Magnetohydrodynamic wave breaking of a magnetized plasma for large Mach numbers

M. B. Isichenko and Ya. L. Kalda

I. V. Kurchatov Atomic Energy Institute
(Submitted 29 November 1988)
Zh. Eksp. Teor. Fiz. **96**, 134–139 (July 1989)

We study in this paper the singularities arising during magnetohydrodynamic wave breaking in a magnetized plasma when we take the following effects into account: finite magnetic pressure, violation of the magnetic-field freezing due to the finite electron inertia, and violation of quasi-neutrality. When the Mach number, $M = v/c_A$ is large, the wave breaking leads to a significant increase of the magnetic field at small scales. We obtain an estimate of the maximum magnetic field near the singularity.

1. INTRODUCTION

Wave breaking, or trajectory intersection, is one of the typical hydrodynamical effects which leads to the appearance of the singularities in various physical quantities and of an enhanced energy dissipation. Since wave breaking sets in both in non-dispersive and in many dispersive systems for a continuum of initial conditions, the study of appropriate typical singularities is of independent interest, like that of wave collapse.

The simplest ideas about wave breaking come from the evolution of a cold beam of independent particles,[1] an evolution described by the equations

$$\partial \mathbf{v}/\partial t+(\mathbf{v}\nabla)\mathbf{v}=0, \tag{1}$$

$$\partial\rho/\partial t+\operatorname{div}\rho\mathbf{v}=0, \tag{2}$$

where $\mathbf{v}$ is the velocity and ρ the density of the particles. In this model, which we shall call the ballistic model, after a finite time (at $t = t_c$) a singularity with an infinite density is formed; in the one-dimensional case

$$n(x,t_c)\sim(x-x_c)^{-2/3},\quad v(x,t_c)\sim(x-x_c)^{1/3}. \tag{3}$$

Three-dimensional singularities were studied in Ref. 2 in connection with a possible explanation of the disk-shaped structure of galaxies, and in Ref. 3 where periodic wave breaking of electron beams in a plasma was invoked for an explanation of the generation of high harmonics of laser radiation.

It is known[4] at the same time that typical singularities in one-dimensional hydrodynamics, which are described by simple Riemann waves, do not lead to a significant increase in the density, the profile of which breaks like the velocity profile, $\propto (x - x_c)^{1/3}$.

The aim of the present paper is the study of wave breaking in a plasma with a large Mach number $M = v/c_A \gg 1$ [$c_A = H(4\pi\rho)^{-1/2}$ is the Alfvén velocity, H the magnetic field, and $\rho = M_i n_i$ the plasma density] when at the initial time the term with the magnetic pressure is a small correction to (1), although it determines the behavior near the singularity.

The increase in the density in almost ballistic wave breaking is accompanied by an increase in the frozen magnetic field, and this may lead to observable effects, for instance, to enhanced cyclotron emission.

We discuss in this paper also other effects which modify the "ballistic" singularity—the violation of the magnetic-field freezing and the loss of quasi-neutrality, which can also occur during wave breaking.

2. ALLOWANCE FOR FINITE PRESSURE

We consider the case of ideal single-fluid magnetohydrodynamics. We neglect the plasma pressure in comparison with the magnetic field pressure.[1)] In a one-dimensional geometry $H = H_z(x,t)$ the motion of the plasma is described by the gasdynamics equations with an adiabatic exponent $\gamma = 2$:

$$\rho(\partial v/\partial t+v\partial v/\partial x)=-\partial p/\partial x,\quad p=H^2/(8\pi), \tag{4}$$

$$\partial H/\partial t+\partial(vH)/\partial x=0. \tag{5}$$

It is convenient, if initially $H/\rho = \text{const}$ ("isentropic flow"), to rewrite (2), (4), (5) in terms of the Riemann invariants:

$$D_\pm(v\pm\alpha c)=0,\ D_\pm=\partial/\partial t+(v\pm c)\partial/\partial x,$$
$$c^2=dp/d\rho=c_A^2,\quad \alpha=2/(\gamma-1). \tag{6}$$

One can obtain an exact solution of the set of gasdynamic Eqs. (6) using a non-linear change of variables $(x,t)\to(a,b)$:

$$x=a+\int_0^t(v+c)_{a=\text{const}}\,dt=b+\int_0^t(v-c)_{b=\text{const}}\,dt. \tag{7}$$

The meaning of the variables (a,b) (which we can call quasi-Lagrangian variables) is that they denote the initial coordinates of points moving with velocities $v \pm c$ and landing at the time t in the point x. In terms of the variables (a,b) we have

$$D_+=(\partial t/\partial b)^{-1}\partial/\partial b,\ D_-=(\partial t/\partial a)^{-1}\partial/\partial a,$$

whence we get at once the solution

$$v(a,b)=(1/2)[v_0(a)+v_0(b)]+(\alpha/2)[c_0(a)-c_0(b)], \tag{8}$$

$$c(a,b)=(1/2\alpha)[v_0(a)-v_0(b)]+(1/2)[c_0(a)+c_0(b)], \tag{9}$$

where $v_0(x)$ and $c_0(x)$ are the initial profiles of the velocity

 0038-5646/89/070073-03$04.00

and of the sound velocity $c = \text{const}\, \rho^{1/\alpha}$, expressed in terms of the density.

The solution (8), (9) is somewhat formal, as we must solve the integral equation

$$t(a,\ b)=(b-a)/[v_0(a)-v_0(b)+c_0(a)+c_0(b)+A], \tag{10}$$

$$A=(\alpha-1)/(2\alpha t(a,b)) \int_a^b \{[v_0(\xi)-c_0(\xi)][t(a,\xi)-t(a,b)]$$
$$+[v_0(\xi)+c_0(\xi)][t(\xi,b)-t(a,b)]\}d\xi \tag{11}$$

obtained by substituting (8) and (9) into (7), in order to change to the variables (x,t). Solving (10) we can find

$$x(a,b)=\frac{b[v_0(a)+c_0(a)+A_1]-a[v_0(b)-c_0(b)+A_2]}{v_0(a)-v_0(b)+c_0(a)+c_0(b)+A}, \tag{12}$$

where A_1 and A_2 are integral terms like (11), which we do not write down for the sake of brevity. We note only that for $\gamma = 3$, i.e., $\alpha = 1$, when Eqs. (6) uncouple, $A = A_1 = A_2 = 0$, and Eqs. (8)–(12) are the solution of the initial gasdynamic problem in parametric form.

The start of wave breaking corresponds to the loss of single-valuedness of one of the functions $a(x,t)$ or $b(x,t)$ [the function $v(a,b)$ is single-valued]. To be specific, we assume that this occurs first for $a(x,t)$. Denoting by an index c the point where the singularity occurs, we can write the condition for wave breaking in the form

$$\partial t(a_c,\ b_c)/\partial a=\partial^2 t(a_c,\ b_c)/\partial a^2=0, \tag{13}$$

whence we get, if we use (10),

$$b_c-a_c=\frac{v_0(a)-v_0(b)+c_0(a)+c_0(b)+A}{v_0'(a_c)+c_0'(a_c)+\partial A/\partial a}, \tag{14}$$

$$v_0''(a_c)+c_0''(a_c)+\partial^2 A/\partial a^2=0. \tag{15}$$

The formalism worked out here turns out to be effective in the limit of interest to us, of large Mach numbers $M_0 = v_0/c_0 \gg 1$. In that case, near the wave breaking point we have $|v_0(a) - v_0(b)| \gg c_0(a) \approx c_0(b)$ and a simple estimate shows that $A \approx A_1 \approx A_2 \approx c_0$ [we grouped the terms in the denominator of (10) just for such a decrease of the integral term (11)]. Expanding the smooth functions $v_0(x), c_0(x)$ in a power series in the vicinity of a_c we get, neglecting terms of order M_0^{-1}, from (9), (14), and (15):

$$v_0''(a_c)=0, \tag{16}$$

$$b_c-a_c=(12c_0(a_c)/v_0(a_c))^{1/3}, \tag{17}$$

$$c(a_c,\ b_c)=c_0(a_c)\,[-3v_0^3(a_c)/(2c_0^2(a_c)v_0(a_c))]^{1/3}. \tag{18}$$

We show in the figure schematically the distributions of v and ρ at the initial time and at the moment of the singularity. In the point C in the figure the tangents to the curves $v(x,t_c)$ and $\rho(x,t_c)$ are vertical. It is interesting to note the paired nature of the singularity considered: besides the point C there is an "almost singular" point C' in which soon afterwards there occurs also a gradient catastrophy [to which corresponds the loss of single-valuedness of $b(x,t)$], but at $t = t_c$ the slope of the tangent in C' differs from the vertical by a small parameter M_0^{-1} The distance between the points C and C' is of the order of $l_0 M_0^{-1}$, where $l_0 = v_0/v_0'$ is the characteristic scale of the problem.

Thus, almost ballistic ($M_0 \gg 1$) wave breaking in gasdynamics leads, according to (18), to an increase in the sound velocity in the vicinity of the singularity to a value $c_{\max} \approx c_0 M_0^{2/3}$ which corresponds to a density

$$\rho_{max} \approx \rho_0 M_0^{4/(3\gamma-3)}. \tag{19}$$

The result (19) has been found before[3] in the case of one-dimensional collisionless kinetics (for which we can put formally $\gamma = 3$).

Returning to MHD wave breaking, we note that near the singularity the assumption of isentropic behavior, used above, does not restrict the applicability of the results. One can say the same also about wave breaking in an oblique magnetic field. In that case H acquires a component H_x perpendicular to the x axis because of the freezing, but which changes insignificantly even for three-dimensional wave breaking (see below). Putting $\gamma = 2$ in (19) we get an estimate for the maximum magnetic field

$$H_{max} \approx H_0(4\pi\rho_0 v_0^2/H_0^2)^{2/3}. \tag{20}$$

It is interesting to note that in the limits of the applicability of single-fluid magnetohydrodynamics $H_{\max}$ is larger the smaller the initial magnetic field H_0.

So far we have considered one-dimensional wave breaking. However, thanks to the effective "switch-off" of two dimensions, when we approach the singularity, in the case of three-dimensional initial conditions (thin "platelets" of enhanced density appear), the quasi-one-dimensional approximation is valid in the general case. With reference to the method used in Ref. 3, we give here the main results of the analysis. When we approach the singularity ($t = t_c$) the region of enhanced density is a strongly flattened ellipsoid (platelet) of thickness $l \approx l_0(\delta t/t_0)^{3/2}$ and diameter $l_0(\delta t/t_0)^{1/2}$ (here $t_0 = l_0/v_0$ is the characteristic time of the problem, $\delta t = t_c - t > 0, \delta t \ll t_0$). After the velocity of sound in the center of the platelet reaches its maximum value $c_{\max} \approx c_0 M_0^{2/3}$ the increase in density stops. By that time the thickness of the platelet has decreased to $l_{\min} \approx l_0 M_0^{-1}$ Later (when $t > t_c$) the platelet starts to increase in size having

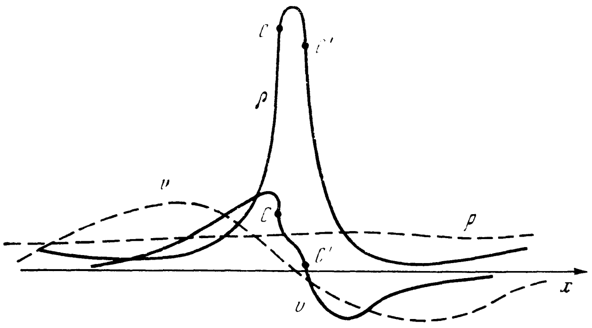

FIG. 1. Distributions of the velocity (v) and of the density (ρ) of a gas initially (dashed lines) and at the time of the singularity (solid lines).

within it a region of multiple-flow motion. The maximum sound velocity $c_{\max}$ occurs at the edges of the platelet in a region which has the shape of an expanding torus which is flattened along its main axis, and the major axis of which is of the order of $l_0(-\delta t/t_0)^{1/2}$, while the minor radii are $r_1 \approx l_0 M_0^{-1}$, $r_2 \approx l_0 M_0^{-2/3}(-t_0/\delta t)^{1/2}$.

3. ALLOWANCE FOR DISPERSION

In this section we discuss the effect of two-fluid behavior, which in the linear case leads to dispersion of magnetosonic waves and in the non-linear case to additional restrictions on the maximum magnetic field which arises during wave breaking. For the sake of simplicity we consider the one-dimensional case.

For transverse magnetosonic waves in a cold plasma there are two characteristic dispersion scales: $d = c/\omega_{pe}$ [$\omega_{pe} = (4\pi n e^2/m)^{1/2}$ is the electron plasma frequency; in this formula c denotes the velocity of light] and $\lambda = c_A/\omega_{pi} = H/(4\pi e n)$ (ω_{pi} is the ion Langmuir frequency). The dispersion equation of such waves has the form

$$\omega(k) = kc_A/(1+k^2a^2+k^2\lambda^2).$$

The scale d characterizes the size at which the electron inertia starts to play a role whereas when the scale λ is reached the quasi-neutrality in the plasma is violated (in that sense λ can be treated as a "magnetic Debye length" differing from the normal Debye length in that the electron thermal pressure is replaced by the magnetic pressure).

It was noted correctly in Ref. 5 that dispersive effects do not prevent the appearance of a singularity, but only lead in some cases to a threshold effect in wave breaking. We shall assume that the motion of the plasma is well above threshold, thus assuming that not only is the Mach number large, but also the initial scale of the plasma flow l_0 is significantly larger than the characteristic dispersive scales d and λ.

We shall, to begin with, assume that during the whole wave breaking process the inequality $d \gg \lambda$ is satisfied, and that the restrictions connected with the finite magnetic pressure are insignificant. In that case the ion motion is ballistic and the plasma is quasi-neutral: $n_e = n_i = n$, but as the characteristic scale decreases near the singularity. $l \propto n^{-1} \to 0$, this scale reaches the more slowly decreasing value $d \propto n^{-1/2}$. The freezing of the magnetic field onto the electrons is then violated, but the freezing of the curl of their generalized momentum is conserved[6]:

$$\partial\Omega/\partial t + \partial(\Omega v)/\partial x = 0, \tag{21}$$

$$\Omega = H - \partial(d^2\partial H/\partial x)/\partial x. \tag{22}$$

Since we have as before by virtue of (21), $\Omega \propto (x - x_c)^{-2/3}$, and $d \propto n^{-1} \propto (x - x_c)^{2/3}$, at $t = t_c$, we conclude from (22) that when $l \approx d$ the growth in the magnetic field stops and afterwards, on the background of the maximum attainable field

$$H_{max} \approx H_0(l_0/d_0)^2 \tag{23}$$

a "cusp" type singularity in the profile of H, $H \propto (x - x_c)^{2/3}$, is formed at $t = t_c$ with a characteristic half-cubic cusp.

Also possible is a regime such that either $d_0 < \lambda_0$ or $d_0 > \lambda_0$ from the start, but during the wave breaking d in the vicinity of the singularity becomes less than $\lambda \propto H/n$ because the latter is conserved due to the freezing, even when $l \gg \lambda$. In that case the freezing of the magnetic field onto the electrons does not prevent changes, but when the characteristic scale becomes comparable with the small scale λ the plasma loses its quasi-neutrality, in agreement with the equation:[7,5]

$$n_e - n_i = \lambda_0^2\partial^2 n_e/\partial x^2. \tag{24}$$

As a result the increase of the electron density and of the magnetic field which is proportional to it is stopped, forming only, according to (24), an insignificant discontinuity in the second derivative of $H \propto n \propto (x - x_c)^{4/3}$ on the background of the maximum magnetic field

$$H_{max} \approx H_0 l_0/\lambda_0. \tag{25}$$

4. CONCLUSION

Summarizing what we have said we are led to the conclusion that initially ballistic wave breaking of a plasma in a magnetic field leads to a local increase in the field when the characteristic scale l decreases to $l_{\min} = \max(l_0 M_0^{-1}, d_0^2/l_0, \lambda_0)$, after which the finite magnetic pressure, the electron inertia, or the violation of quasi-neutrality prevents a further growth in the magnetic field which reaches near the singularity a maximum value

$$H_{max} = \min(H_0 M_0^{4/3}, H_0(l_0/d_0)^2, H_0 l_0/\lambda_0). \tag{26}$$

Further wave breaking leads to multiple flow and turbulization of the plasma, which is characteristic for the "flickering" structure of collisionless shock waves.[5]

In the case when the characteristics of the plasma flow are controlled by the large parameters $M_0 \gg 1$, $l_0 \gg d_0, \lambda_0$, the wave breaking is accompanied by bursts of strong magnetic fields (26) concentrated in small scales $l_{\min}$. Such a situation may turn out to be typical of the interaction of the solar wind with the Earth's magnetosphere, where magnetic-field bursts may cause local flashes of electromagnetic radiation generated when electron cyclotron waves fuse into transverse waves.[8] The peaking in the power of the radiation may in this case be caused by an adiabatic increase in the amplitude of the oscillations when the magnetic field grows, and also by the appearance of the dominant dipole radiation component due to the strong inhomogeneity of the density and of the magnetic field near the singularity.[9]

[1)] Having a strong inequality is not of principal importance when we study singularities, since the magnetic pressure increases faster ($\gamma = 2$) than the gas dynamic pressure ($\gamma = \frac{5}{3}$).

[1] B. B. Kadomtsev, *Kollektivnye yavleniya v plazme (Collective Phenomena in a Plasma)*, Nauka, Moscow, 1977.

[2] Ya. B. Zel'dovich, Astrofizika **6**, 119 (1970) [English translation in Astrophysics **6**].

[3] M. B. Isichenko and V. V. Yan'kov, Zh. Eksp. Teor. Fiz. **87**, 1914 (1984) [Sov. Phys. JETP **60**, 1101 (1984)].

[4] L. V. Ovsyannikov, *Lektsii po osnovam gazovoĭ dinamiki (Lectures on the Principles of Gas Dynamics)* Nauka, Moscow, 1971.

[5] V. V. Krasnosel'skikh, Zh. Eksp. Teor. Fiz. **89**, 498 (1985) [Sov. Phys. JETP **62**, 282 (1985)].

[6] A. S. Kingsep, K. V. Chukbar, and V. V. Yan'kov, Vopr. teor. plazmy **16**, 209 (1987) [English translation in Rev. Plasma Phys. **16**, Plenum, in press].

[7] R. Z. Sagdeev, Vopr. teor. plazmy **4**, 3 (1964) [Rev. Plasma Phys. Plenum, **4**, 23 (1966)].

[8] K. D. Cole and O. A. Pokhotelov, Plasma Phys. **22**, 595 (1980).

[9] M. B. Isichenko and E. B. Tatarinova, Fiz. Plazmy **13**, 87 (1987) [Sov. J. Plasma Phys. **13**, 46 (1987)].

Translated by D. ter Haar