Quasiclassical scattering and glory effect in the gravitational field of a black hole

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The differential cross section for backward scattering ($\theta \to \pi$) of spinless particles with nonzero rest mass on the background of a Schwarzschild geometry is calculated by quantum-mechanical methods. It is shown that a glory exists if the gravitational radius exceeds the particle wavelength.

1. One of the remarkable results from the theory of scattering by black holes is the prediction in Ref. 1 of a glory effect in the backward scattering of light rays. In the framework of geometrical optics, Ford and Wheeler[1] showed that the differential scattering cross section in a Schwarzchild field grows like $(\pi - \theta)^{-1}$ as $\theta \to \pi$. Later, an analogous result was obtained for massive probe particles as well.[2] Glory scattering by a black hole (BH) is due to the fact that particles with impact parameters slightly greater than the radius of stability of a circular orbit can execute one or considerably more than one (half)-revolutions about the black hole before moving away to infinity.

However, it is known from quantum mechanics that backward (forward) scattering may not be classical if the classical angle of deviation tends to π (or 0) at certain finite values of the impact parameter. In this case it is necessary to take account of interference phenomena associated with the finiteness of the particle wavelength. Thus, a quantum-mechanical investigation of the problem of the scattering of particles by a black hole is of interest. The long-wavelength case ($\lambda \gg R_G$) has been considered in sufficient detail for waves and particles with various spins, both in the approximation of a weak scattering field and in a more rigorous approach (see, e.g., Ref. 4 and the citations contained therein). In Ref. 5, the opposite case of short-wavelength scattering of electromagnetic waves by a Schwarzschild BH was considered. It was noted that when polarization properties are taken into account the glory effect may be absent—the strictly backward differential scattering cross section for photons is equal to zero, this being true for both the short-wavelength and the long-wavelength case. A more exact analysis of the glory for massless waves was carried out in Refs. 6 and 7.

In the present paper we consider, on the basis of the Klein-Gordon equation, the scattering of massive spinless particles by a nonrotating (Schwarzschild) BH in the short-wavelength case:

$$G(E^2-\mu^2c^4)^{1/2}M/\hbar c^3 \gg 1. \tag{1}$$

With the assumption (1), everywhere up to the event horizon $R_G = 2GM/c^2$ the particle wavelength is short in comparison with the characteristic length scale of the nonuniformity of the gravitational field, and, therefore, it is convenient to use the phase-shift theory of scattering and the quasiclassical approximation. In this connection, we note Ref. 8, in which it was shown that an exact analysis of glory scattering on the basis of path integrals in flat space leads to a result coinciding with the quasiclassical result.

2. Writing the Klein-Gordon equation in the Schwarzschild metric and performing a separation of variables, we obtain the wave function of a particle with energy ω, orbital angular momentum l, and angular-momementum component m along the z axis ($\theta = 0$) (see Ref. 4):

$$\Phi_{\omega,l,m} = \frac{1}{r} R_{\omega,l}(r) Y_l^m(\theta,\varphi) e^{-i\omega t}, \tag{2}$$

where the Y_l^m are spherical harmonics. (Here and below, we use a system of units with $c = \hbar = G = 1$.) The radial part satisfies the equation

$$\frac{d^2R}{dr^{*2}} + W_l(r^*)R=0, \qquad \frac{dr^*}{dr} = \left(1-\frac{2M}{r}\right)^{-1}, \tag{3}$$

with

$$W_l(r)=\omega^2-\left(1-\frac{2M}{r}\right)\left[\mu^2+\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]. \tag{4}$$

Choosing at $r\to\infty$ ($\omega > \mu$) a solution of the form

$$R_{\omega,l}(r)=2\sin\left(kr-\frac{\pi l}{2}+\delta_l\right), \qquad k=(\omega^2-\mu^2)^{1/2}, \tag{5}$$

and, near the horizon ($r^* \to -\infty$), a solution satisfying the capture condition $R \propto \exp(i\omega r^*)$, it is not difficult to show that the phase shift δ_l is complex, with

$$2\,\mathrm{sh}(2\,\mathrm{Im}\,\delta_l)=\frac{1-|S_l|^2}{|S_l|}=T_l\exp(2\,\mathrm{Im}\,\delta_l), \tag{6}$$

where $S_l = \exp(2i\delta_l)$, and T_l is the coefficient of absorption of an incident partial wave. Expanding the wave function of a stationary scattering state in the partial waves (2), we obtain the standard expression for the elastic-scattering amplitude. For angles $\theta \neq 0$ it has the form

$$f(\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}P_l(\cos\theta). \tag{7}$$

Here the imaginary part of the phase shift determines the total absorption cross section:

$$\sigma_A=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-|S_l|^2)=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)T_l. \tag{8}$$

3. Restricting ourselves next by the condition (1), we note that in this case the function $W_l(r)$ takes the form

$$W_l(r)=\omega^2-\left(1-\frac{2M}{r}\right)\left(\mu^2+\frac{L^2}{r^2}\right)\equiv\omega^2-U_L^2(r). \tag{9}$$

Here, for $l \gg 1$ we have made the replacement

$$l(l+1)\to(l+{}^1/_2)^2\equiv L^2. \tag{10}$$

The function $U_L(r)$ coincides with the effective potential energy of a classical probe particle in a Schwarzschild field (see, e.g., Ref. 9). In the classical limit, when

$$L<L_c=\omega M\left[\frac{(1+8v^2)^{3/2}+8v^4+20v^2-1}{2v^2}\right]^{1/2}=\omega Mf, \tag{11}$$

where $v = (1-\mu^2/\omega^2)^{1/2}$, the particles fall into the black

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hole, and a state with $L = L_c$ corresponds to an unstable circular orbit.

For $|L - L_c| \ll L_c$ the function $W_l(r)$ can be represented in the form

$$W_L(r) = W_{L_c}(r) + \delta W_L(r), \tag{12}$$

where

$$\delta W_L(r) = -\frac{L^2 - L_c^2}{r^2}\left(1 - \frac{2M}{r}\right) \approx -2\frac{L - L_c}{L_c}\frac{L_c^2}{r^2}\left(1 - \frac{2M}{r}\right). \tag{13}$$

Using the approximation (12), (13) and the WKB method, one can calculate the transmission coefficient T_l for orbital angular momenta close to the critical value (i.e., near the top of the potential barrier):

$$T_l = \{1 + \exp[\pi(L^2 - L_c^2)\beta^{-1/2} L_c^{-1}]\}^{-1}, \tag{14}$$

where the coefficient β depends only on the velocity of the particle at infinity:

$$\beta(v) = \left(1 - \frac{12\mu^2 M^2}{L_c^2}\right)^{1/2} = \left[\frac{5 + 44v^2 + 32v^4 - 3(1 + 8v^2)^{3/2}}{8(1 - v^2)}\right]^{1/2}. \tag{15}$$

It varies slowly from 0.5 at $v = 0$ to 1 at $v = 1$. The same result is obtained if one uses a parabolic approximation of the potential near the top of the barrier. It follows from (14) that for $L - L_c \gg 1$ the transmission coefficient is exponentially small, while for $L_c - L \ll 1$ it is practically equal to 1. The quasiclassical total absorption cross section takes the form

$$\sigma_A = \frac{\pi M^2}{v^2}\frac{(1+8v^2)^{3/2} + 8v^4 + 20v^2 - 1}{2v^2}\left(1 - \frac{2}{L_c}\right). \tag{16}$$

The factor in front of the bracket is the exact expression for the classical cross section for capture of probe particles in a Schwarzschild field, with well known nonrelativistic and ultrarelativistic limits. The additional term in the brackets is of order $(R_G\omega)^{-1}$ and is a quantum correction. It should be noted that in the first of Refs. 10 an attempt was made, for the first time, to calculate the glory for spinless particles of nonzero mass.

4. The relation (14) implies that for $L > L_c$ we can use the usual WKB formula to calculate the real part of the scattering phase shifts:

$$\operatorname{Re}\delta_l = \lim_{r\to\infty}\left\{\int_{r_1}^{r} [W_l(r)]^{1/2} dr' - \int_{L/k}^{r} (k^2 - L^2/r^2)^{1/2} dr\right\}. \tag{17}$$

The imaginary part of the phase shift here is small. The second integral in (17) gives the phases of the free motion, while the first contains the divergent part typical of long-range potentials. The exact form of this part is

$$\delta(r) = 2\eta\ln(2kr), \quad \eta = 2kM + \varkappa/k, \quad \varkappa = \mu^2 M. \tag{18}$$

The first integral in (17) reduces to an elliptic integral, and analytical calculations are possible only in the limiting cases $L \gg L_c$ and $0 < L - L_c \ll L_c$. However, in these particular cases it is possible to use a perturbation method from the outset to estimate the contribution of the small terms in the potential (9).

In the first of these cases ($L \gg L_c$) the term proportional to r^{-3} in the effective potential can be treated as a perturbation[11]:

$$W_L(r) = W_L^{(0)}(r) + W_L^{(1)}(r), \tag{19}$$

where

$$W_L^{(0)}(r) = k^2 + \frac{2\varkappa}{r} - \frac{L^2}{r^2}, \tag{20}$$

$$W_L^{(1)}(r) = \frac{2ML^2}{r^3}. \tag{21}$$

Without dwelling on the details of the calculation, we write out the result, omitting the logarithmically divergent part:

$$\operatorname{Re}\delta_l = -\eta\ln(L^2 + \varkappa^2/k^2)^{1/2} + \frac{\varkappa}{k} - \frac{L}{2}\left[\arcsin\frac{\eta}{(L^2 + \varkappa^2/k^2)^{1/2}} + \arcsin\frac{\varkappa}{k(L^2 + \varkappa^2/k^2)^{1/2}}\right]. \tag{22}$$

It is easy to convince oneself that in the nonrelativistic limit ($k \gg \mu$), with the replacement $\mu M \to -eQ$, the formula (22) goes over into the expression for the WKB phase shifts in a Coulomb field. In the relativistic case ($k \gg \mu$) this same formula gives the partial phase shifts, calculated in Ref. 12, for scalar massless waves in a Schwarzschild metric.

In the second case ($0 < L - L_c \ll L_c$), when the impact parameters of the particles are close to the critical values, we make use of the representation of the function $W_l(r)$ in the form (12), (13), regarding δW_l as a perturbation. Then

$$\operatorname{Re}\delta_l = \delta_c + \delta_L^{(1)}, \tag{23}$$

where

$$\delta_c = -\eta\ln\left[\frac{3kM}{2\beta-1}\frac{(3\beta)^{1/2} + (2\beta-1)^{1/2}}{(3\beta)^{1/2} - (2\beta-1)^{1/2}}\right] + \frac{3kM}{2\beta-1} + \left(\frac{\pi}{2} - \beta^{1/2}\right)L_c + 2\omega M\ln\left[\frac{(2\beta+2)^{1/2} - (3\beta)^{1/2}}{(2\beta+2)^{1/2} + (3\beta)^{1/2}}\frac{(2\beta+2)^{1/2} + (2\beta-1)^{1/2}}{(2\beta+2)^{1/2} - (2\beta-1)^{1/2}}\right], \tag{24}$$

$$\delta_L^{(1)} = \left(\frac{L^2 - L_c^2}{2L_c}\right)\left(\frac{1}{2\beta^{1/2}}\ln\frac{L^2 - L_c^2}{2CL_c^2} + \frac{\pi}{2} - \frac{1}{2\beta^{1/2}}\right). \tag{25}$$

Here C is a function that depends only on the velocity of the particles at infinity:

$$C(v) = 4(6\beta)^3/(2-\beta)[(3\beta)^{1/2} + (2\beta-1)^{1/2}]^4. \tag{26}$$

It varies slowly in the range from 32 at $v = 0$ to 15.5 at $v = 1$ [see (15)]. In Eqs. (23)–(25), as above, we have omitted logarithmically divergent terms.

5. The subsequent calculation of the scattering amplitude does not differ fundamentally from that in the case of flat space, and this is also true for the glory. However, before giving the results we remark that in our case the calculational method is applicable only for the condition $L_c \gtrsim 10^2$ (or, correspondingly, for $\omega R_G \gtrsim 10^2$). In the opposite case the phase shift $\delta_L^{(1)}$ can be of the order of 1, and the angle of deviation changes sharply. In this case it would be necessary to take account of interference of waves with $l \sim l_c$. Such interference phenomena have been analyzed numerically for scalar waves with $\lambda \sim R_G$ in Ref. 13.

Since the phase shifts (23)–(25) give a nonunique relationship between the angle of deviation and the impact parameter (the angle of deviation tends to infinity as $L\to L_c$), introducing the number n of revolutions of the particle we find that the scattering angle is equal to

$$\theta(L)=-2n\pi-\frac{2d(\mathrm{Re}\,\delta_l)}{dl}=-(2n+1)\pi-\frac{1}{\beta^{1/2}}\ln\frac{L^2-L_c^2}{2CL_c^2}. \tag{27}$$

For $L\gg L_c$ the angle of deviation is a one-to-one function of the orbital angular momentum, and for relativistic particles the phase shifts (22) always correspond to small angles of deviation:

$$\theta(L)=-\frac{2d(\mathrm{Re}\,\delta_l)}{dl}\approx\frac{2\eta}{L}. \tag{28}$$

At the same time, for nonrelativistic particles with angular momenta $L_c\ll L\ll\varkappa/k$ the angle of deviation can be large ($1\lesssim\theta<\pi$). For this it is necessary that the condition $v\ll 1/4$ be fulfilled.

Expanding the phase shift δ_l in a series about the value $L_n(\theta)$ determined from (27), we represent the scattering amplitude (7) in the form of a sum over n and go over from the summation to integration over the continuous parameter L:

$$f(\theta)=\frac{L_c}{ik}\exp(2i\delta_{l_c})P_{l_c}(\cos\theta)+\sum_{l=l_c+1}^{\infty}(2l+1)e^{2i\delta_l}P_l(\cos\theta)=f_A(\theta)+\sum_{n=0}^{N}f_n(\theta), \tag{29}$$

$$f_n(\theta)=\frac{1}{ik}\int L\exp\left\{i\left[2\delta_{L_n}-\theta(L-L_n)-\frac{(L-L_n)^2}{2L_n'(\theta)}\right]\right\}P_l(\cos\theta)\,dL. \tag{30}$$

In order of magnitude, the maximum number N of revolutions is equal to $(\ln L_c)/2\pi\beta^{1/2}$. The moment $L_0(\theta)$ is determined from (27) and (22) for small angles. But if $\theta\to\pi$, or $n\gg 1$, then $L_n(\theta)$ can be determined directly from (27):

$$(L_n(\theta)-L_c)/L_c=C(v)\exp\{-[\theta+(2n+1)\pi]\beta^{1/2}\}. \tag{31}$$

The value of the phase shift $\delta_{L_n}(\theta)$ is then equal to

$$\delta_{L_n}(\theta)=\delta_c-L_c\left(\frac{\theta}{2}+n\pi+\frac{1}{2\beta^{1/2}}\right). \tag{32}$$

First we shall consider scattering through small angles. Using the asymptotic representation of the Legendre polynomials in terms of Bessel functions $J_0(L\theta)$ and taking into account that, by virtue of (28), $\theta L\approx 2\eta\gg 1$, we find

$$f_0(\theta)=-[2\eta/k\theta^2]\exp\left\{-i\left[2\eta\left(\ln\frac{2\eta}{\theta}-1\right)-2kM+\frac{\pi}{4}\right]\right\},\quad \theta\ll 1. \tag{33}$$

When $f_n(\theta)$ with $n\gg 1$ is integrated over a range of angles $\theta\ll|L_n(\theta)|^{-1/2}$ one can take the Bessel function outside the integral. We obtain

$$f_{n\geq 1}(\theta)\approx\frac{1}{ik}(2\pi C\beta^{1/2}L_n^3)^{1/2}\times\exp\left[-\frac{(2n+1)\pi\beta^{1/2}}{2}\right]J_0(L_n\theta),\quad \theta\ll 1. \tag{34}$$

It can be seen that with increase of n the amplitudes $f_n(\theta)$ decrease exponentially rapidly. Therefore, the small-angle scattering is classical and is determined by the amplitude (33), which has a Rutherford character:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta\gg 1}=|f_0(\theta)_1|^2=\left[\frac{(2k^2+\mu^2)R_G}{k^2\theta^2}\right]^2,\quad \theta\ll 1. \tag{35}$$

We remark that if one calculates the phase shifts using the scheme (19)–(21), keeping terms $\sim L_c^2/L$, differs from the Rutherford cross section by a term of order $L_c^2/k^2\theta^3$ (Ref. 11). In the nonrelativistic case the angular distribution of the scattering will be of the Rutherford form up to rather large angles.

In the range of intermediate angles $(\theta,\pi-\theta)\gg L_c^{-1}$ the amplitudes $f_n(\theta)$ corresponding to the phase shifts (23)–(25) are integrated with the aid of the corresponding representation of the Legendre polynomials:

$$f_n(\theta)=-\frac{L_c}{k}\left(\frac{C\beta^{1/2}}{\sin\theta}\right)^{1/2}\times\exp\left\{-\frac{[\theta+(2n+1)\pi]\beta^{1/2}}{2}\right\}\exp\left\{i\left[2\delta_{L_n}+\theta L_n+\frac{\pi}{4}\right]\right\}. \tag{36}$$

We stress that in the case $n=0$ this formula is valid only for sufficiently large angles, such that the expression (31) is small in comparison with 1. The corresponding cross section is determined by the amplitude f_0 and coincides, to within a factor approximately equal to 1.4, with the classical cross section calculated in Ref. 2:

$$\frac{d\sigma}{d\Omega}\approx C\beta^{1/2}\frac{L_c^2}{k^2(\pi-\theta)}\exp(-2\pi\beta^{1/2}). \tag{37}$$

However, for angles close to π, the expressions (36) and (37) are not valid. Using the representation of the Legendre polynomials in the form $(-1)^l J_0[L(\pi-\theta)]$, we find that for angles $\pi-\theta\ll|L_n'(\pi)|^{-1/2}$ the amplitudes $f_n(\theta)$ have the form

$$f_n(\theta)\approx(-1)^{L_n}\frac{L_n}{ik}(2\pi L_n'(\pi))^{1/2}J_0[L_n(\pi-\theta)]\exp(2i\delta_{L_n}). \tag{38}$$

With increase of n the amplitudes f_n decrease exponentially fast, and so the principal contribution to the backward scattering will be given by the amplitude $f_0(\theta)$. Restoring the dimensions of the constants c, $\hbar$, and G, we write the differential cross section for backward scattering in the form

$$\frac{d\sigma}{d\Omega}=\frac{\pi}{4}\frac{R_G^2}{1-(\mu c^2/E)^2}\frac{ER_G}{\hbar c}AJ_0^2\left[(\pi-\theta)\frac{ER_G}{2\hbar c}f\right], \tag{39}$$

where f and A are dimensionless functions of the velocity of the particles at infinity. The function $f(v)$ is defined in (11), and $A(v)$ has the form

$$A=f^3C\beta^{1/2}\exp(-2\pi\beta^{1/2})=\frac{4f^3(6\beta)^3\beta^{1/2}\exp(-2\pi\beta^{1/2})}{(2-\beta)[(3\beta)^{1/2}+(2\beta-1)^{1/2}]^2}. \tag{40}$$

The limiting values of this function are equal to 17 at $v=0$ and 4.06 at $v=1$, in agreement with the result of Ref. 7.

Unlike the classical cross section, determined by formula (37), the quantum cross section is finite when $\theta \to \pi$. Nevertheless, the total number of particles scattered backward (more precisely, in the range of angles $0 \leqslant \pi - \theta \leqslant \alpha L$, where $L_c^{-1/2} \ll \alpha \ll 1$) is, by formula (39), twice as large as in the classical case. By virtue of well known properties of Bessel functions, the angular distribution (39) possesses a series of peaks at $\pi - \theta = 0, 3.83/L_c, 7.01/L_c$, etc., alternating with minima. The ratios of the intensities at the maxima are 1:0.161:0.090:0.062, etc. We note also that the differential cross section (39) obtained in the case of relativistic particles exceeds the Rutherford cross section in order of magnitude by a factor of R_G/λ. In the range of angles $L_c^{-1} \ll \pi - \theta \ll L_c^{-1/2}$ the amplitudes (36) and (38) go over into each other [as, correspondingly, do the cross sections (37) and (39)], and the angular distribution of the scattering becomes classical.

Thus, the scattering of spinless particles by a black hole, like the scattering of massless waves, possesses a glory. In forward scattering the glory effect is masked by the Rutherford scattering, and so the wave and spin properties of the particles are manifested only in the the backward scattering.

The orbital angular momentum value L_0 corresponding to the first ring of the glory is determined from formula (27):

$$L_0(\theta) = \frac{3 \cdot 6^{1/2} \omega M}{(1+\beta)^{1/2}(2-\beta)} \times \left[1 + \frac{96\beta^3 \exp(-2\pi\beta^{1/2})}{(2-\beta)\{\beta^{1/2} + [(2\beta-1)/3]^{1/2}\}^4} \right], \qquad k \gg \mu \tag{41}$$

and coincides with the value obtained by Ford and Wheeler[1] and Darwin[14] in the ultrarelativistic limit $\omega \gg \mu (\beta \to 1)$. We shall consider more carefully the nonrelativistic limit $\omega \to \mu$, $\beta \to \frac{1}{2}$. In this case, in view of the small value of the exponent the expression (31) becomes large (≈ 0.38), and the formula (41) yields a rough estimate for L_0 (≈ 5.5). Developing an iterative scheme on the basis of the original formula (27), we find after a few steps that $L_0 = 4.65 \mu M$. The glory cross section for nonrelativistic particles is inversely proportional to the square of the momentum and to the Compton wavelength of the particles: $|f_0(\theta - \pi)|^2 \approx 10^3 \mu M^3 / v^2$. From this we easily obtain the result that the glory for particles of nonzero rest mass is observable against the background of the Rutherford scattering for $v \geqslant 10^{-2} (\lambda_c / R_G)^{1/2}$.

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