (2), we then have

$$\hat{E}_j^{(-)}(x)|z\rangle = V_j(x)|z\rangle,\ \langle z|\hat{E}_i^{(+)}(x) = V_i^*(x)\langle z|, \quad (8)$$

where

$$V_i(x) = \frac{i(\hbar c)^{1/2}}{2\pi}\int d^3k\, k^{1/2} e_{i\alpha}(\mathbf{k}) \exp[i(\mathbf{kr}-ckt)] z_\alpha(\mathbf{k}). \quad (9)$$

Making use of Eqs. (9), (1), and (5), we finally obtain

$$dW_N = [\nu|\mathbf{n}\mathbf{V}(x_1)|^2 dt_1][\nu|\mathbf{n}\mathbf{V}(x_2)|^2 dt_2]\dots[\nu|\mathbf{n}\mathbf{V}(x_N)|^2 dt_N] = dW_1(x_1)\dots dW_1(x_N). \quad (10)$$

Equation (10) holds, in particular, for $\mathbf{r}_1 = \mathbf{r}_2 = \dots = \mathbf{r}_N = \mathbf{r}$, meaning that the probability dW_N of detecting N photons at the point $\mathbf{r}$ during a set of nonoverlapping time intervals is equal to the product of probabilities of those same events, or in other words, the individual detections are statistically independent. Inasmuch as $dW_1 \sim dt$, fundamental probability theory says that the probability of detecting m events in a finite time T is given by the Poisson distribution

$$W_m(z,T) = \frac{1}{m!}\left[\nu\int_0^T |\mathbf{n}\mathbf{V}(\mathbf{r},t)|^2 dt\right]^m \times \exp\left[-\nu\int_0^T |\mathbf{n}\mathbf{V}(\mathbf{r},t)|^2 dt\right]. \quad (11a)$$

Taking advantage of Eq. (8), this expression can be rewritten in the form

$$W_m(z,T) = \langle z|\hat{W}_m^{(G)}|z\rangle = \mathrm{Sp}(\hat{\rho}_z \hat{W}_m^{(G)}), \quad (11b)$$

where

$$\hat{W}_m^{(G)} = \frac{1}{m!} : \left[\nu\int_0^T \hat{E}^{(+)}(t)\hat{E}^{(-)}(t)dt\right]^m \times \exp\left[-\nu\int_0^T \hat{E}^{(+)}(t)\hat{E}^{(-)}(t)dt\right]:, \quad (12)$$

$$\hat{E}^{(\pm)}(t) \equiv n_i \hat{E}_i^{(\pm)}(\mathbf{r},t) \quad (13)$$

(the superscript G refers to the Glauber theory) and the notation: $\hat{O}$: signifies that the operator $\hat{O}$ is in normal-ordered form.

The preceding argument identifies Eqs. (11b) and (12) as the probability of detecting m in a finite time given a field in a coherent state. In order to proceed to the general case, we represent the arbitrary statistical operator $\hat{\rho}$ in terms of $\hat{\rho}_z$, using the so-called Glauber-Sudarshan P-representation, which takes the form

$$\hat{\rho} = \iint \frac{d^2z}{\pi} P(z)\hat{\rho}_z, \quad d^2z = d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z). \quad (14)$$

In the Glauber theory, one makes the valid but nevertheless supplementary assumption that even for an arbitrary state of the field, the probability $W_m(T)$ of recording the arrival of m photons in a time T may be expressed in terms of this same quantity $W_m(z,T)$ for a coherent state:

$$W_m(T) = \iint \frac{d^2z}{\pi} P(z) W_m(z,T), \quad (15)$$

as in Eq. (14). Here $W_m(z,T)$ is to be interpreted as the conditional probability of detecting m photons for the case in which the field is in a coherent state $\hat{\rho}_z$, and according to (14), $P(z)$ is the probability density of the system being in the state $\hat{\rho}_z$. In fact, however, it is a quasiprobability, and can even be negative (see Ref. 13, for example). In the Glauber theory, therefore, the route from (11b) to (15) actually involves an arbitrary assumption. It would be a difficult one to justify if, for example, $P(z)$ were to take on negative values. Nevertheless, following the usual practice, we shall use Eq. (15) as a vehicle with which to exhibit the modifications to the conventional theory necessitated by the violation of causality.

Substituting (11b) into (15) and making use of (14), we arrive at the central equation of the Glauber theory for the probability of detecting m photons in time T:

$$W_m(T) = \iint \frac{d^2z}{\pi} P(z)\,\mathrm{Sp}(\hat{\rho}_z \hat{W}_m^{(G)}) = \mathrm{Sp}\left\{\rho : \frac{1}{m!}\left[\nu\int_0^T \hat{E}^{(+)}(t)\hat{E}^{(-)}(t)dt\right]^m \times \exp\left[-\nu\int_0^T \hat{E}^{(+)}(t)\hat{E}^{(-)}(t)dt\right]:\right\}. \quad (16)$$

We have reproduced one of the possible derivations leading to Eq. (16) because we shall shortly follow a similar line of reasoning as it applies to a slightly different situation.

At first glance, it might seem that the derivation of (16) to be found in Ref. 3 [Eqs. (8.96)–(8.100)] does not rely on the assumption (15). This is in fact not the case, as instead of Eq. (8.98) of Ref. 3, which in the notation used in that book appears as

$$\langle :(sT\hat{a}^+\hat{a})^{p-r}(1-sT\hat{a}^+\hat{a})^r:\rangle,$$

the correct expression is

$$\langle :(sT\hat{a}^+\hat{a})^{p-r}:\rangle\langle 1-sT\hat{a}^+\hat{a}\rangle^r.$$

These two expressions are identical if the field is in a coherent state. Equation (8.100) of Ref. 3 is therefore valid only for such a state, and the assumption (15) is necessary if one is to generalize it to an arbitrary state.

2. CAUSALITY VIOLATION IN GLAUBER'S THEORY OF PHOTODETECTION

We now show that the probability $dW_1(t)$ given by (6) violates causality (according to (10), this is a problem shared by all of the probabilities dW_N). Making use of (1) and (8), we have for a coherent state $|z\rangle$

$$dW_1(t) = \nu n_i n_j \langle z|\hat{E}_i^{(+)}(\mathbf{r},t)\hat{E}_j^{(-)}(\mathbf{r},t)|z\rangle dt = \nu n_i n_j V_i^*(t)V_j(t)dt = \nu|V(t)|^2 dt = \nu[(\mathrm{Re}\, V)^2 + (\mathrm{Im}\, V)^2]dt,\ V(t) \equiv n_i V_i(\mathbf{r},t). \quad (17)$$

Here we point out that according to (9), the function $V(t)$ is expanded as a Fourier integral solely over positive frequencies $\omega = ck \geqslant 0$. Thus, if $t = t' - t''$, where $t'' \geqslant 0$, there will be an additional damping factor $\exp(-ckt'')$ in the integrand appearing in (9). Since the integral in Eq. (9) converges when $t''=0$, it also converges for all $t''>0$, i.e., $V(t)$ is an analytic function of t in the lower half of the complex t plane. Then[12] Re V and Im V are related by a Hilbert transform:

$$\text{Im } V(t)=\frac{1}{\pi}\fint_{-\infty}^{+\infty}\frac{\text{Re } V(t')}{t'-t}dt' \tag{18}$$

(the bar through the integral sign denotes the Cauchy principal value).

According to (4) and (13), the electric field $E(t)$ takes the form

$$E(t)=2\,\text{Re } V(t). \tag{19}$$

From Eq. (9), we see that $V(\mathbf{r},t)$ and $E(\mathbf{r},t)$ satisfy the wave equation. We examine a plane-wave solution of this equation for the electric field, where the wave has a sharp leading edge:

$$E(\mathbf{r},t)=\theta(ct-z)f(z-ct),\qquad \theta(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases} \tag{20}$$

Equation (18) then yields

$$\text{Im } V(t)=\frac{1}{2\pi}\int_{z/c}^{\infty}\frac{f(z-ct')}{t'-t}dt'.$$

Clearly, Im$V(t)$ is then nonzero for any t, including $t<z/c$. Since according to (17), $dW_1(t)$ contains a term proportional to Im$V(t)|^2$, the probability of detecting a photon becomes nonzero even before the arrival of the electric field at the detector. Figure 1 shows the value of $dW_1(t)$ corresponding to a finite-duration wavetrain of the form

$$E(z,t)=\theta(ct-z)\theta(z-ct+l)\sin(kz-\omega t+\varphi)$$

for $kl=\pi$ and $kl=4\pi$, with $\varphi=0$, $\varphi=\pi/2$. We have plotted $\zeta=kz-\omega t$ along the horizontal axis; the signal occupies the segment $\zeta\leqslant 0$. it is quite evident that $dW_1>0$ for $\zeta>0$—i.e., prior to the arrival of the wave at the detector. The shorter the wave, the stronger the virtual precursor, and the amount of advance is in order of magnitude, equal to the wavelength (in space) or period of oscillation (in time).

Note that there is a second relation analogous to (18) that expresses ReV in terms of ImV, so that if ImV is localized, ReV must necessarily turn out not to be, and the probability $dW_1(t)$ defined by (6) cannot in principle be localized in time. Only the electric field E and functions thereof can be specified by local quantities.

Mathematically, the structure of (18) is reminiscent of the Kramers-Kronig dispersion relation. The latter applies to the spectral components $\varepsilon(\omega)$ and results directly from the causality requirement, however, while (18) applies to the time-dependence of the fields and results from artificial constraints imposed on the spectrum of the signal.

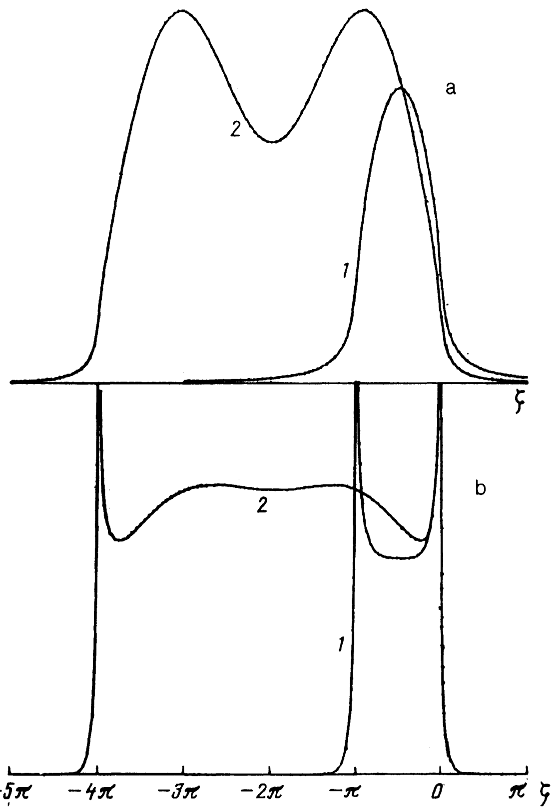

FIG. 1. Photon counting rate as a function of $\zeta=kz-\omega t$ for different pulse lengths: 1) $kl=\pi$, and 2) $kl=4\pi$. a) Pulse with no jump at $\zeta=0$ ($\varphi=0$); b) pulse with a jump at $\zeta=0$ ($\varphi=\pi/2$). A nonzero photocount rate for $\zeta>0$ represents a nonphysical precursor that violates causality.

3. ELIMINATING THE BREAKDOWN OF CAUSALITY FROM THE THEORY OF PHOTOCOUNTS

Let us now analyze the derivation of the equations of the Glauber theory in order to ascertain where it is that the assumptions leading to the violation of causality are being made. Clearly even Eq. (1) suffers from this problem. If we turn to Glauber's derivation[1] of Eq. (1), we see easily that the original field operators appear in the form $\hat{E}=\hat{E}^{(+)}+\hat{E}^{(-)}$, along with the atomic dipole moment operator $\hat{d}$. If an atom in a lower level $|g\rangle$ makes a transition to an upper level $|e\rangle$ due to interaction with the field, a temporal phase factor exp $(i\omega_0 t)$ will appear; $\omega_0>0$ is the frequency of the atomic transition. At the same time, the expansion of the operator $\hat{E}^{(+)}$ contains the factor $\exp(ickt)$, and likewise $\hat{E}^{(-)}$ contains the factor $\exp(-ickt)$, $k>0$. Obviously, there will be a factor $\exp[i(\omega_0+ck)t]$ in the expansion of $\hat{E}^{(+)}\langle e|\hat{d}|g\rangle$, and $\exp[i(\omega_0-ck)t]$ in the expansion of $\hat{E}^{(-)}\langle e|\hat{d}|g\rangle$. If the interaction is prolonged ($\omega_0 T\gg 1$), integration of the term $\hat{E}^{(+)}\langle e|\hat{d}|g\rangle$ with respect to t will therefore yield small quantities; these have been discarded from the Glauber theory, which is equivalent to working in the "rotating wave" approximation.

This is precisely the point at which the violation of causality makes its way into the theory. For $\omega_0 T\gg 1$, the wave advance that results is relatively insignificant (compared to T), but if $\omega_0 T\sim 1$, it can be of the same order of magnitude as the signal duration. This can occur, for example, in the realm of femtosecond optical pulses.

One more important point must be noted here. Eliminating the term $\hat{E}^{(+)}$ from the sum $\hat{E}=\hat{E}^{(+)}+\hat{E}^{(-)}$ in the interaction Hamiltonian automatically guarantees that the operators $\hat{E}^{(-)}$ and $\hat{E}^{(+)}$ in Eq. (1) appear in normal-ordered form. Thus, in addition to the convenience that this provides in calculating mean values for coherent states of the field, it also gets rid of infinities associated with the divergence of the commutators $[\hat{E}_i^{(-)};\hat{E}_j^{(+)}]$.

This makes it clear how the calculations must be modified in order to avoid problems with causality violation. The terms $\hat{E}^{(+)}$ must be retained, and the correlation functions (1) must be replaced by functions which, up to a multiplicative factor to be specified later, are of the form

$$G_{i_1\ldots i_{2n}}(x_1,\ldots,\ x_{2n})=c_n\,\mathrm{Sp}\,\{\hat{\rho}{:}\hat{E}_{i_1}(x_1)\ldots\hat{E}_{i_{2n}}(x_{2n}){:}\}. \tag{21a}$$

The normal ordering in (21) eliminates from the commutators $[\hat{E}_i^{(-)};\hat{E}_j^{(+)}]$ infinite contributions that do not depend on the state of the field.

In fact, we require only those correlation functions like (21) in which the arguments x are in pairwise correspondence:

$$G_{i_1\ldots i_{2n}}(x_1,x_1,x_2,x_2,\ldots,x_n,x_n)$$
$$=c_n\,\mathrm{Sp}\,\{\hat{\rho}{:}\hat{E}_{i_1}(x_1)\hat{E}_{i_2}(x_1)\ldots\hat{E}_{i_{2n-1}}(x_n)\hat{E}_{i_{2n}}(x_n){:}\}. \tag{21b}$$

Consider now how replacing (1) with (21b) changes subsequent results of the Glauber theory. Most importantly, instead of (6), we obtain for $n=1$ (assuming $c_1=1/2$, so that the term $\hat{E}^{(+)}\hat{E}^{(-)}$ enters with the same numerical coefficient as before)

$$dW_1={}^1/_2\nu n_in_j\,\mathrm{Sp}\{\hat{\rho}{:}\hat{E}_i(x)\hat{E}_j(x){:}\}dt$$
$$=\nu\,\mathrm{Sp}\{\hat{\rho}[\hat{E}^{(+)}(x)\hat{E}^{(-)}(x)+{}^1/_2(\hat{E}^{(+)}(x))^2+{}^1/_2(\hat{E}^{(-)}(x))^2]\}$$

[in the notation of (13)]. There is an additional term here, $\frac{1}{2}[(\hat{E}^{(+)})^2+(\hat{E}^{(-)})^2]$, over and above those appearing in (6), which has been discarded in the Glauber theory. Let $\hat{\rho}=\hat{\rho}_z=|z\rangle\langle z|$. Making use of (8) once again, we obtain

$$dW_1=\nu[V^*V+{}^1/_2V^2+{}^1/_2V^{*2}]dt={}^1/_2\nu(V+V^*)^2dt$$

or

$$dW_1={}^1/_2\nu E^2(\mathbf{r},t)dt. \tag{22}$$

We see then, that the count rate dW_1/dt is determined by the square of the (real) electric field, and satisfies the requirements of causality. Let us now turn to the probability of detecting n photons arriving in non-overlapping time intervals $(t_1,t_1+dt_1),\ldots,(t_N,t_N+dt_N)$ at a point $\mathbf{r}$.

Here Eq. (5) must be used, with the function (21b) inserted instead of the Glauber correlation function (1). Choosing the numerical coefficient accordingly, we obtain

$$\frac{dW_n}{dt_1\ldots dt_n}=\left(\frac{\nu}{2}\right)^n\mathrm{Sp}\{\hat{\rho}{:}\hat{E}^2(\mathbf{r},t_1)\ldots\hat{E}^2(\mathbf{r},t_n){:}\}. \tag{23}$$

Now we again examine the case in which $\hat{\rho}=\hat{\rho}_z=|z\rangle\langle z|$. We may take advantage of the general formula for the mean of a normal-ordered operator over coherent states, which in the single-mode case takes the form

$$\langle z|{:}F(\hat{a},\hat{a}^+){:}|z\rangle=F(z^*,z),$$

which can be recast in an obvious manner for the multiple-mode case. Taking $V_i\equiv V(\mathbf{r},t_i)$, we then obtain from (23)

$$\frac{dW_n}{dt_1\ldots dt_n}=\left(\frac{\nu}{2}\right)^n(V_1^*+V_1)^2(V_2^*+V_2)^2\ldots(V_n^*+V_n)^2$$
$$=\frac{\nu}{2}E^2(t_1)\ldots\frac{\nu}{2}E^2(t_n).$$

We have thus obtained a relation that is identical in form with (10):

$$dW_n(t_1,\ldots,t_n)=dW_1(t_1)\ldots dW_1(t_n), \tag{24}$$

where $dW_1(t)$ is now given by Eq. (22). Clearly Eq. (10), upon which (16) is based, relies only on the normal ordering of the field operators, and is equally valid in the present case, which corrects the Glauber theory. Having traced out the path between Eqs. (10) and (16), we can be sure that the same considerations apply to the case at hand as well. We are left, then, with an equation analogous to (16):

$$W_m(T)=\mathrm{Sp}(\hat{\rho}\hat{W}_m). \tag{25}$$

In accordance with (22), instead of the operator $\hat{W}_m^{(G)}$ as in (12), we have the operator

$$\hat{W}_m(T)=\frac{1}{m!}{:}\,\hat{N}^m\exp(-\hat{N}){:}\,,\qquad \hat{N}=\frac{\nu}{2}\int_0^T\hat{E}^2(t)\,dt, \tag{26}$$

where

$$\hat{E}(\mathbf{r},t)=n_iE_i(\mathbf{r},t)=n_i[\hat{E}_i^{(+)}(\mathbf{r},t)+\hat{E}_i^{(-)}(\mathbf{r},t)].$$

Based on the foregoing discussion, it is easily appreciated that Eq. (25) is free of the problems afflicting (16). We shall also have occasion to make use of the generating function for the factorial moments:

$$Q(\lambda)=\sum_{m=0}^{\infty}(1-\lambda)^mW_m=\mathrm{Sp}\{\hat{\rho}:\exp(-\lambda\hat{N}){:}\}. \tag{27a}$$

Differentiating this equation with respect to λ, we find that

$$Q^{(k)}(\lambda)=(-1)^k\sum_{m=0}^{\infty}m(m-1)\ldots(m-k+1)(1-\lambda)^{m-k}W_m$$
$$=(-1)^k\,\mathrm{Sp}\{\hat{\rho}{:}\hat{N}^k\exp(-\lambda\hat{N}){:}\}.$$

Taking $\lambda=0$, we have

$$Q^{(k)}(0)=(-1)^k\langle m(m-1)\ldots(m-k+1)\rangle=(-1)^k\,\mathrm{Sp}\{\hat{\rho}{:}\,\hat{N}^k{:}\},$$

i.e., the factorial moments of the number of photons detected equal the mean values of the corresponding powers of the operator $\hat{N}$:

$$\langle m(m-1)\ldots(m-k+1)\rangle=\mathrm{Sp}(\hat{\rho}{:}\hat{N}^k{:}). \tag{27b}$$

We shall demonstrate below that in the appropriate limiting case, Eq. (25) yields the same results as (16), but that for short pulses or brief observation times T, the results given by (26) can be vastly different.

4. PHOTOCOUNT STATISTICS FOR DIFFERENT STATES OF THE RADIATION FIELD

In deriving (25), we have actually relied on the fact that the photocounts for a coherent state of the field will conform to Poisson statistics. The only difference between the present and previous theories is that the parameter of this distribution-the mean number of photocounts detected during the observation time—is given by

$$\bar{N}(T)=\frac{\nu}{2}\int_0^T E^2(t)\,dt,$$

where the square of the electric field appears in the integrand, rather than the absolute square of the analytic signal, as in the Glauber formula.

Under other circumstances, however, the difference can be more substantial. As a second example, let us consider an n-photon state of the field:

$$|n\rangle=\frac{1}{(n!)^{1/2}}\left[\int F_\alpha(\mathbf{k})\hat{a}_\alpha^+(\mathbf{k})\,d^3k\right]^n|0\rangle. \quad (28)$$

This is not the most general possible n-photon state, but it encompasses a wide range of electromagnetic fields, with the spectral function $F_\alpha(\mathbf{k})$ determining both the spatial and temporal shape of the field. Thus, if $F_\alpha(\mathbf{k})$ is nonzero only within some sector parallel to a given vector $\mathbf{k}_0$, the state (28) describes a plane wave that propagates in the $\mathbf{k}_0$-direction and that has a bounded spectrum; in particular, such a wave might be a short pulse. In the present case, if $F_\alpha(\mathbf{k})$ is nonzero only on a sphere of radius $|\mathbf{k}_0|$ and has a special form, then the state (28) can be a monochromatic Gaussian beam (with frequency $\omega=c|\mathbf{k}_0|$).

One feature characteristic of the state (28) is that its temporal and spatial shape are independent of the number of photons. In this regard, the state is reminiscent of the familiar field modes in an optical resonator, for example, Just as in the latter, one can introduce creation and annihilation operators[14] for the state (28):

$$\hat{a}^+=\int F_\alpha(\mathbf{k})\hat{a}_\alpha^+(\mathbf{k})\,d^3k,\quad \hat{a}=\int F_\alpha^*(\mathbf{k})\hat{a}_\alpha(\mathbf{k})\,d^3k, \quad (29)$$

having required that those operators satisfy the canonical commutation relations

$$[\hat{a},\hat{a}^+]=\int F_\alpha^*(\mathbf{k})F_\alpha(\mathbf{k})\,d^3k=1. \quad (30)$$

One then obtains the same equations for $|n\rangle$ as for a simple harmonic oscillator:

$$\hat{a}^+|n\rangle=(n+1)^{1/2}|n+1\rangle,\ \hat{a}|n\rangle=n^{1/2}|n-1\rangle,$$
$$|n\rangle=(n!)^{-1/2}(\hat{a}^+)^n|0\rangle. \quad (31)$$

A coherent state of the field (7) can be expressed in terms of the operators (29) (see Ref. 14):

$$|z\rangle=\exp\left\{-\frac{|z|^2}{2}+z\hat{a}^+\right\}|0\rangle$$
$$=\exp\left(-\frac{|z|^2}{2}\right)\sum_n\frac{z^n}{(n!)^{1/2}}|n\rangle, \quad (32)$$

with

$$z_\alpha(\mathbf{k})=zF_\alpha(\mathbf{k}). \quad (33)$$

It can easily be shown that

$$[\hat{a}_\alpha(\mathbf{k}),\hat{a}^+]=F_\alpha(\mathbf{k}),$$

thereby yielding

$$[\hat{a}_\alpha(\mathbf{k}),(\hat{a}^+)^n]=nF_\alpha(\mathbf{k})(\hat{a}^+)^{n-1},\ \hat{a}_\alpha(\mathbf{k})|n\rangle$$
$$=(n!)^{1/2}F_\alpha(\mathbf{k})|n-1\rangle, \quad (34)$$

$$\hat{E}^{(-)}(\mathbf{r},t)|n\rangle=n^{1/2}V(\mathbf{r},t)|n-1\rangle, \quad (35)$$

where $V(\mathbf{r},t)=n_i V_i(\mathbf{r},t)$, and from (9),

$$V_i(\mathbf{r},t)=z\frac{i(\hbar c)^{1/2}}{2\pi}\int d^3k\,k^{1/2}e_{i\alpha}(\mathbf{k})\exp[i(\mathbf{kr}-ckt)]F_\alpha(\mathbf{k}). \quad (36)$$

Note that for all spectrally band-limited fields ($\Delta\omega\ll\omega_0$, where ω_0 is the mean signal frequency), $V(\mathbf{r},t)$ oscillates rapidly at a frequency $\sim\omega_0$.

The next step is to use the corrected formula (25) for the n-photon state (28) to calculate $W_m(T)$. It is most convenient first to find the expected value of $:\hat{N}^m:$

$$J_{nm}\equiv\langle n|:\hat{N}^m:|n\rangle=\langle n|:(:\hat{N}:)^m:|n\rangle.$$

We may represent $:\hat{N}:$ in the form

$$:\hat{N}:=\hat{M}+\hat{L}_++\hat{L}_-,\quad \hat{M}=\nu\int_0^T\hat{E}^{(+)}(\mathbf{r},t)\hat{E}^{(-)}(\mathbf{r},t)\,dt,$$

$$\hat{L}_+=\frac{\nu}{2}\int_0^T[\hat{E}^{(+)}(\mathbf{r},t)]^2dt,\quad \hat{L}_-=(\hat{L}_+)^+. \quad (37)$$

Then

$$:(:\hat{N}:)^m:=\sum_{p=0}^m\sum_{j=0}^p\frac{m!}{(m-p)!\,(p-j)!\,j!}\hat{L}_+^j:\hat{M}^{m-p}:\hat{L}_-^{p-j}. \quad (38)$$

From (35) and (37) we obtain

$$\hat{L}_-|n\rangle=\frac{\nu}{2}\int_0^T[\hat{E}^{(-)}(\mathbf{r},t)]^2dt|n\rangle$$
$$=\frac{\nu}{2}n^{1/2}(n+1)^{1/2}\int_0^T V^2(t)\,dt|n-2\rangle,$$

or, with the notation

$$\mathcal{L}(\mathbf{r},t)=\frac{\nu}{2}\int_0^T V^2(t)\,dt, \quad (39)$$

we have

$$\hat{L}_-|n\rangle=n^{1/2}(n-1)^{1/2}\mathcal{L}|n-2\rangle. \quad (40)$$

Iteratively applying the operator $\hat{L}$, we obtain

$$\hat{L}_-^k|n\rangle=\theta\left(n-2k+\frac{1}{2}\right)\left[\frac{n!}{(n-2k)!}\right]^{1/2}\mathcal{L}^k|n-2k\rangle, \quad (41a)$$

where we have introduced a factor

$$\theta\left(n-2k+\frac{1}{2}\right)=\begin{cases}1, & 2k\leqslant n\\ 0, & 2k>n.\end{cases}$$

Taking the conjugate of (41a) yields

$$\langle n|\hat{L}_+^j=\theta\left(n-2j+\frac{1}{2}\right)\left[\frac{n!}{(n-2j)!}\right]^{1/2}(\mathcal{L}^*)^j\langle n-2j|. \quad (41b)$$

Assuming that $k=p-j$ in (41a), we obtain

$$\langle n|:\hat{N}^m:|n\rangle=\sum_{p=0}^m\sum_{j=0}^p\frac{m!}{(m-p)!\,(p-j)!\,j!}$$
$$\times\theta\left(n-2j+\frac{1}{2}\right)\left[\frac{n!}{(n-2j)!}\right]^{1/2}(\mathcal{L}^*)^j\theta\left(n-2p+2j+\frac{1}{2}\right)$$
$$\times\left[\frac{n!}{(n-2p+2j)!}\right]^{1/2}\mathcal{L}^{p-j}\langle n-2j|:\hat{M}^{m-p}:|n-2p+2j\rangle. \quad (42)$$

Now we note that $\hat{M}$ and $:\hat{M}^{m-p}:$ contain as many creation operators on the left as there are annihilation operators on the right. We then obtain a nonvanishing result for $\langle n-2j|:\hat{M}^{m-p}:|n-2p+2j\rangle$ only when $n-2j = n-2p+2j$, i.e., when $p=2j$. The sum over p in (42) can therefore be dropped, putting $p=2j$, and the sum over j can run from 0 to $[m/2]$, the integer part of $m/2$:

$$\langle n|:\hat{N}^m:|n\rangle=\sum_{j=0}^{[m/2]}\frac{m!}{(m-2j)!\,(j!)^2}\theta\left(n-2j+\frac{1}{2}\right)\times\frac{n!}{(n-2j)!}(\mathscr{L}^*\mathscr{L})^j\langle n-2j|:\hat{M}^{m-p}:|n-2j\rangle. \quad (43)$$

It remains to calculate the matrix element

$$\langle l|:\hat{M}^q:|l\rangle=v^q\int_0^T dt_1\ldots\int_0^T dt_q\langle l|\hat{E}^{(+)}(t_1)\ldots\hat{E}^{(+)}(t_q)\hat{E}^{(-)}(t_1)\ldots\hat{E}^{(-)}(t_q)|l\rangle.$$

Applying Eq. (35), we find

$$\langle l|:\hat{M}^q:|l\rangle=\frac{l!}{(l-q)!}\theta\left(l-q+\frac{1}{2}\right)\left[v\int_0^T|V(t)|^2dt\right]^q.$$

Substituting this expression into (43), we obtain

$$J_{nm}=\langle n|:\hat{N}^m:|n\rangle=\frac{n!\,m!}{(n-m)!}\theta\left(n-m+\frac{1}{2}\right)p^m\sum_{j=0}^{[m/2]}\frac{(\alpha^2/4)^j}{(j!)^2(m-2j)!}, \quad (44)$$

where

$$p=v\int_0^T|V^2(t)|dt,\quad \alpha^2=\left|\int_0^T V^2(t)\,dt\right|^2\Big/\left[\int_0^T|V^2(t)|dt\right]^2. \quad (45)$$

The J_{nm} can be expressed more compactly if we make use of a well-known identity for Legendre polynomials:

$$\sum_{j=0}^{[m/2]}\frac{m!}{(j!)^2(m-2j)!}\gamma^j=(1-4\gamma)^{m/2}P_m\left(\frac{1}{(1-4\gamma)^{1/2}}\right),$$

where

$$P_m(x)=(2^m m!)^{-1}(d/dx)^m[(x^2-1)^m].$$

Then the J_{nm} become

$$J_{nm}=\langle n|:\hat{N}^m:|n\rangle=\theta\left(n-m+\frac{1}{2}\right)\frac{n!}{(n-m)!}\times[p(1-\alpha^2)^{1/2}]^m P_m\left(\frac{1}{(1-\alpha^2)^{1/2}}\right). \quad (46)$$

Turning now to Eqs. (39), (40), and their conjugates, it can be shown that the quantities $\mathscr{L}$, $\mathscr{L}^*$, and α^2 reflect the contribution made by the non-Glauber operators $\hat{L}_+$ and $\hat{L}_-$. Consequently, the Glauber theory corresponds to the limit $\alpha^2\to 0$, so we naturally devote special attention to those effects related to the finiteness of α. For a quasimonochromatic field $\alpha^2 \ll 1$, since the integral in the numerator of α^2 contains a rapidly oscillating factor.

According to (27b), the J_{nm} are the factorial moments of the number of photocounts for an n-photon field. Let us denote the conditional moments of the quantity m (assuming that the field contains n photons) by $\langle m/n\rangle$, $\langle m^2/n\rangle$, etc. Then recalling that $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$, we find from (46) that for $m=1,2$.

$$\langle m/n\rangle=\theta(n-{}^1/_2)np,$$

$$\langle m(m-1)/n\rangle=\langle m^2/n\rangle-\langle m/n\rangle=\theta(n-{}^3/_2)n(n-1)p^2(1+{}^1/_2\alpha^2). \quad (47)$$

It is clear from the first of these equations that p is the probability of detecting a single photon. Combining the second equation with the first, we obtain (for $n\geqslant 2$)

$$\langle m^2/n\rangle-\langle m/n\rangle^2=np(1-p)+{}^1/_2\alpha^2 n(n-1)p^2. \quad (48)$$

Notice that in the present instance, when $\alpha=0$, (i.e., in the Glauber theory), the mean squared fluctuation in the number of photocounts, $np(1-p)$, is binomially distributed and, as is to be expected for the state $|n\rangle$ in the Glauber theory, is a sub-Poissonian quantity. If the finiteness of the signal is taken into account, the fluctuations are enhanced. If $\alpha^2>2/(n-1)$, the fluctuations in the number of photocounts in an n-photon state becomes larger than for a Poisson distribution. Thus, even if α^2 is very small, the contribution of the second term in (48) can be sizable. In particular, if fluctuations in an n-photon state in the theory with $\alpha=0$ are sub-Poissonian, taking the finiteness of α into account will always lead to super-Poissonian fluctuations for large n. This can be quite important in experiments designed to detect sub-Poissonian states.

Now let us find the value of $\langle n|W_m|n\rangle = W(m/n)$, the conditional probability of detecting m photons when the field is in an n-photon state. According to (25), (26), we have

$$W(m/n)=\frac{1}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\langle n|:\hat{N}^{m+k}:|n\rangle.$$

By virtue of (46), we obtain

$$W(m/n)=\frac{n!}{m!}\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\frac{(m+k)!}{(n-m-k)!}p^{m+k}\times\sum_{j=0}^{[(m+k)/2]}\frac{(\alpha^2/4)^j}{(j!)^2(m+k-2j)!}$$

or, more compactly,

$$W(m/n)=\frac{n!}{m!\,(n-m)!}(p\sqrt{1-\alpha^2})^m\sum_{k=0}^{n-m}\frac{(n-m)!}{k!\,(n-m-k)!}\times[-p(1-\alpha^2)^{1/2}]^k P_{m+k}\left(\frac{1}{(1-\alpha^2)^{1/2}}\right). \quad (49)$$

Let us analyze this equation. For $\alpha=0$, inasmuch as $P_{m+k}(1)=1$, the summation in (49) gives $(1-p)^{n-m}$, yielding the familiar formula for the binomial distribution:

$$W(m/n)=\frac{n!}{m!\,(n-m!)}p^m(1-p)^{n-m}. \quad (50)$$

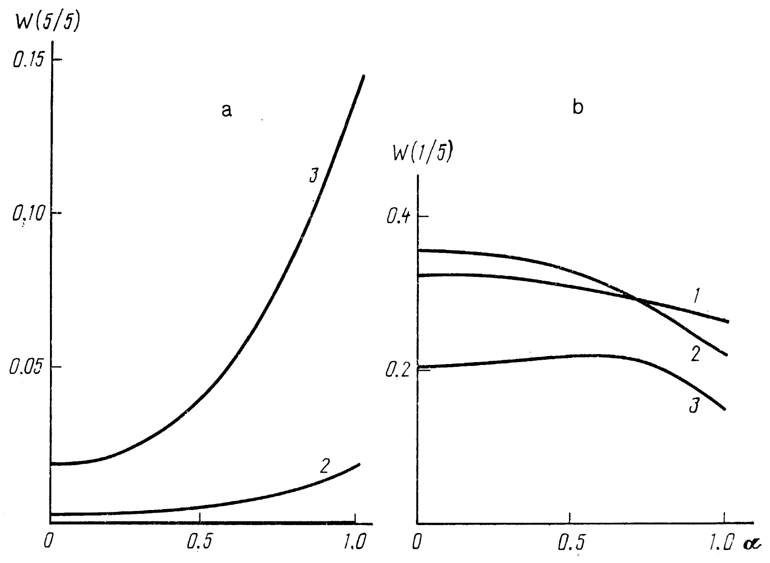

FIG. 2. Dependence of $W(m/5)$ on α: a) $m=5$; b) $m=1$. Curves 1, 2, and 3 correspond to $p=0.1, 0.3$, and 0.45. At $\alpha=1$, the probability $W(5/5)$ is approximately an order of magnitude greater than at $\alpha=0$ (panel a, curve 3).

When α becomes finite, the probability $W(m/n)$ is no longer binomial. According to (48), this is manifested as an increase in the size of the fluctuations in the number of photons detected as α increases. The probability of detecting all n photons also increases. According to (49), we have

$$W(n/n)=p^n[(1-\alpha^2)^{1/2}]^n P_n(1/(1-\alpha^2)^{1/2}). \quad (51)$$

If $\alpha\to 1$, we may retain just the leading term in $P_n(1/(1-\alpha^2)^{1/2})$, which depends on the nth power of the argument. This leads to the result

$$\lim_{\alpha\to 1} W(n/n)=\frac{(2n)!}{2^n(n!)^2}p^n\approx\frac{2^n}{(\pi n)^{1/2}}p^n. \quad (52)$$

Equation (52) is of interest in two respects. First, it is clear that the entire theory makes sense only when $p\leqslant 1/2$. The same constraint also follows from the requirement that $0\leqslant W(m/n)\leqslant 1$ for other values of m. This is a perfectly natural constraint, since when we derived Eq. (26) [as in the derivation of (16) in the Glauber theory], we worked to only the first nonvanishing order of perturbation theory. Second, comparing Eq. (50) for the case $m=n$ with (52), we see that $W(n/n)_{|\alpha=0}=p^n$, while for $\alpha=1$ there appears here a large numerical factor $2^n/(\pi n)^{1/2}$.

In Fig. 2, we have plotted the α-dependence of $W(m/5)$ for several values of m and p. It is clear from these curves that as α increases, the probability of detecting all n photons rises dramatically [in accordance with (52)], due to the reduced probability of detection for $0<m<n$. There is thus an additional bunching of photocounts, which can be treated as a consequence of the "tendency of bosons to clump together."

To estimate the magnitude of α, consider a pulse of light with frequency ω and duration τ:

$$V(t)={}^1/_2E_0e^{-i\omega t}\theta(t)\theta(\tau-t),$$

which is recorded over some time T.[1)] For the case in which $\tau\gg T$, we easily find from (45) that

$$\alpha^2=\left(\sin\frac{\omega T}{2}\Big/\frac{\omega T}{2}\right)^2, \quad p=\frac{\nu T}{4}|E_0|^2.$$

Conversely, when $\tau\ll T$, we obtain

$$\alpha^2=\left(\sin\frac{\omega\tau}{2}\Big/\frac{\omega\tau}{2}\right)^2, \quad p=\frac{\nu\tau}{4}|E_0|^2.$$

Thus, α is determined by the smaller of the two quantities τ and T.

5. CONCLUDING REMARKS

To summarize, our basic conclusions are as follows. The rotating-wave approximation that is actually employed in the theory of photodetection leads to a violation of causality in that theory. Allowance for the creation and annihilation of virtual pairs of photons makes it possible to eliminate this deficiency, and without overly complicating the theory, it leads to a set of corrected equations for photocount statistics. For coherent states, the latter yield Poisson statistics, as in the former theory, but with a different parameter. Matters are more complicated for highly nonclassical states, such as those with a fixed number of photons. In the Glauber theory, the parameter α, which describes the contribution of those processes that restore causality, is identically zero, and the probability of detecting photons is given by a binomial distribution. If one allows for the finiteness of the parameter α, then for small n, the probability distribution changes and fluctuations increase by a small quantity proportional to α^2. But if n is very large, the corrections to the probability of detecting a large number of photons will differ radically from the predictions of the Glauber theory, even when α is very small.

We close by noting one more important point. The operator $\hat{W}_m$, whose mean value gives the probability of detecting m photons, is diagonal in the $|n\rangle$ basis for the single-mode case in the Glauber theory. This means that if a state of the field $|\Psi\rangle$ is of the form $|\Psi\rangle=\Sigma\Psi_n|n\rangle$, the probability of detecting m photons will be

$$W=\sum_{n=m}^{\infty}W(m/n)|\psi_n|^2,$$

which yields to a simple probabilistic interpretation. For the corrected theory, the latter relation no longer holds. It would clearly be of interest to carry out experiments designed to detect the indicated statistical features of the photocounts, with the hope that states with a fixed number of photons could be created through nondestructive quantum measurements.[8]

[1)] The function V used to evaluate α and p is not an analytic signal, but this is not an essential consideration here.

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Translated by Marc Damashek