On the theory of turbulent viscosity

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By analogy with the virtual turbulent diffusion coefficient (i.e., the time rate of change of dispersion of a cloud of Lagrangian particles), a new definition of a coefficient of effective turbulent viscosity, ν_e, is introduced. This coefficient describes the nonlinear effect of momentum exchange in nonstationary turbulent flows, without resorting to any closure hypotheses. Exact expressions have been obtained for ν_e with the Riemann and Burgers equation as examples. These expressions describe, in particular, the effects of turbulent amplification of the effects of molecular viscosity. A comparison is made with estimates obtained earlier on the basis of the traditional definition of ν_e in terms of the interaction of the mean velocity field with the fluctuations, a definition used to close the statistical description of mean fields.

INTRODUCTION

It is known that the appearance of turbulence in a medium causes an extraordinary enhancement of heat, momentum, mass transfer, etc. This leads to the necessity of introducing physical parameters of the turbulent state such as the effective turbulent heat transfer coefficient, the effective turbulent viscosity, and the effective turbulent diffusion coefficient. The values of these coefficients exceed by many orders the values of the corresponding molecular coefficients (Refs. 1, 2). In particular, in the theory of turbulent diffusion such a physical characteristic of mixing processes is the virtual turbulent diffusion coefficient D_E defined in terms of the rate of flowing apart of a cloud of Lagrangian particles relative to the center of gravity of the cloud (e.g., along the x_1 axis) [Refs. 1, 3, 4]:

$$D_e=\frac{1}{2}\frac{d}{dt}\left\langle\left[\frac{1}{q_0}\int d^n\mathbf{x}x_1{}^2q(\mathbf{x},t)-\left(\frac{1}{q_0}\int d^n\mathbf{x}x_1q(\mathbf{x},t)\right)^2\right]\right\rangle\equiv\frac{d}{dt}\frac{\langle\overline{(x_1-\bar{x}_1)^2}\rangle}{2}. \quad (1)$$

where n is the dimensionality of space, $q(\mathbf{x},t)$ is the distribution function of the admixture concentration field, $q_0=\int d^n\mathbf{x}q(\mathbf{x},t)$ is the total mass of the admixture ($q_0=$ const for a conservative admixture); the angle brackets in Eq. (1) denote statistical averaging over the ensemble of realizations of the random velocity field which effects the turbulent transport of the admixture, and the bar denotes an average over space. The quantity D_E characterizes the intensity of the turbulent mixing of the admixture independently of the possibility of introduction of a closed statistical description of the mean field $\langle q\rangle$, e.g., in the form of an equation of the type of the diffusion equation (this is what the Boussinesq, Reynolds, Taylor, and Prandtl hypotheses consist of, Ref. 1). Such a closure is by far not always achievable, in particular, because of the violation of locality of the representation of the fluctuation field in terms of the gradient of the mean field.[5] For example, the domain where this closure hypothesis is applicable is determined exactly by a comparison of the semiempirical diffusion coefficient with D_E in Eq. (1).[1,6]

In spite of the similarity between the processes of turbulent mass and momentum transport, until now the notion of turbulent viscosity is most often associated only with the possibility of introducing the most varied (see Ref. 7) closed descriptions of the average velocity field, in particular, in the form of an equation of the diffusion type (Refs. 1, 8–10). There exist, however, papers of the type of Refs. 11–13, in which the turbulent viscosity was already considered as an independent physical phenomenon characterizing the mode--mode interactions,[11] or as the response of the fluid in turbulent motion to an external disturbance having the form of a random force (Refs. 12, 13). However, in the papers of this trend a closed-form definition of the turbulent viscosity coefficient can be introduced only on the basis of the use of formal statistical closure hypotheses (of the type of the direct-interaction approximation and its modifications[11]), for which the domain of applicability is not completely determined, as far as the physical parameters of the system are concerned. Indeed, in the limit of large Reynolds numbers characteristic for the turbulent state, the nonlinearity of the equations of fluid dynamics leads to a strong interaction between the motions of different scales (and not only scales which are close in magnitude), guaranteeing an intensive momentum exchange between these motions, i.e., being responsible for turbulent viscosity proper. Accordingly, the information on such a *de facto* infinite-dimensional system can be distorted in a predictable way by means of the introduction of any statistical closure hypothesis.[14]

Therefore this paper proposes a new approach to the quantitative definition of the turbulent viscosity, an approach which does not introduce any statistical hypotheses for $t>0$. Thus, for a most complete accounting for all the nonlinear effects in the turbulent exchange of momentum we define by analogy with Eq. (1) a virtual tensor of turbulent viscosity ν_e^{ij} in the form

$$\nu_e^{ij}=\frac{1}{2}\frac{d}{dt}\left\langle\left[\frac{1}{P_j}\int d^n\mathbf{x}x_i{}^2u_j(\mathbf{x},t)-\left(\frac{1}{P_j}\int d^n\mathbf{x}x_iu_j\right)^2\right]\right\rangle\equiv\frac{d}{dt}\overline{\langle(\mathbf{x}-\bar{\mathbf{x}})^2\rangle}, \quad (2)$$

where $P_j=\int d^n\mathbf{x}u_j(\mathbf{x},t)$ is the invariant component of the total momentum for $P_j\neq0$ [for $P_j\equiv0$ one may consider in Eq. (2) an integration over that region V of space where part of the total momentum

$$\tilde{\mathbf{P}}(t)=\int_V d^n\mathbf{x}\mathbf{u}(\mathbf{x},t)$$

does not vanish identically], $\mathbf{u}(\mathbf{x},t)$ is the random velocity field, satisfying the equations of fluid dynamics which conserve the total momentum $\mathbf{P}$. In the presence of external forces $\mathbf{f}(\mathbf{x},t)$ we still have $\mathbf{P}=$ const if $\int d^n\mathbf{x}\cdot\mathbf{f}(\mathbf{x},t)=0$. The quantity ν_e^{ij} describes the rate of change of the dispersion of the distribution of the jth component of the velocity fluid $\mathbf{u}$ (i.e., the broadening of the corresponding "wave packet") along the axis labeled by i, with no summation understood over repeated indices for all $i,j=(1,2,...,n)$. In the same manner as in Eq. (1), the angle brackets in Eq. (2) denote a statistical averaging over the ensemble of realizations of the turbulent velocity field $\mathbf{u}(\mathbf{x},t)$ and the superior bar denotes the appropriate spatial averaging. The statistical averaging procedure in Eq. (2) may take on an explicit character if for the modeling of the turbulent regime one uses, e.g., the method of chaotization of integrable problems.[15–19] In particular, an exact solution for ν_e^{ij} can be obtained from Eq. (2) if an exact solution is known for a $\mathbf{u}(\mathbf{x},t)$ realization that depends explicitly on a random initial field $\mathbf{u}(\mathbf{x},\alpha)$ with a specified normalized distribution density $\rho(\alpha)$ ($\int d\alpha\rho(\alpha)=1$) of the parameters α of this field. The statistical averaging in Eq. (2) corresponds to averaging over α with respect to the distribution $\rho(\alpha)$. The definition (2) may, of course, be also used in the case when the realizations of the field $\mathbf{u}(\mathbf{x},t)$ are defined approximately, or one uses for them a representation derived from a numerical or a normal experiment. We also note that for small Reynolds numbers, when the nonlinear terms in the equations of hydrodynamics can be neglected and only the molecular mechanism of momentum transfer is realized, the definition (2) coincides exactly with that of the molecular viscosity coefficient for the appropriate solution $\mathbf{u}(\mathbf{x},t)$ of the Cauchy problem in infinite space. The same correspondence with the molecular kinetic coefficient applies to the definition of D_e in Eq. (1) for the case when the transport of $q(\mathbf{x},t)$ is realized only on account of the molecular diffusion.

In the present paper we restrict our attention to the Riemann and Burgers equations, since exact nonstationary solutions are well known for these equations and thus there is the possibility to obtain exact expressions for the turbulent viscosity coefficient (2). Principal attention is paid to purely nonlinear effects (without taking account of the initial stochastic factors) of intensification of the process of momentum exchange in turbulent motion. For this, one essentially considers deterministic initial conditions corresponding to the Riemann and Burgers equations in the absence of external forces. Indeed, in the transition to turbulence, it is the nonlinearity of the equations of hydrodynamics that is responsible not only for the limited predictability and stochastization of the motion (Ref. 14), but also for establishment of new macroscopic couplings which guarantee the appearance of a more intensive macroscopic momentum transport mechanism replacing the purely molecular one. On the contrary, by itself the stochasticity of the motion, which is characteristic also for large equilibrium systems, cannot completely determine the nature of this purely nonequilibrium organized (collective) process of turbulent momentum exchange (Ref. 20).

An investigation of the Riemann and Burgers equations not only makes it significantly easier to understand the physical meaning of the new definition of turbulent viscosity introduced in Eq. (2), but is also of independent interest in connection with the use of these equations for the analysis of many physical processes in nonlinear acoustics, plasma physics, radiophysics, and fluid dynamics (Refs. 2, 9, 21–23). In particular, this refers also to the problem of justification of the possibility of closing the appropriate statistical description of the mean fields, and the problem of the role of molecular viscosity in the process of formation of the phenomenon of turbulent viscosity.

1. THE EFFECTIVE VISCOSITY IN RIEMANN TURBULENCE

1. One of the most important exact nonstationary solutions of the Euler equations of hydrodynamics is a simple Riemann wave described by the corresponding Riemann equation[21]

$$\partial u/\partial t+u\partial u/\partial x=0, \tag{3}$$

where u is the one-dimensional velocity field. The equation (3) and its higher-dimensional generalization

$$\frac{\partial u_i}{\partial t}+u_l\frac{\partial u_i}{\partial x_l}=0 \qquad (i=1,2,\ldots,n) \tag{3'}$$

also describes the inertial motion of a fluid or a beam of noninteracting particles.[16,23]

The equations (3) and (3′) correspond to distributed nondissipative systems for which one can, nevertheless, introduce a concept of turbulent viscosity based on the definition (2). There exists a certain similarity to the problem posed in Ref. 24, of the determination of an effective friction coefficient in terms of the parameters of an exact time-reversible solution, but for a discrete Hamiltonian system of coupled oscillators. We are making use in the sequel of the notation $\nu_e^{1,1}\equiv\nu_e$ corresponding, in particular to the one-dimensional representation (2). Carrying out directly in this definition of ν_e a differentiation with respect to t making use of Eq. (3), it is easy to obtain the following general exact representation for ν_e, valid for arbitrary values of the time t:

$$\nu_e=\frac{E_0}{P_0}(x_{0E}-x_{0P})+\frac{t}{P_0^2}\left[\frac{P_0}{3}\int_{-\infty}^{\infty}dxu_0^3(x)-E_0^2\right], \tag{4}$$

where

$$E_0=\frac{1}{2}\int_{-\infty}^{\infty}dxu_0^2(x),\qquad P_0=\int_{-\infty}^{\infty}dxu_0(x),$$

$$x_{0E}=\frac{1}{2E_0}\int_{-\infty}^{\infty}dxxu_0^2(x),\qquad x_{0P}=\frac{1}{P_0}\int_{-\infty}^{\infty}dxxu_0(x)$$

and use has been made of the invariance of the quantities

$$I_m=\int_{-\infty}^{\infty}dxu^m(x,t),\qquad m=1,2,\ldots$$

for the velocity field $u(x,t)$ in Eq. (3). In Eq. (4) $u_0(x)$ denotes an arbitrary deterministic initial velocity field [that is why there are no angle brackets in Eq. (4)], corresponding to nonzero values of the total energy E_0 and the total momentum P_0. In particular, for symmetric initial conditions $u_0(x)=u_0(-x)$ the free term vanishes and $\nu_e=C_0\bar{\varepsilon}t$, where C_0 is a positive constant depending on the form of the function $u_0(x)$, $\bar{\varepsilon}\equiv E_0/\lambda$ is the partial average of the energy of the initial velocity field u_0, and λ is a characteristic scale of the spatial variation of $u_0(x)$. [The positivity of

$$C_{0\bar{\varepsilon}}\equiv\frac{1}{P_0^2}\left[\frac{P_0}{3}\int_{-\infty}^{\infty}dxu_0^3(x)-E_0^2\right]$$

follows from the Hölder inequality

$$\int dxf(x)g(x)<\left(\int dxf^k\right)^{1/k}\left(\int dxg^{k'}\right)^{1/k'}$$

for $k>1$ and $1/k+1/k'=1$ (Ref 25).]

We note that such a time-dependence [see also Eq. (7)] for the turbulent viscosity coefficient also arises with its traditional definition ν_T, related to the possibility of a closed statistical description of the mean velocity field in the form of a diffusion equation obtained on the basis of averaging of Eq. (3), Refs. 8–10. Indeed, in these papers the very introduction of the effective viscosity ν_T is considered to be possible for Riemann turbulence only if one considers an interaction of the signal (the mean field $\langle u\rangle$) with the noise (fluctuations) in the limit when the amplitude and the scale of the signal differ significantly from the corresponding characteristics of the noise. In these conditions it follows from Eq. (3) that $\langle u\rangle$ satisfies the relation

$$\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}=\nu_T\frac{\partial^2\langle u\rangle}{\partial x^2}\qquad(5)$$

in the case when the signal is weakly perturbed by noise, and the relation

$$\frac{\partial\langle u\rangle}{\partial t}=\nu_T\frac{\partial^2\langle u\rangle}{\partial x^2},\qquad(5')$$

when, on the contrary, the signal perturbs the noise regime weakly (Ref. 8). In either case $\nu_T=2t\bar{\varepsilon}$, where $\bar{\varepsilon}$ is the initial noise energy (Refs. 8–10). In addition, in Refs. 8 and 9 this expression for ν_T is considered valid only for times t shorter than the breaking time t_0 of a simple wave (for the noise component), whereas for $t>t_0$ it is assumed that ν_T does not depend on t at all. This is the fundamental distinction between the conclusions of Refs. 8–10 and the expression for ν_e in Eq. (4), which is valid not only for arbitrary signal-to-noise ratios, but also for arbitrary times t. At the same time it should be noted that the character and the rate u of spreading of a wave packet of the field, computed on the basis of the definition of ν_e in Eq. (4), corresponds to the evolution regime (5′) in infinite space with $\nu_T=\nu_e$.

2. We now consider the expression (2) for the case of the n-dimensional generalization (3′) of the Riemann equation, when the initial velocity field $\mathbf{u}_0(\mathbf{x})$ represents N nonoverlapping packets of finite extent and amplitudes $\mathbf{a}_k$, $(k=1,2,...,N)$, localized in the appropriate intervals $\mathbf{x}\in[\mathbf{c}_k,\mathbf{d}_k]$, $\mathbf{d}_k>\mathbf{c}_k$:

$$u_{0i}(\mathbf{x})=\sum_{k=1}^{N}a_{i,k}(x_1-c_{1,k})(d_{1,k}-x_1)P_k(\tilde{\mathbf{x}})\theta(\mathbf{x}-\mathbf{c}_k)\theta(\mathbf{d}_k-\mathbf{x}),$$

$$i=1,2,\ldots,n,\qquad(6)$$

where

$$P_k(\tilde{\mathbf{x}})\equiv\prod_{j=2}^{n}(x_j-c_{j,k})^{\mu}(d_{j,k}-x_j)^{\mu},$$

where n is the dimensionality of space, and $\tilde{\mathbf{x}}\equiv(x_2,x_3,...x_n)$. Assume, for simplicity, that $a_{1,k}\neq0$, and $a_{i,k}=0$ for $i=2,3,...n$, $k=1,2,...N$. For $N=1$, for example, this can always be achieved by means of an appropriate choice of the coordinate system. The Appendix contains an exact solution (A1) of Eq. (3′) corresponding to Eq. (6) with this condition. The effective viscosity tensor (2) corresponding to the solution (A1) has only one nonvanishing component $\nu_e^{1,1}\equiv\nu_e$, the magnitude of which, just as the solution (4), has the same expression also in the region of uniqueness of the solution (A1), for

$$t<t_0=\min_{(k,\tilde{\mathbf{x}})}1/a_{1,k}P_k(\tilde{\mathbf{x}})(d_{1,k}-c_{1,k}),$$

and for $t\geqslant t_0$ after the breaking of the wave. This agreement holds because the use of the invariance of the total momentum P in the calculation of ν_e in the region of nonuniqueness of the solution (A1) for $t>t_0$ leads to the need of modifying the definition of ν_e for $t>t_0$ to the form (A2). The Appendix also contains an expression for ν_e, (A3), for the case where all the quantities $a_{1,k}$, $\mathbf{d}_k$, $\mathbf{c}_k$ in Eq. (6) are different for different values of the index k. In particular, for $a_{1,k}\equiv a$, $d_{i,k}-c_{i,k}\equiv d_i-c_i$ for all $k=1,2,...,N$, $i=1,2,...,n$ in (A3) one obtains exactly the same expression as in the case $N=1$:

$$\nu_e=C_0\bar{\varepsilon}t,\qquad(7)$$

where

$$C_0=\frac{6}{7}\left[\frac{(\mu+1)(2\mu+1)}{3\mu+1}\right]^{n-1}\left[1-\frac{7}{10}\left(\frac{(\mu+1)(3\mu+1)}{(2\mu+1)^2}\right)^{n-1}\right],$$

$$\mu>-\frac{1}{3},\qquad\bar{\varepsilon}=\frac{a^2(d_1^4-c_1^4)(d_2-c_2)^{2\mu}\ldots(d_n-c_n)^{2\mu}}{60(2\mu+1)^{n-1}},$$

where $\bar{\varepsilon}$ is the spatial average of the energy for $t=0$. The expression (7) follows from the general definition $\bar{\varepsilon}\equiv E_0/V_0$ where E_0 and V_0 have the following form for the initial velocity field (6):

$$E_0=\sum_{k=1}^{N}\frac{a_{1,k}^2(d_{1,k}-c_{1,k})^5(d_{2,k}-c_{2,k})^{2\mu+1}\ldots(d_{n,k}-c_{n,k})^{2\mu+1}}{60(2\mu+1)^{n-1}},$$

$$V_0=\sum_{k=1}^{N}(d_{1,k}-c_{1,k})\ldots(d_{n,k}-c_{n,k}).$$

In form the solution (7) coincides with Eq. (4) (for $u_0(x)=u_0(-x)$, but now the quantity C_0 turns out to depend essentially on the dimensionality n of space and on the exponent μ of smoothness of the initial velocity field (6). In addition, it follows from (A3) that for $N>1$ and different $A_{1,k}$ and $\mathbf{d}_k-\mathbf{c}_k$ (for various k) we have $\nu_e\to$const as $t\to0$, in the same manner as in Eq. (4) and (A3) [or Eq. (7)], although the method of obtaining (4) is not related to the necessity of using an explicit representation of the solution for u, including the region of nonuniqueness.

3. Making use of an exact solution of the Riemann equation (3′) represented in the form (A1) one can determine the character of the evolution of enstrophy (the mean-square gradient of the velocity field) as it approaches the singularity at $t=t_0$, which determines the instant of appearance of a bifurcation of the solutions of the equations of hydrodynamics of an ideal fluid.

Thus, in the one-dimensional case $n=1$ we have

$P_k(\tilde{\mathbf{x}}) \equiv 1$ in Eqs. (6) and (A1) and the magnitude of the enstrophy

$$\Omega = \int_{-\infty}^{\infty} dx \left(\frac{\partial u}{\partial x}\right)^2,$$

which determines the rate of dissipation of the total energy in the presence of molecular viscosity [see Eq. (13) below], has the form (for $a_k = a$ and $d_k - c_k = b$, at all $k = 1,2,...,N$):

$$\Omega(t) = \frac{N}{t^2}\left[-b + \frac{1}{2at}\ln\left|\frac{1+abt}{1-abt}\right|\right] \tag{8}$$

for $t \leqslant t_0 = 1/ab$. Thus, Eq. (8) implies that over a finite time $t_0 = 1/ab$ there occurs an explosive unbounded growth of the enstrophy

$$\Omega(t) \approx \frac{Nb^3a^2}{2}\ln\frac{1}{|1-t/t_0|}$$

as $t \to t_0$, a growth due to the nonlinear breaking of a simple Riemann wave (A1) and to the loss of uniqueness of this exact nonstationary solution of the hydrodynamic Euler equations. It should be stressed that the explosive growth of Ω in Eq. (8) takes place logarithmically rather than as a power law, as happens, for instance in the collapse (coalescence) of point vortex dipoles (i.e., infinitesimal vortex rings) in the hydrodynamics of an ideal incompressible fluid (Refs. 19, 26).

At the same time for $n > 1$ the solutions (A1) no longer lead to an explosive growth of the enstrophy. This is due in final analysis to the quasi-two-dimensional character of the solution (A1) for $n \geqslant 2$. Indeed, for $\mathbf{u}(\mathbf{x},t)$ in Eq. (A1) only one component of the velocity field, u_1, is different from zero, so that the effect of stretching of the vortex lines is manifestly absent, an effect which leads in the three-dimensional case to an explosive enstrophy intensification (Refs. 1, 19, 26) limited only by the influence of dissipative factors (Refs. 27, 26).

4. We note that in the construction of the turbulent regime on the basis of the method of chaotization of exactly integrable problems (Ref. 15–19) one can obtain for the Riemann equation (3) or (3′) only expressions for $\langle \nu_e \rangle$, statistically averaged with a prescribed probability measure for the initial velocity field and corresponding to the representations of ν_e in Eqs. (4), (A3), and (7). However, the qualitative behavior of $\langle \nu_e \rangle$ as a function of time agrees exactly with the non-averaged ν_e [although this representation of ν_e in Eqs. (4), (A3) and (7) corresponds to a procedure of spatial averaging, contained in the definition (2)]. We note also that the applicability of these expressions for $\langle \nu_e \rangle$ does not depend in fact on the level and the scale of fluctuations of the random field $u(x,t)$ corresponding to the initial condition $u(x,t=0)$ determined by the random parameters α. However, the measure $\rho(\alpha)$ is chosen in such a way that the total momentum P should not vanish for any values of α (of course, if $P \not\equiv 0$).

5. Concluding this section, it should be noted that the method used in the present paper (in essence related to the method of moments[28]) can be used not only for the analysis of the turbulent transport of momentum (i.e., viscosity), but also for the investigation of other macroscopic kinetic characteristics of the turbulent flow. For instance, for the description of the process of redistribution of the density of turbulent energy (related in particular with the phenomenon of intermittency in turbulence[1]) one must carry out in the definition (2) an averaging over space, taking into account the density distribution of the turbulent energy $u^2(\mathbf{x},t)$, in place of the distribution of $\mathbf{u}(\mathbf{x},t)$. In distinction from $\mathbf{u}$, the distribution of $\mathbf{u}^2$ is already positive definite, which is not of principal interest and reflects only the correspondence of $\mathbf{u}^2(\mathbf{x},t)$ to a nonnegative probability distribution for the coordinates of the Lagrangian particles.[1] For nondissipative systems of the type (3) and (3′), as well as for the case when the external forces compensate exactly the energy dissipation, the distribution $\mathbf{u}^2(\mathbf{x},t)$ defines an invariant measure corresponding to conservation of the total energy. In particular, for the Riemann equation (3) the corresponding effective energy diffusion (or the intermittency)

$$\Pi_e = \frac{1}{2}\frac{d}{dt}\left[\frac{1}{2E_0}\int_{-\infty}^{\infty} dx x^2 u^2(x,t) - \left(\frac{1}{2E_0}\int_{-\infty}^{\infty} dx x u^2(x,t)\right)^2\right] \tag{9}$$

has for arbitrary values of t the form

$$\Pi_e = \frac{1}{3E_0}\left(\int_{-\infty}^{\infty} dx x u_0^3(x) - \frac{1}{2E_0}\int_{-\infty}^{\infty} dx x u_0^2 \int_{-\infty}^{\infty} dx u_0^3\right) + tb_0, \tag{10}$$

derived in the same manner as Eq. (4), by differentiating Eq. (9) making use of Eq. (3) and of the invariance of the quantities

$$I_m = \int_{-\infty}^{\infty} dx u^m(x,t), \quad m = 1, 2, \ldots$$

for Eq. (3). In Eq. (10) we have

$$b_0 = \frac{1}{4E_0}\int_{-\infty}^{\infty} dx u_0^4 - \frac{1}{9E_0^2}\left(\int_{-\infty}^{\infty} dx u_0^3\right)^2 > 0,$$

which follows from the Hölder inequality.[25] The expression (10) does not differ qualitatively from (4) and signifies that the nonlinear turbulent processes of transformation of energy and momentum density have much in common in the Riemann turbulence case under consideration. We also remark that the quantity Π_e is the analog of the rate of variation of the effective width of a light beam (in a moving coordinate frame) in thermal self-action in a weakly absorbent moving medium.[29] Indeed, the calculation of the effective broadening in Ref. 29 also makes use of the method of moments, where in place of $\mathbf{u}^2(\mathbf{x},t)$ the spatial averaging is done in terms of the energy density $I(x,y,z = ct)$ of the light beam. In the process of evolution with respect to $z = ct$ for I, as well as for the solution $u(x,t)$ of the Riemann equation (3), one observes (Refs. 29, 30) effects of the appearance of nonuniqueness (in the distribution of the level lines of $I(x, y,z = ct)$ in the (x,y) plane) suggestive of the phenomenon of breaking of a simple wave in the nonlinear self-action of u. Moreover, for small t (or z) the rate of variation of the effective width of the light beam increases linearly in time (see Ref. 29), similar to the linear growth in Eq. (10). It is therefore possible that in itself the process of thermal self-action of a light beam can be modeled by Eq. (3) or (3′), e.g., in the nondissipative approximation of geometric optics (in Ref. 30 as $\varepsilon \to 0$) describing a beam of noninteracting "particles."

2. BURGERS TURBULENCE

In the theory of turbulent diffusion an essential role may be played by interaction effects between molecular and turbulent diffusion (Refs. 1, 3, 4). Similar enhancement effects of the molecular viscosity can be investigated for the case of the Burgers model of turbulence, in which a realization of the velocity field satisfies the Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \qquad (11)$$

where ν is the molecular viscosity coefficient. As before, Eq. (11) corresponds to the total-momentum conservation law

$$P=\int_{-\infty}^{\infty} dxu(x,t)=P_0=\text{const.}$$

Utilizing the one-dimensional representation of the definition (2) and differentiating in it directly with respect to time, taking account of (11), we obtain for the virtual turbulent viscosity coefficient the expression

$$\nu_e=\nu+\left\langle\frac{E}{P_0}(x_E-x_P)\right\rangle, \qquad (12)$$

where

$$x_E=\frac{1}{2E}\int_{-\infty}^{\infty} dxxu^2(x,t)\equiv\frac{U}{E},$$

$$x_P=\frac{1}{P_0}\int_{-\infty}^{\infty} dxxu\equiv\frac{R}{P_0}, \qquad E=\frac{1}{2}\int_{-\infty}^{\infty} dxu^2,$$

and E is the total energy which for $\nu\neq0$ is no longer an invariant of the motion, since

$$\frac{dE}{dt}=-\nu\Omega\equiv-\nu\int_{-\infty}^{\infty} dx\left(\frac{\partial u}{\partial x}\right)^2. \qquad (13)$$

The angle brackets in Eq. (12) may be omitted only when the initial velocity field $u(x,t=0)$ is deterministic and the effective viscosity ν_e is determined only by the joint effect of the nonlinearity of Eq. (1) and the molecular viscosity.

1. In particular, for the initial condition

$$u(x,0)=A\delta(x) \qquad (14)$$

the exact solution of Eq. (11) has the form[22]

$$u(x,t)=\left(\frac{\nu}{t}\right)^{1/2}(\exp\text{Re}-1)\exp(-x^2/4\nu t)\Big/\left[\pi^{1/2}+(\exp\text{Re}-1)\int_{x/(4\nu t)^{1/2}}^{\infty} du\exp(-u^2)\right] \qquad (15)$$

where $\text{Re}=A/2\nu$ is the Reynolds number. Substitution of (15) directly into the one-dimensional form of the definition (2) leads to the expression

$$\nu_e=2\nu[\overline{x^2}-(\bar{x})^2], \qquad (16)$$

where the bar denotes spatial averaging with the weight function

$$\tilde{\rho}(x)=e^{-x^2}(e^{\text{Re}}-1)\Big/\left[\pi^{1/2}+(e^{\text{Re}}-1)\int_x^{\infty} due^{-u^2}\right],$$

$$\int_{-\infty}^{\infty} dx\tilde{\rho}=\text{Re}/(e^{\text{Re}}-1).$$

In the limit $|\text{Re}|\gg1$ we obtain from (16)

$$\nu_e={}^1/_9\nu[|\text{Re}|+O(1/|\text{Re}|)], \qquad (16a)$$

i.e., $\nu_e\to(|A|/18)\text{sign}\,t$ for $|\text{Re}|\to\infty$. For $A>0$ the limit $\text{Re}\to\infty$ corresponds to the transition to the Riemann equation, when the solution (15) has the form

$$u(x,t)=\frac{x}{t}\theta(x)\theta((2At)^{1/2}-x),$$

where $\theta(x)$ is the Heaviside step function[22] and for $A<0$ and $\text{Re}\to-\infty$ it follows from (15) that

$$u(x,t)=-\frac{|x|}{t}\theta(-x)\theta((2|A|t)^{1/2}+x).$$

For $\text{Re}\to\infty$, the viscosity ν_e in Eq. (16a) does not depend at all either on ν nor on t. The difference of this result from (4) is explained by the fact that in order to obtain (4) a sufficient degree of smoothness of the initial field had to be assumed. Indeed, for the condition (14) one cannot even correctly define the initial energy

$$E_0=\int_{-\infty}^{\infty} dx\frac{u^2(x,0)}{2},$$

which enters into Eq. (4), since the integrand would contain a product of delta functions. In the preceding section it was already remarked that in Ref. 9 for $\nu=0$ a constant turbulent viscosity coefficient had been obtained for times t exceeding the wave breaking time t_0.

In the opposite limit $|\text{Re}|\ll1$ we obtain from Eq. (16)

$$\nu_e\approx\nu(1+O(\text{Re})). \qquad (16b)$$

Thus, it follows from Eqs. (16a) and (16b) that for large values of Re the magnitude of ν_e may significantly exceed the value of the molecular kinetic coefficient ν, and for $\text{Re}\to0$ it coincides with ν.

Another exact solution of the Burgers equation (11) in infinite space is the N-wave[22]:

$$u(x,t)=\frac{x}{t}\frac{(|a|/t)^{1/2}e^{-x^2/4\nu t}}{1+(|a|/t)^{1/2}e^{-x^2/4\nu t}}. \qquad (17)$$

This antisymmetrized representation of (17) describes also an exact solution of (11) in a half-space under the same initial conditions:

$$u(x,0)=-2\nu\frac{d\ln\varphi(x)}{dx},$$

$$\varphi(x)=1+2(\pi\nu|a|)^{1/2}\delta(x)$$

and the boundary conditions $u(x=0,t)=0$ and $u(x,t)\to0$ for $x\to\infty$. Since for the solution (17) the total momentum is

$$P=\int_{-\infty}^{\infty} dxu\equiv0, \qquad P_1(t)=\int_0^{\infty} dxu=2\nu\ln(1+(|a|/t)^{1/2}),$$

we shall consider ν_e in Eq. (2) for the half space $x\geqslant0$ with the appropriate normalization to the magnitude of P_1. We then obtain for ν_e the expression (A4) given in the Appendix. It follows, in particular, from (A4) that, as $t\to0$

$$\nu_e\approx\nu\ln|a/t|, \qquad (18)$$

and for $t\to\infty$ we get $\nu_e=(4-\pi)\nu/2$. The difference of ν_e from ν in the limit $t\to\infty$ is due, in this case, to considering the effective viscosity only in the half-space $x\geqslant0$ rather than the whole space. Indeed, for the diffusion equation

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial x^2}$$

[i.e., Eq. (11) without the nonlinear term] with the boundary conditions

$$u(0,t)=0,\quad u(x,t)\to 0$$

for $x\to\infty$ and the initial condition

$$u(x,0)=u_0\frac{\partial\delta(x)}{\partial x}$$

the solution has the form

$$u(x,t)=\frac{u_0 x\exp(-x^2/4\nu t)}{4\nu t(\pi\nu t)^{1/2}}.$$

For such a solution the expression $\nu_e=(4-\pi)\nu/2$ obtained above follows from the definition (2) or (1). Thus the presence of boundaries leads to a modification of the effective transport coefficients (1) and (2) compared to unbounded space.

The unbounded growth of ν_e in Eq. (18), as $t\to 0$ corresponds to an intensification of the action of the effective viscosity on account of the nonlinear effects and is tied to a singularity in the enstrophy

$$\Omega=\int_{-\infty}^{\infty}dx(\partial u/\partial x)^2$$

for $u(x,t)$ from Eq. (17), since for $t\to 0$

$$\Omega\approx 2(2\pi\nu)^{1/2}/t^{5/2}. \tag{19}$$

At the same time, for $t\to\infty$ the relation $\nu_e\sim\nu$ is related to the process of smoothing of the gradients of the field u, which has the effect that $\Omega\approx|a|(2\pi\nu)^{1/2}/t^{5/2}$ for $t\to\infty$. We also note that in the construction of the turbulent Burgers regime, the realizations of which are described by the exact solutions (15) or (17), the introduction of the appropriate statistical averaging for (16a) or (18) does not lead to a qualitative difference between $\langle\nu_e\rangle$ *and* ν_e [for example, in Eq. (16a) $|A|$ is simply replaced by $\langle A^2\rangle^{1/2}$] as was already pointed out in the previous section for Riemann turbulence. In addition, the above analysis of the nonlinear momentum transport in the Burgers N-wave may have some applications, related in particular to effects of transformation of a wide class of noise pulses into an N-wave at large distances from the entry into the nonlinear medium.[31]

2. For sufficiently smooth initial fields, in distinction from the examples considered in the preceding subsection, it is convenient to make use of a representation of ν_e in the form (12). In this case, in addition to the equality (13) it is convenient to use the relations

$$\frac{d^2R}{dt^2}=\frac{dE}{dt}\quad\frac{d^2U}{dt^2}$$

$$=\nu\int_{-\infty}^{\infty}dx\left[x\left(\frac{\partial u}{\partial x}\right)^3-3u\left(\frac{\partial u}{\partial x}\right)^2\right]. \tag{20}$$

On the basis of Eqs. (20), (13), and (12) one can consider the problem of the influence of an arbitrarily small molecular viscosity on the properties of turbulent viscosity in comparison with the conclusions obtained in Sec. 1 for ν_e within the framework of Riemann turbulence. We restrict our attention to corrections containing only the first power of ν. In the approximation one may use for the field $u(x,t)$ in the right-hand sides of Eqs. (13) and (20) any nondissipative solution of the Riemann equation. In particular, for arbitrarily smooth initial velocity fields we have obtained the explicit expressions (A6) and (A7) for the functions U, E, and R which determine the quantity ν_e in (12) in the form of a power series in t, with all the coefficients given as functionals of the initial field $u_0(x)$ and its derivatives. Thus, for the initial field $u_0(x)=a_0\exp(\alpha x^2)$ for $t<t_0=e^{1/2}/a_0(2\alpha)^{1/2}$ (t_0 is the wave breaking time) the first terms in this power series in t are

$$\nu_e=a_1t+\nu[1-a_2t^2+O(t^4)]+O(\nu^2), \tag{21}$$

where $a_1=a_0^2(8-3\sqrt{3})/24\sqrt{3}$, $a_2=\alpha a_0^2(32-9\sqrt{3}/24\sqrt{3})$. For an extrapolation analysis of the representation (21) one may use the Shanks nonlinear transformation (Refs. 16, 22), which in many respects resembles the rational Padé approximants (Ref. 33). Extrapolation reduces Eq. (21) to the form

$$\nu=\frac{\nu a_1+t(\nu^2a_2+a_1^2)}{a_1+\nu a_2t} \tag{22}$$

Analytic continuation of Eq. (22) into the region $t>t_0$ in the limit $t\to\infty$ leads to

$$\nu_e\approx\nu+\frac{a_0^2d}{\alpha\nu}=\frac{a_0}{\alpha^{1/2}}\left(\frac{1}{\mathrm{Re}}+d\,\mathrm{Re}\right), \tag{23}$$

where $d\approx 0.01$ and $\mathrm{Re}=a_0/\nu\alpha^{1/2}$ is the corresponding Reynolds number. We note that in the limit $t\to\infty$, taking into account the following terms of the expansion of ν_e in powers of t, the result of applying the Shanks transformation does not in fact change the structure of the expression (23), since only the numerical value of the constant d changes (d has a tendency to increase as more and more terms of the power series in t are taken into account). We also note that the dependence of ν_e on the Reynolds number in a certain interval of Re qualitatively resembles the dependence of the resistance coefficients C on Re in the results of observing the flow around bodies of different shapes.[2] Of course, Eq. (23) cannot in principle describe that region of sufficiently large values of Re where the geometry of the body may manifest itself in an essential way, and one observes a sharp reduction of C in a certain interval of Reynolds numbers (the resistance crisis, Ref. 2). In this connection it is interesting to use the definition (2) of the effective viscosity for the realizations of the velocity field $\mathbf{u}(\mathbf{x},t)$, corresponding to just such streamlining problems, with the purpose of discovering similar critical changes in ν_e as Re increases.

Substituting into the right-hand sides of Eqs. (20) and (23) the explicit form of the representation (A1) of the solution, we obtain for ν_e the expression (agreeing with the case $N=1$)

$$\nu_e=\frac{3ta^2b^4}{700}+\nu\left[\frac{3}{4}-\frac{3}{4a^2b^2t^2}+\frac{3}{40}\left(4-abt+\frac{5}{a^3b^3t^3}\right)\ln(1+abt)+\frac{3}{40}\left(4+abt-\frac{5}{a^3b^3t^3}\right)\ln|1-abt|\right]+O(\nu^2), \tag{24}$$

where $a_{1,k}=a, d_k-c_k=b$ for all $k=1,2,\ldots,N$. The quantity ν_e in Eq. (24) has no singularity at the time of the breaking of the wave $t_0 = 1/ab$. For $t \gg t_0$ the equation (24) has the same form as Eq. (7), with $C_0 = 9/35$ and $\bar{\varepsilon} = a^2 b^4/60$. However, the derivative $d\nu_e/dt$ in (24) already has a logarithmic singularity [similar to the explosive growth of the enstrophy Ω in Eq. (8)] as $t \to t_0 = 1/ab$. In principle such an effect is absent for $\nu = 0$, when Eq. (24) takes the form (7).

We note that the energy damping coefficient $\gamma = |E^{-1} dE/dt|$, which in this case is proportional to $(d\nu_e/dt)/ab^3$ also has a logarithmic singularity as $t \to t_0$. The energy can therefore flow into localized singularities, which is generally characteristic for shock waves (Refs. 2, 21), and some vortex systems (Refs. 19, 26). Indeed, even in the limit $\nu \to 0$ the rate of dissipation of the total energy $\varepsilon_0 \equiv dE/dt$ may remain finite on account of the explosive intensification of the enstrophy Ω for $t \to t_0$ (see Eq. (8) and Refs. 19, 26). At the same time, for nonzero ε_0 dimensional and scaling considerations[1] allow one to determine in the inertial range the energy spectrum of Burgers turbulence, $\widetilde{E}(k) \sim \varepsilon_0^{2/3} k^{-1}$, corresponding to a cascade process of energy transfer through the spectrum of wave numbers k. For point vortices in the plane the self-energy spectrum is also proportional to k^{-1} (Ref. 34), and for acoustic turbulence a steeper decay regime of decrease with the growth of k was noted in Ref. 35.

3. In conclusion we note that the consideration in the present paper of nonlinear effects in the formation of the turbulent viscosity can be meaningfully extended [even on the basis of Eq. (1)] to an estimate of the influence of nonlinear kinetics on the character of transport of admixtures (see, e.g., Ref. 36). In addition, one must stress that a general property of the effective coefficients (1) and (2) of turbulent exchange is their antisymmetry with respect to time reversal, i.e., for $t \to -t$ we have in Eq. (1) $D_0 \to -D_0$ and in Eq. (2) $\nu_e \to -\nu_e$. Zel'dovich[37] was the first to call attention to the importance of such an antisymmetry of the macroscopic characteristics of strongly nonequilibrium turbulence, since this effectively enhances the corresponding antisymmetry property of the molecular kinetic coefficients (Refs. 3, 4, 37) [for instance, the mean-square displacement of a Brownian particle through molecular diffusion, $\langle x^2 \rangle \sim Dt$ cannot be negative for any value of t]. We note that the turbulent viscosity coefficients ν_e introduced in Refs. 2 and 20 [in Ref. 2 $\nu_T \sim \nu |\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}}|$—see the analogous expression (16a)] exhibit the indicated antisymmetry, since for them is characteristic the proportionality to the coefficient ν of molecular viscosity. Moreover, in Ref. 20 the definition of ν_T depends directly on the level of fluctuations in the system and the appropriate calculations, in the same manner as in Refs. 11–13, inevitably leads to the necessity of using some closure hypotheses, even for the simplest systems of the type of Eqs. (3) or (11). This is apparently related to the inadequacy of the mathematical description of nonlinear systems on the basis of the representation of the velocity field as a linear superposition of a mean field and fluctuations. In this respect the definitions (1) and (2) can more fully reflect the nonlinear properties of the turbulent exchange, as these definitions consider the field as a whole.

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APPENDIX

1. For the initial condition (6) the equation (3') has the following exact solution (for $a_{1,k} \neq 0$, $a_{i,k} = 0$ and $i = 2,3,\ldots,n$):

$$u_i(\mathbf{x}, t) = 0, \quad i = 2, 3, \ldots, n,$$

$$u_1(\mathbf{x}, t) = \frac{1}{2t} \sum_{k=1}^{N} \theta(\tilde{\mathbf{x}} - \tilde{\mathbf{c}}_k)\,\theta(\tilde{\mathbf{d}}_k - \tilde{\mathbf{x}}) \Big\{ 2x_1 - c_{1,k} - d_{1,k} - \frac{1}{a_{1,k} t P_k(\tilde{\mathbf{x}})} + B\Big[\Big(c_{1,k} + d_{1,k} + \frac{1}{a_{1,k} t P_k(\tilde{\mathbf{x}})} \Big)^2 - 4 \Big(\frac{x_1}{a_{1,k} t P_k(\tilde{\mathbf{x}})} + c_{1,k} d_{1,k} \Big)^2 \Big]^{1/2} \Big\} \tilde{A}, \tag{A1}$$

where $B = 1$ for $t \leqslant t_0$ and $B = \pm 1$ for $t > t_0$,

$$\tilde{A} = \theta(x_1 - c_{1,k})\,\theta(d_{1,k} - x_1), \quad t \leqslant t_0,$$

$$\tilde{A} = \begin{cases} \theta(x_1 - c_{1,k}), & B = 1, \quad t > t_0 \\ +\theta(x_1 - d_{1,k}), & B = -1, \quad t > t_0 \end{cases}.$$

$$t_0 = \min_{(k,\mathbf{x})} \frac{1}{a_{1,k}(d_{1,k} - c_{1,k}) P_k(\tilde{\mathbf{x}})}, \quad \tilde{\mathbf{x}} \equiv (x_2, \ldots, x_n),$$

$$\tilde{\mathbf{c}}_k \equiv (c_{2,k}, \ldots, c_{n,k}), \quad \tilde{\mathbf{d}}_k \equiv (d_{2,k}, \ldots, d_{n,k}).$$

2. For $t > t_0$ in Eq. (A1) one can define ν_e in the form

$$\nu_e = \frac{1}{2} \frac{d}{dt} \sum_{k=1}^{N} \int d\tilde{\mathbf{x}} \Big[\frac{1}{P} \Big(\int_{c_{1,k}}^{d_{1,k}} dx_1 x_1^2 u_1^+ + \int_{d_{1,k}}^{x_0} dx_1 x_1^2 (u_1^+ - u_1^-) \Big) - \frac{1}{P^2} \Big(\int_{c_{1,k}}^{d_{1,k}} dx_1 x_1 u_1^+ + \int_{d_{1,k}}^{x_0} dx_1 x_1 (u_1^+ - u_1^-) \Big)^2 \Big], \tag{A2}$$

where

$$P = \sum_{k=1}^{N} \int d\tilde{\mathbf{x}} \Big[\int_{c_{1,k}}^{d_{1,k}} dx_1 u_1^+ + \int_{d_{1,k}}^{x_0} dx_1 (u_1^+ - u_1^-) \Big] = P_0 = \mathrm{const}$$

for arbitrary $t > t_0$. For $t \leqslant t_0$

$$P = \sum_{k=1}^{N} \int d\tilde{\mathbf{x}} \int_{c_{1,k}}^{d_{1,k}} dx_1 u_1^+$$

coincides with the value of P for $t > t_0$ in the same manner as for ν_e. In Eq. (A2) x_0 is the value of x_1 for which the radicand in (A1) vanishes, and the superscripts plus or minus on u_1 in Eq. (A2) correspond to the sign of the square root in (A1). Thus the modification of the expression for the invariant total momentum P for $t > t_0$ of necessity leads to a corresponding change in the definition of ν_e.

Taking into account (A1) we obtain from Eqs. (2) and (A2) for arbitrary values of t

$$\nu_e=\frac{1}{120}\sum_{k=1}^{N} a_{1,k}^2(d_{1,k}-c_{1,k})^5(d_{2,k}-c_{2,k})^{2\mu+1}\ldots(d_{n,k}-c_{n,k})^{2\mu+1}$$

$$\times\frac{1}{P}\left[\frac{d_{1,k}+c_{1,k}}{(2\mu+1)^{n-1}}+2ta_{1,k}(d_{1,k}-c_{1,k})^2\right.$$

$$\left.\times(d_{2,k}-c_{2,k})^{\mu}\ldots(d_{n,k}-c_{n,k})^{\mu}\right]$$

$$-\frac{1}{144P^2}\sum_{k=1}^{N}a_{1,k}(d_{1,k}-c_{1,k})^3(d_{2,k}-c_{2,k})^{\mu+1}$$

$$\ldots(d_{n,k}-c_{n,k})^{\mu+1}\left[\frac{c_{1,k}+d_{1,k}}{(\mu+1)^{n-1}}+\frac{ta_{1,k}(d_{1,k}-c_{1,k})^2}{5(2\mu+1)^{n-1}}(d_{2,k}-c_{2,k})^{\mu}\right.$$

$$\left.\ldots(d_{n,k}-c_{n,k})^{\mu}\right]\sum_{m=1}^{N}\left[\frac{a_{1,m}^2(d_{1,m}-c_{1,m})^5}{5(2\mu+1)^{n-1}}(d_{2,m}\right.$$

$$\left.-c_{2,m})^{2\mu+1}\ldots(d_{n,m}-c_{n,m})^{2\mu+1}\right],\qquad \text{(A3)}$$

where

$$P=\frac{1}{6(\mu+1)^{n-1}}\sum_{k=1}^{N}a_{1,k}(d_{1,k}-c_{1,k})^3$$

$$\times(d_{2,k}-c_{2k})^{\mu+1}\ldots(d_{n,k}-c_{n,k})^{\mu+1}.$$

3. From Eqs. (2) and (7) we obtain for ν_e

$$\nu_e=\nu\left\{-2+\left[4+\frac{2(a/t)^{1/2}}{(1+(a/t)^{1/2})\ln(1+(a/t)^{1/2})}\right]\right.$$

$$\left.\times\frac{B}{\ln(1+(a/t)^{1/2})}\right\}\qquad \text{(A4)}$$

where

$$B=\frac{\pi^2}{6}+\frac{1}{2}\ln^2(a/t)^{1/2}-\frac{1}{(a/t)^{1/2}}+\frac{1}{2^2a/t}+\ldots$$

$$+\frac{(-1)^n}{n^2(a/t)^{n/2}}+\ldots,\quad a>t,$$

$$B=(a/t)^{1/2}-\frac{a/t}{2^2}+\ldots+\frac{(-1)^{n+1}}{n^2}(a/t)^{n/2}+\ldots,\quad a<t.$$

4. In order to determine the functions U, E, and R in Eq. (12) to accuracy $O(\nu)$ we substitute into the right-hand side of Eq. (13) and (20) the general solution of the Riemann equation (3) [or Eq. (11) with $\nu=0$ which in the uniqueness region $t<t_0=-1/\min_{(x)}du_0(x)/dx$ has the form

$$u(x,t)=\int_{-\infty}^{\infty}d\zeta u_0(\zeta)\delta(\zeta-x+u_0(\zeta)t)\left(1+t\frac{du_0(\zeta)}{d\zeta}\right),\qquad \text{(A5)}$$

where $u_0(x)$] is a smooth initial fields, δ is the delta function, $P=$ const for (A5) since $u_0\to0$ for $|x|\to\infty$. Then, for example,

$$\frac{d^2R}{dt^2}=\frac{dE}{dt}\approx-\nu\left(\int_{-\infty}^{\infty}dx\left(\frac{\partial u}{\partial x}\right)^2\right)\Big|_{\nu=0}$$

$$=-\nu\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}d\zeta_1u_0(\zeta)u_0(\zeta_1)\frac{\partial}{\partial x}\delta(\zeta-x+u_0(\zeta)t)$$

$$\times\frac{\partial}{\partial x}\delta(\zeta_1-x+u_0(\zeta_1)t)\left(1+t\frac{du_0(\zeta_1)}{d\zeta_1}\right)\left(1+t\frac{du_0(\zeta)}{d\zeta}\right)$$

$$=-\nu\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}d\zeta_1\frac{du_0(\zeta)}{d\zeta}\frac{du_0(\zeta_1)}{d\zeta_1}$$

$$\times\delta(\zeta-x+u_0(\zeta)t)\delta(\zeta_1-x+u_0(\zeta_1)t)$$

$$=-\nu\int_{-\infty}^{\infty}d\zeta_1\int_{-\infty}^{\infty}d\zeta\frac{du_0(\zeta)}{d\zeta}\frac{du_0(\zeta_1)}{d\zeta_1}\delta(\zeta-\zeta_1+t(u_0(\zeta)-u_0(\zeta_1)))$$

$$=-\nu\int_{-\infty}^{\infty}d\zeta_1\int_{-\infty}^{\infty}d\zeta\frac{du_0(\zeta)}{d\zeta}\frac{du_0(\zeta_1)}{d\zeta_1}$$

$$\times\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp[ik(\zeta-\zeta_1+t(u_0(\zeta)-u_0(\zeta_1)))]$$

$$=-\nu\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-\infty}^{\infty}d\zeta_1\int_{-\infty}^{\infty}d\zeta\frac{du_0(\zeta)}{d\zeta}\frac{du_0(\zeta_1)}{d\zeta_1}$$

$$(u_0(\zeta)-u_0(\zeta_1))^n\frac{\partial^n\delta}{\partial\zeta^n}(\zeta-\zeta_1)$$

$$=-\nu\int_{-\infty}^{\infty}d\zeta\frac{du_0(\zeta)}{d\zeta}Q(t,u_0(\zeta)).\qquad \text{(A6)}$$

Similarly, we obtain the expression

$$\frac{d^2U}{dt^2}=\nu\int_{-\infty}^{\infty}dx\left(x\frac{du_0(x)}{dx}-3u_0-2tu_0\frac{du_0}{dx}\right)Q^2(t,u_0(x)),\qquad \text{(A7)}$$

where,

$$Q=\sum_{n=0}^{\infty}\frac{t^n}{n!}(-1)^n\sum_{k=0}^{n}(-1)^k\frac{C_n^ku_0^k}{(n-k+1)}\frac{d^{n+1}u_0^{n+1-k}}{dx^{n+1}},$$

$$C_n^k=\frac{n!}{k!(n-k)!}.$$

The expressions for U, E, and R are obtained by elementary integrations of the right-hand sides of Eqs. (A6) and (A7) with respect to t.

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