Point spectrum in the problem of stability of self-similar scalar collapse

V. M. Malkin and V. N. Khudik

Nuclear Physics Institute, Siberian Division, USSR Academy of Sciences
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Long-lived growing perturbations (unstable quasimodes) exist in all self-similar scalar-collapse regimes. The quasimodes, however, have no time to evolve in the absence of small-scale perturbations of sufficient intensity, and this complicates the question of the existence of true unstable modes, a question not investigated to this day. We investigate the spectrum of the true eigenvalues in all central-symmetry self-similar regimes of a scalar collapse. We prove the existence of self-similar solutions that are stable against arbitrarily small perturbations, and formulate a very stringent necessary condition for the stability of self-similar solutions of general form.

1. INTRODUCTION

The evolution of a high-frequency scalar field ψ that simulates the electric fields of Langmuir waves is described by the following equations proposed in Ref. 1:

$$(i\,\partial/\partial t+\Delta-n)\psi=0, \tag{1.1}$$

$$(\partial^2/\partial t^2-\Delta)n=\Delta|\psi|^2. \tag{1.2}$$

These equations, as well as their vector operands,[2] have collapsing solutions. In the ultrasonic adiabatic approximation (which becomes more and more accurate as the collapse develops) we obtain from (1.1) and (1.2) the following equations for the time envelope $\bar{\psi}$ of the field ψ (a is the dimension of the caviton):

$$-a^{-2}+\Delta-n)\bar{\psi}=0, \tag{1.3}$$

$$(\partial^2/\partial t^2)n=\Delta|\bar{\psi}|^2. \tag{1.4}$$

$$\frac{d}{dt}\int d^3\mathbf{r}\,|\bar{\psi}|^2=0. \tag{1.5}$$

They admit of the self-similar substitution

$$\bar{\psi}(\mathbf{r},t)=a^{-3/2}E(\boldsymbol{\rho}),\quad n(\mathbf{r},t)=a^{-2}u(\boldsymbol{\rho}),$$
$$\boldsymbol{\rho}=(\mathbf{r}-\mathbf{r}_s)/a,\quad a\propto(t_s-t)^{2/3}. \tag{1.6}$$

As $t\to t_s$ the caviton dimension a tends to zero and the entire energy trapped in the caviton is localized at the point $\mathbf{r}_s$, a situation corresponding to a strong wave collapse. The functions $E(\boldsymbol{\rho})$ and $u(\boldsymbol{\rho})$ satisfy the equations

$$(-1+\Delta-u)E=0, \tag{1.7}$$

$$\left(\frac{7}{3}+\frac{2}{3}\rho\frac{\partial}{\partial\rho}\right)\left(\frac{4}{3}+\frac{2}{3}\rho\frac{\partial}{\partial\rho}\right)u=\Delta E^2. \tag{1.8}$$

The Laplace operator Δ is written here now in terms of the variables $\boldsymbol{\rho}$, while the field E is chosen to be real (this is always possible). That Eqs. (1.7) and (1.8) have solutions was proved in Ref. 3 with a computer example. It became possible later to determine the structure of the entire set of self-similar solutions (1.6) and find some of them analytically.[4] A computer investigation[5] of the evolution of the centrosymmetric solution of Eqs. (1.1) and (1.2) revealed an evolution of the self-similar solution obtained in Ref. 3. In Ref. 6, at the same time, a general criterion

$$E^2(0)>14/9, \tag{1.9}$$

is obtained for the instability of the self-similar solutions and coincides with the necessary condition for their existence, i.e., it attests to instability of all the self-similar solutions within the framework of the system (1.3)–(1.5). The analysis in Ref. 6 has shown that allowance for the small ("acoustic") term Δn [discarded in the transition from (1.2) to (1.5)] and the corresponding corrections to the self-similar solution (1.6) transform the unstable eigenmodes of the linearized equations (1.3)–(1.5) into quasimodes. The latter retain their form and no longer increase up to the instant of singularity formation, but only when the caviton is contracted by a number of times that is finite albeit large relative to the Mach parameter. This number is estimated to equal the ratio of the caviton dimension a at the instant of quasimode formation to the minimum wavelength λ of the perturbations that are modulationally unstable at the same instant of time (shorter-wavelength perturbations are stabilized by the acoustic term Δn, in the absence of which $\lambda=0$). The quasimode instability becomes manifested in those cases when the density n has perturbations of some small spatial scale λ which are so strong, even in the start of their modulational instability, that they attain an amplitude $\sim 1/\lambda^2$ before the cavity is contracted to a size[1)] $\sim\lambda$. The opposite condition was satisfied with a tremendous margin in the numerical computations,[5] where the initial level of the small-scale perturbations was determined by the computer round-off error. The results of Ref. 5 therefore do not contradict Ref. 6, but point to stability of the self-similar solution obtained in Ref. 3 for infinitesimally small centrosymmetric perturbations. The problem of the stability, in the small, of all the remaining self-similar solutions and also of the solution of Ref. 3 to multipole perturbations, has not been studied before. We undertake here to fill this gap.

2. BASIC EQUATIONS

Stipulating that the functions $\bar{\psi}$ and n be represented in the vicinity of the caviton center ($\rho=0$) by expansions of the type

$$\bar{\psi}=\sum_{l=0}^{\infty}\rho^l\sum_{m=-l}^{l}Y_{l,m}\left(\frac{\boldsymbol{\rho}}{\rho}\right)\sum_{k=0}^{\infty}\bar{\psi}_{k,l,m}(t)\rho^{2k}, \tag{2.1}$$

where $Y_{l,m}$ are spherical harmonics, we can solve the problem posed in the Introduction within the framework of the

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system (1.3)–(1.5) (see Ref. 6). Functions that have expansions of type (2.1) will henceforth be called regular at the caviton center.

Linearization of the system (1.3)–(1.5) against the background of the self-similar solution (1.6) leads to the following equations for the eigenfunctions (see Ref. 6):

$$(-1+\Delta-u)\tilde{E}=(\tilde{u}+\tilde{\Omega})E, \tag{2.2}$$

$$\alpha\int d^3\rho\, E\tilde{E}=0, \tag{2.3}$$

$$\left(\frac{7}{3}+\frac{2}{3}\rho\frac{\partial}{\partial\rho}-\alpha\right)\left(\frac{4}{3}+\frac{2}{3}\rho\frac{\partial}{\partial\rho}-\alpha\right)\tilde{u}=2\Delta(E\tilde{E}). \tag{2.4}$$

Growing perturbations correspond to eigenvalues α located in the left-hand half plane:

$$\operatorname{Re}\alpha<0. \tag{2.5}$$

All the eigenvalues are located in the region

$$\operatorname{Re}\alpha<4/3, \tag{2.6}$$

outside of which the function $\tilde{u}$ increases like $\rho^{3\alpha/2-2}$ as $\rho\to\infty$. For each self-similar solution there exist the eigenmodes:

$$\begin{gathered}\alpha=-2/3,\quad \tilde{E}=\mathbf{c}\nabla E,\quad \tilde{u}=\mathbf{c}\nabla u,\quad \tilde{\Omega}=0\quad (\mathbf{c}=\text{const});\\ \alpha=-1,\quad \tilde{E}/\tilde{\Omega}=3/4E+1/2\boldsymbol{\rho}\nabla E,\quad \tilde{u}/\tilde{\Omega}=u+1/2\boldsymbol{\rho}\nabla u;\\ \alpha=0,\quad \tilde{E}/\tilde{\Omega}=1/2\boldsymbol{\rho}\nabla E,\quad \tilde{u}/\tilde{\Omega}=u+1/2\boldsymbol{\rho}\nabla u,\end{gathered} \tag{2.7}$$

generated by the symmetries of Eqs. (1.3)–(1.5) with respect to spaciotemporal shifts and to stretching of the coordinates:

$$\mathbf{r}\to g\mathbf{r},\quad a\to ga,\quad \bar{\psi}\to\bar{\psi},\quad n\to g^{-2}n.$$

The presence in the left-hand half plane of even one eigenvalue that differs from -1 and $-2/3$ means instability of the investigated self-similar solution. When searching for such eigenvalues it is desirable to avoid a thorough "combing" of the left-hand α half plane, which is a laborious and unreliable method of instability checking. We can attempt to construct a function analytic in the left half plane and having zeros that coincide with the eigenvalues α, and in which poles are absent or easily counted. The number of unstable eigenmodes is explicitly expressed in terms of the logarithmic derivative of such a function on a contour Γ passing upwards along the imaginary α axis and closed by an infinite semicircle on the left. The function having the required properties is easiest to introduce in this case if the investigated self-similar solution has central symmetry. The perturbations can then be easily classified in accordance with their multipolarity.

For perturbations with an "orbital momentum" $l\geqslant1$ the condition (2.3) is satisfied automatically, the value of $\tilde{\Omega}$ in Eq. (2.2) is zero, and the system (2.2)–(2.4) can be written in the form

$$\begin{gathered}\left[-1+\frac{1}{\rho}\frac{d^2}{d\rho^2}\rho-\frac{l(l+1)}{\rho^2}-u\right]\tilde{E}=\tilde{u}E,\\ \left(\frac{7}{3}+\frac{2}{3}\rho\frac{d}{d\rho}-\alpha\right)\left(\frac{4}{3}+\frac{2}{3}\rho\frac{d}{d\rho}-\alpha\right)\tilde{u}\\ =2\left[\frac{1}{\rho}\frac{d^2}{d\rho^2}\rho-\frac{l(l+1)}{\rho^2}\right]E\tilde{E}.\end{gathered} \tag{2.8}$$

The solution of Eqs. (2.8), which is regular at the center of the caviton and is normalized by the condition

$$\lim_{\rho\to0}\frac{\tilde{E}}{\rho^l}=1, \tag{2.9}$$

depends on a single complex parameter α. As $\rho\to\infty$ there exists an analytic α-dependent limit

$$\lim_{\rho\to\infty}\rho e^{-\rho}\tilde{E}(\rho,\alpha)=I_l(\alpha). \tag{2.10}$$

The eigenvalues $\bar{\alpha}$ corresponding to modes with orbital momentum $l\geqslant1$ coincide with the roots of the equation

$$I_l(\alpha)=0, \tag{2.11}$$

located in the region (2.6). The poles of the function $I_l(\alpha)$ are those values of α for which Eqs. (2.8) have a nontrivial solution, regular at the caviton center, satisfying the condition

$$\lim_{\rho\to0}(\tilde{E}/\rho^l)=0. \tag{2.12}$$

Analysis of the recurrence relations between the coefficients of expansions, similar to (2.1), of the functions $\tilde{E}$ and $\tilde{u}$ shows that such a solution exists only at the points $\alpha=\alpha^{\pm}_{l+2k}$, $k=0,1,2,\ldots$:

$$\alpha_p^{\pm}=2/3p+4/3+1/2\pm[2E^2(0)+1/4]^{1/2}. \tag{2.13}$$

For the Mth term of the sequence of centrosymmetric self-similar solutions available for each method of populating the caviton, and numbered in increasing order of $E^2(0)$, the latter satisfies the inequalities

$$2/3M(4/3M+1)<E^2(0)<2/3(M+1/2)[4/3(M+1/2)+1]. \tag{2.14}$$

With the aid of (2.13) and (2.14) it is easy to find the number P_l of the poles of the function $I_l(\alpha)$, located in the left half plane:

$$P_l=\begin{cases}0, & l\geqslant2M-1\\ M-[(l+1)/2], & l<2M-1\end{cases}. \tag{2.15}$$

The number Z_l of the zeros of this function in the same location is given by

$$Z_l=P_l+J_l,\qquad J_l=\frac{1}{2\pi i}\int_\Gamma d\alpha\frac{d}{d\alpha}\ln I_l(\alpha). \tag{2.16}$$

The integral of the logarithmic derivative $I_l(\alpha)$ along the infinite semicircle of the contour Γ is zero, inasmuch as for $|\alpha|\to\infty$ ($\operatorname{Re}\alpha<0$) the function $I_l(\alpha)$ tends to a finite limit. By using the property $I_l(\alpha^*)=I_l^*(\alpha)$, we can rewrite the integral along the imaginary axis, which remains in J_l, in the form

$$J_l=\frac{1}{\pi}\int_0^\infty d\beta\frac{d}{d\beta}\arg I_l(i\beta). \qquad (2.17)$$

The number of unstable eigenmodes with fixed "orbital momentum" $l\geqslant 2$ is equal to $(2l+1)Z_l$, and the number of those with $l=1$ is $3(Z_1-1)$, so that at $l=1$ there exists an eigenvalue $\alpha=-2/3$ that has no bearing on the stability problem.

For perturbations with $l=0$ the condition (2.3) is not trivial, and $\widetilde{\Omega}$ in (2.2) differs from zero. For each complex value of α the spherically symmetric solution of (2.2) and (2.4) which is regular at the center of the caviton depends linearly on two parameters. The condition that $\widetilde{E}$ decrease as $\rho\to\infty$ determines uniquely the ratio of these parameters, and after introducing some normalization, for example

$$\left.\frac{d\widetilde{E}}{d\rho^2}\right|_{\rho=0}=\left.\frac{dE}{d\rho^2}\right|_{\rho=0}, \qquad (2.18)$$

it determines also the function $\widetilde{E}(\rho,\alpha)$. Analytically dependent on α, together with $\widetilde{E}(\rho,\alpha)$, is the integral in (2.3)

$$I(\alpha)=\int d^3\rho E(\rho)\widetilde{E}(\rho,\alpha). \qquad (2.19)$$

The nonzero eigenvalues α corresponding to spherically symmetric modes are roots, located in the (2.6) region, of the equation

$$I(\alpha)=0. \qquad (2.20)$$

It is easy to show with the aid of Eqs. (2.7), for the eigenmode corresponding to zero α, that

$$I(0)=-\frac{3}{4}\int d^3\rho\, E^2(\rho)<0. \qquad (2.21)$$

The asymptote of the function $I(\alpha)$ for $|\alpha|\to\infty$ and $\operatorname{Re}\alpha<0$ can also be calculated analytically:

$$\widetilde{E}(\rho,\alpha)=E(\rho)\,[1+\alpha^{-2}O(1)],$$
$$I(\alpha)=\int d^3\rho E^2(\rho)\,[1+\alpha^{-2}O(1)]. \qquad (2.22)$$

The numbers Z_0 of zeros and P of poles of the function $I(\alpha)$, which are located in the left-hand α half plane, are related by

$$Z_0=P+J, \qquad (2.23)$$

where

$$J=\frac{1}{\pi}\int_0^\infty d\beta\frac{d}{d\beta}\arg I(i\beta).$$

The difference between the signs of $I(0)$ and $I(i\infty)$ is evidence that J is odd. To calculate P it should be noted that for each pole value of α Eqs. (2.2) and (2.4) have a nontrivial solution with $l=0$. This solution is regular at the caviton center, decreases as $\rho\to\infty$, and satisfies the condition

$$\left.\frac{d\widetilde{E}}{d\rho^2}\right|_{\rho=0}=0. \qquad (2.24)$$

The spherically symmetric solution of Eqs. (2.2) and (2.4), which is regular at the center of the caviton and has the property (2.24), has been determined for any value of α accurate to multiplication by an arbitrary constant. Normalizing this solution by the equation

$$\widetilde{E}(0)=1, \qquad (2.25)$$

we can introduce the analytic function

$$I_0(\alpha)=\lim_{\rho\to\infty}\rho e^{-\rho}\widetilde{E}(\rho,\alpha).$$

Its zeros are located at the poles of the function $I(\alpha)$, so that the latter can be calculated from the equation

$$P=P_0+\frac{1}{\pi}\int_0^\infty d\beta\frac{d}{d\beta}\arg I_0(i\beta). \qquad (2.26)$$

At the poles of the function $I_0(\alpha)$, the number of which in the left-hand α half plane is designated here by P_0, Eqs. (2.2) and (2.4) have a nontrivial solution with $l=0$, which is regular at the caviton center and satisfies the conditions

$$\widetilde{E}(0)=0,\quad \left.\frac{d\widetilde{E}}{d\rho^2}\right|_{\rho=0}=0. \qquad (2.27)$$

This solution exists only for $\alpha=\alpha_{2k}^{\pm}$, $k=1,2,...$ [cf. (2.13)]. The number, coinciding with P_0, of the sequence α_{2k}^{-} $(k=1,2,...)$ can be easily expressed in terms of the number M of the self-similar solution

$$P_0=M-1. \qquad (2.28)$$

Combining Eqs. (2.23), (2.26), and (2.28) we get

$$Z_0=M-1+\frac{1}{\pi}\int_0^\infty d\beta\frac{d}{d\beta}\arg[I(i\beta)I_0(i\beta)]. \qquad (2.29)$$

The number of unstable eigenmodes with $l=0$ is equal to Z_0-1, since the zero of the function $I(\alpha)$ at the point $\alpha=-1$, has no bearing on the stability problem.

3. EXACT SPECTRUM IN SELF-SIMILAR REGIMES WITH POPULATED GROUND STATE

The results of the numerical investigation of the stability of the centrosymmetric self-similar solutions with populated ground state and with not very high a number M are gathered in Table I. The values of α_0^- and Z_l were calculated from the equations of the preceding section. Knowing the number of unstable modes of each type, it was simpler to find the eigenvalues corresponding to them, which are given in the last column of Table I. It was easy to verify the stability, against multipole perturbations with $l\gg 1$, of the self-similar solutions with $M\sim 1$ by using the fact that at $l\gg 1$ the "centrifugal potential" $l(l+1)/\rho^2$ crowds out the perturbation into the region $\rho\gg 1$, where the self-similar solution is already close to zero.

As seen from Table I, a centrosymmetric self-similar solution with populated ground state and minimum value of the field E at the caviton center is stable to arbitrary small perturbations. Similar solutions with $M=2$ and 3 are unstable, and the largest growth exponent in both cases is reached in the dipole eigenmodes and exceeds the maximum growth exponent $-\alpha_0^-$ of the quasimodes. To assess the degree to which the above properties are common, it is useful to analyze the limiting case $M\gg 1$.

Self-similar solutions with populated ground states and $E(0)\gg 1$ were obtained in Ref. 4. Inside the caviton, i.e., inside the region where $u\leqslant -1$, they satisfy the approximate expression

TABLE I.

M	α_0-	l	Z_l	α
1	−0.186	0	1	
		1	1	
		⩾2	0	
2	−1.634	0	2	−1.548
		1	2	−2.453
		2	1	−1.258
		3	1	−0.407
		⩾4	0	
3	−3.029	0	3	−3.112, −1.444
		1	3	−4.226, −2.143
		2	2	−2.804, −1.291
		3	2	−1.896, −0.511
		4	1	−1.11
		5	1	−0.38
		⩾6	0	

$$\Delta E^2 \approx -{}^{28}/_9. \qquad (3.1)$$

It is seen from (3.1) that the scale of variation of the field E is large compared with unity:

$$|\nabla \ln E^2|^{-1} \sim E(0) \sim M \gg 1.$$

The approximation (3.1) is not valid only at distances $\delta \sim M^{1/3}$ from the caviton boundary. In the δ-vicinity of the boundary the field E decreases, roughly speaking, by one-half, remaining of the order of $E_b \sim (M\delta)^{1/2} \sim M^{2/3}$. Outside the caviton the decreases of E is exponential. In the region $\delta \lesssim \rho - R \ll R \sim M$, where R is the distance from the center to the caviton boundary along a specified line, the scale of E exceeds unity noticeably as before:

$$-\ln \frac{E}{E_b} \sim \left(\frac{\rho - R}{\delta}\right)^{3/2}.$$

With further increase of ρ this scale, according to estimates, remains unchanged and of the order of unity:

$$-\ln(E/E_b) \sim \rho - R, \quad \rho - R \gg R.$$

The absence of zeros of the field E allows us to replace E by a singularity-free function

$$y = \tilde{E}/E.$$

It is easy to express $\tilde{u}$ in terms of this function by using Eqs. (1.7) and (2.2):

$$\tilde{u} = -\tilde{\Omega} + E^{-2}\,\nabla(E^2 \nabla y). \qquad (3.2)$$

Substitution of (3.2) in (2.4) yields a closed equation for y, in the form

$$\nabla[E^2(E^2 - \Lambda)\nabla y] + yE^2\Delta E^2 + \Lambda\tilde{\Omega}E^2$$
$$= E^2\left[\frac{2}{3}\left(\frac{11}{3} - 2\alpha\right)\rho\frac{\partial}{\partial\rho} + \frac{4}{9}\left(\rho\frac{\partial}{\partial\rho}\right)^2\right][E^{-2}\nabla(E^2\nabla y)], \qquad (3.3)$$

$$\Lambda \equiv {}^1/_2({}^7/_3 - \alpha)({}^4/_3 - \alpha). \qquad (3.4)$$

Comparing various terms in the left-hand side of (3.3), we can suggest the following estimates of the quantities Λ and $\tilde{\Omega}$ corresponding to global eigenmodes:

$$\Lambda \sim M^2, \quad \tilde{\Omega}\Lambda \sim y_{\mathrm{char}} \qquad (3.5)$$

(y_{char} is the characteristic value of the function y). The left-hand side of (3.3), roughly estimated for such modes, is M times larger than the right-hand side. In the region $|\rho - R| \ll R$ [where E is already much smaller than $E(0)$ and the scale of variation of E is still much larger than unity], the first term of the left-hand side of (3.3) can be estimated to be larger not only than the right-hand side, but also than all the remaining terms. This means in fact that the function y remains practically constant in the direction normal to the caviton boundary:

$$\nabla_n y\,|_{|\rho - R| \ll R} = o(y/\Lambda). \qquad (3.6)$$

The condition obtained makes it possible to pose a boundary-value problem in only one region of validity of Eq. (3.1). Integration of (3.3) over the interior of the caviton with allowance for (3.1) and the orthogonality condition

$$\int d^3\rho\, yE^2 = 0 \qquad (3.7)$$

yields $\Lambda\tilde{\Omega} \approx 0$. A more thorough analysis, based on integration of (3.3) over all of space, shows that

$$\Lambda\tilde{\Omega} = o(y_{\mathrm{char}}/M).$$

Neglecting small corrections, Eq. (3.3) has in the interior of the caviton the form

$$\nabla[E^2(E^2 - \Lambda)\nabla y] - {}^{28}/_9\, E^2 y = 0. \qquad (3.8)$$

The solutions of this equation are extremals of the functional

$$\Lambda[y] = \frac{\int d^3\rho\,(E^4|\nabla y|^2 + {}^{28}/_9 E^2|y|^2)}{\int d^3\rho\, E^2|\nabla y|^2}, \qquad (3.9)$$

whence it follows directly that Λ is real and positive. The functional (3.9) is bounded from above on the set of functions y satisfying the orthogonality condition (3.7). The maximum $\Lambda_{\max}$ of this functional corresponds to the minimum eigenvalue α:

$$\alpha_{min} \approx -(2\Lambda_{max})^{1/2}.$$

With the aid of trial functions it is easy to show that

$$\Lambda_{max} > \max_{\rho} E^2(\rho) = E^2(0)$$

and correspondingly $\alpha_{\min} < \alpha_0^-$, in full accord with the eigenmode instability property noted in Table I.

The above proof of the instability of solutions with $E^2(0) \gg 1$ contains no assumptions whatsoever about the caviton symmetry. For a quantitative determination of $\alpha_{\min}$ and comparison with the values indicated in Table I it is useful to consider in greater detail centrosymmetric self-similar solutions with a populated ground state and with a strong field $E(0)$. Inside the caviton such solutions are approximately described by

$$E^2(\rho) \approx E^2(0) - {}^{14}/_{27}\rho^2. \qquad (3.10)$$

Being interested in the most unstable eigenmodes, we can put

$$\Lambda = E^2(0)(1+\varkappa^2), \quad \varkappa > 0. \qquad (3.11)$$

It is convenient also to introduce a new variable ξ:

$$\rho = ({}^{27}/_{14})^{1/2} E(0)\xi,$$

which is equal to unity on the boundary of the caviton. With these improvements, Eq. (3.8) can be written for perturbations with "orbital momentum" l in the form

$$\frac{d}{d\xi}\xi^2(1-\xi^2)(\xi^2+\varkappa^2)\frac{d}{d\xi}y + [6\xi^2 - l(l+1)(\xi^2+\varkappa^2)](1-\xi^2)y = 0. \qquad (3.12)$$

It is convenient to normalize the solution, regular as $\xi \to 0$, of Eq. (3.12) by the condition

$$\lim_{\xi\to 0}\frac{y}{\xi^l} = 1. \qquad (3.13)$$

The boundary condition

$$\left.\frac{dy}{d\xi}\right|_{\xi\to 1-0} = 0 \qquad (3.14)$$

distinguishes a discrete set of real values of $\varkappa$. All correspond to zero or unity "orbital momenta," since the coefficient of y in (3.12) is negative for $l \geqslant 2$ in the interval (0,1), meaning that condition (3.14) cannot be met. The largest values of $\varkappa$ for centrosymmetric and dipole eigenmodes can be numerically determined:

$$\varkappa_{max} = 0.224 \quad (l=0),$$
$$\varkappa_{max} = 0.763 \quad (l=1). \qquad (3.15)$$

Assuming $\varkappa^2 \ll 1$, we can simplify noticeably Eq. (3.12) in two regions that cover the entire interval $0 < \xi < 1$:

$$\frac{d}{d\xi}\xi^2(\xi^2+\varkappa^2)\frac{d}{d\xi}y + [6\xi^2 - l(l+1)(\xi^2+\varkappa^2)]y = 0, \quad 0 < \xi \ll 1, \qquad (3.16)$$

$$\frac{d}{d\xi}\xi^4(1-\xi^2)\frac{d}{d\xi}y + [6 - l(l+1)]\xi^2(1-\xi^2)y = 0, \quad \varkappa \ll \xi < 1. \qquad (3.17)$$

The solution of (3.6) is expressed in terms of a hypergeometric function

$$y = \xi^l F(b_l^*, b_l, {}^3/_2 + l, -\xi^2/\varkappa^2),$$
$$b_l = {}^1/_4[2l+3+i(15-4l(l+1))^{1/2}] \qquad (3.18)$$

and has in the region $\varkappa \ll \xi \ll 1$ the asymptotic form

$$y \approx 2\xi^l \operatorname{Re}\left[\left(\frac{\varkappa^2}{\xi^2}\right)^{b_l^*}\frac{\Gamma({}^3/_2+l)\Gamma(b_l - b_l^*)}{\Gamma(b_l)^2}\right]$$
$$\approx G_l\varkappa^l\left(\frac{\varkappa}{\xi}\right)^{3/2}\sin\left(2\operatorname{Im}b_l\ln\frac{\xi}{\varkappa} + g_l\right), \qquad (3.19)$$
$$g_0 \approx 1.279, \quad g_1 \approx 3.101.$$

The solution of (3.17) has in the same region the asymptotic form

$$y \approx \frac{\text{const}}{\xi^{3/2}}\sin(2\operatorname{Im}b_l\ln\xi + \bar{g}_l),$$
$$\bar{g}_0 \approx 1{,}079, \quad \bar{g}_1 \approx 0{,}813. \qquad (3.20)$$

From the condition that (3.19) and (3.20) be equal it follows that

$$\varkappa \approx c_l \exp\left\{-\frac{2\pi j}{[15-4l(l+1)]^{1/2}}\right\}, \quad j = 1, 2, \ldots,$$
$$c_0 \approx 0.219, \quad c_1 \approx 0.523. \qquad (3.21)$$

The assumption $\varkappa^2 \ll 1$ is satisfied with fair accuracy even for $j = 1$. Retention of the small correction $\varkappa^2$ in (3.11) is not an exaggeration of the accuracy of Eq. (3) for $j \ll \ln M$. The number of eigenmodes that are more unstable than the quasimodes ($\Lambda > E^2(0)$) is thus of the order of $\ln M$. According to Table I, for $M = 3$ there are one mode each with $l = 0$ and 1. The corresponding instability exponents α differ by only 1% from those calculated from Eqs. (3.15), (3.11), and (3.4).

4. POINT SPECTRUM IN SELF-SIMILAR REGIMES WITH UNPOPULATED GROUND STATE

The results of the numerical investigation of the stability of centrosymmetric self-similar solutions with populated first ($N = 1$) or second ($N = 2$) excited states and with not too large a number M are gathered in Table II. All these solutions except the first ($N = 1$, $M = 1$) are unstable, with a pronounced tendency of the number of unstable eigenmodes and of the maximum values of the exponent $-\operatorname{Re}\alpha$ to increase with the number M. It is useful to supplement Table II by an investigation of the limiting case $N \gg 1$.

Centrosymmetric self-similar solutions with higher-index populated bound states were obtained in Ref. 4. In view of the low "binding energy" of the highly excited states with $N \gg E(0) \sim M$ compared with the depth of the caviton, the shape of the latter is practically independent of N. In this sense, the function

$$\hat{u}(\hat{\rho}) = \rho_*^2 u(\rho), \quad \hat{\rho} = \rho/\rho_*, \qquad (4.1)$$

where $\rho_*^{-2} \gg 1$ is the maximum caviton depth, is universal. The asymptote of $\hat{u}(\hat{\rho})$ for $\hat{\rho} \gg 1$ is given by

$$\hat{u}(\hat{\rho}) \approx -\frac{3}{2}\frac{E^2(0)}{\hat{\rho}^2}. \qquad (4.2)$$

For the Mth self-similar solution, the quantity $E^2(0)$ differs from the left-hand boundary of the interval (2.14) only by an increment that is exponentially small in the parameter

TABLE II.

N	M	α_0^-	l	z_l	α
1	1	$-1.64\cdot10^{-3}$	0	1	
			1	1	
			$\geqslant 2$	0	
	2	$-{}^4/_3-8.60\cdot10^{-4}$	0	4	$-1.316,\ -0.43\pm3.3i$
			1	4	$-1.14,\ -0.25\pm3.3i$
			2	3	$-1.80\pm0.26i,\ -0.077$
			$\geqslant 3$	0	
	3	$-{}^8/_3-3.38\cdot10^{-4}$	0	5	$-2.600,\ -1.326,\ -1.32\pm4.3i$
			1	5	$-2.538,\ -1.7,\ -1.20\pm3.4i$
			2	3	$-1.39\pm0.055i,\ -0.010$
			3	3	$-2.96\pm0.11i,\ -0.83$
			4	3	$-1.28,\ -0.5\pm2.31i$
			$\geqslant 5$	0	
2	1	$-2.10\cdot10^{-5}$	0	3	$-0.118\pm1.95i$
			1	3	$-1.17,\ -0.041$
			$\geqslant 2$	0	
	2	$-{}^4/_3-7.83\cdot10^{-6}$	0	4	$-1.331,\ -0.86\pm3.62i$
			1	6	$-0.77,\ -0.029\pm1.37i$
			2	3	$-30.2,\ -1.33,\ -0.015$
			3	2	$-0.078\pm1.86i$
			$\geqslant 4$	0	
	3	$-{}^8/_3-0.26\cdot10^{-6}$	0	5	$-2.647,\ -1.88\pm4.89i,\ -1.33$
			1	5	$-2.064,\ -1.72\pm4.63i,\ -1.54$
			2	6	$-1.33,\ -0.95,\ -0.52+7.29i,\ -0.22+1.5i$
			3	3	$-45.3,\ -2.02,\ -0.70$
			4	3	$-1.33\pm2.54i,\ -0.029$
			$\geqslant 5$	0	

$N/M \gg 1$, accurate to which we have

$$E^2(0)\approx {}^2/_3 M({}^4/_3 M+1).$$

The behavior of the field E in the region $\rho_* \ll \rho \ll 1$ is described approximately by the following two equations:

$$E\approx A\left(\frac{\rho_*}{\rho}\right)^{1/2}\sin\left\{\left[\frac{3}{2}E^2(0)-\frac{1}{4}\right]^{1/2}\ln\frac{\rho}{\rho_*}+\varphi_*\right\}$$
$$\approx\frac{B}{\rho^{1/2}}\sin\left\{\left[\frac{3}{2}E^2(0)-\frac{1}{4}\right]^{1/2}\ln\rho+\varphi\right\},\quad 0<\varphi,\ \varphi_*<\pi. \tag{4.3}$$

The first equation (with fixed values of A and φ_*) pertains to the solution of (1.7) and (1.8) which is regular at the caviton center, while the second (with fixed phases φ) pertains to the solution that decreases as $\rho\to\infty$. The condition that the two equations in (4.3) be equal defines a discrete set of ρ_*:

$$\rho_*^{-2}\equiv\max_\rho[-u(\rho)]=C\exp\left\{\frac{2\pi N}{({}^3/_2E^2(0)-{}^1/_4)^{1/2}}\right\}. \tag{4.4}$$

Here $C\sim1$ is a number that depends on M. According to (4.1)–(4.4), the depth of the caviton in the central region ($\rho\sim\rho_*$) decreases exponentially, while the field E in the energy-containing region ($\rho\sim1$) decreases exponentially with increase of the parameter $N/M\gg1$.

Simple regularities can also be tracked in the disposition of the eigenvalues α for $N/M\gg1$. In particular, there are always eigenvalues that agree with the negative terms of the sequence α_p, apart from corrections exponentially small in the parameter N/M [see (2.13)]. (This can be verified by using the recurrence relations between the coefficients of the expansions of $\tilde{E}$ and $\tilde{u}$ in powers of ρ^2 near the caviton center.) The proximity of $E^2(0)$ to the left-hand boundary of the interval (2.14) allows us to rewrite (2.13) in the form

$$\alpha_p^-\approx{}^2/_3[p+2(1-M)]. \tag{4.5}$$

For $M\geqslant2$ the quantity $\alpha_0^-\leqslant-4/3$ is negative and differs from $-2/3$ and from -1. The corresponding self-similar solution is therefore unstable. The case $M=1$ calls for a more detailed examination. We can, for this purpose, simplify Eq. (2.2) for the unstable eigenmode $\tilde{E}$ in the regions $\rho\gg\rho_*$ and $\rho\ll1$:

$$(-1+\Delta+{}^7/_3\rho^2)\tilde{E}\approx\tilde{\Omega}E,\quad \rho\gg\rho_*, \tag{4.6}$$

$$(\Delta-u)\tilde{E}\approx\tilde{u}E,\quad \rho\ll1. \tag{4.7}$$

We have used in Eq. (4.6) the asymptotic form (4.2) with $E^2(0)=14/9$, and the term $\tilde{u}E$ was left out because the function $\tilde{u}$ decreases at Re $\alpha<0$ more rapidly than $u(\rho)$, whereas $\tilde{E}$ and E decrease at equal rates. Equation (4.7) was obtained by taking into account the inequalities $|u|\gg1$ and $|u\tilde{E}|\gg|\tilde{\Omega}E|$, which hold in the region $\rho\ll1$. The solution $\tilde{E}$ of Eq. (4.7) and of the second equation of (2.8), which is regular at the center of the caviton and is normalized by condition (2.9) (if $l\geqslant1$) or (2.18) (if $l=0$), depends analytically on the complex parameter α. For $l=0$ this solution has in the region $\rho_*\ll\rho\ll1$ the asymptotic form

$$\tilde{E}\approx\tilde{A}(\alpha)\left(\frac{\rho_*}{\rho}\right)^{1/2}\sin\left[\frac{5}{2\sqrt{3}}\ln\frac{\rho}{\rho_*}+\tilde{\varphi}_*(\alpha)\right],$$
$$-\pi<\mathrm{Re}\,\tilde{\varphi}_*(\alpha)\leqslant\pi. \tag{4.8}$$

The centrosymmetric solution $\tilde{E}/\tilde{\Omega}$ of Eq. (4.6), which decreases as $\rho\to\infty$, was determined accurate to addition of the function E multiplied by an arbitrary constant. This constant is obtained from the orthogonality condition $\int d^3\rho E\tilde{E}=0$, the main contribution to which is made by the region $\rho\sim 1$. The solution determined in this manner has in the region $\rho_*\ll\rho\ll 1$ the asymptotic form

$$\tilde{E}=\tilde{\Omega}\tilde{B}\rho^{-1/2}\sin\left(\frac{5}{2\sqrt{3}}\ln\rho+\tilde{\varphi}\right) \tag{4.9}$$

with known amplitude $\tilde{B}$ and known phase $\tilde{\varphi}$. The eigenvalues α are determined from the conditions that Eqs. (4.8) and (4.9) be equal, are zeros of the analytic function

$$\chi(\alpha)=\tilde{A}(\alpha)\sin[\tilde{\varphi}_*(\alpha)-\tilde{\varphi}-(\varphi_*-\varphi)]. \tag{4.10}$$

This function has no poles in the left–hand α plane, so that we can calculate the number of zeros of $\chi(\alpha)$ located there by using the equation

$$Z_0=\frac{1}{\pi}\int_0^\infty d\beta\frac{d}{d\beta}\arg\chi(i\beta). \tag{4.11}$$

Numerical calculation yields $Z_0=3$. In addition to -1, the eigenfunction are

$$\alpha\approx-0.235\pm i1.973. \tag{4.12}$$

Centrosymmetric self-similar solutions with $N\gg 1$ are thus unstable also if $M=1$. It is useful to note that dipole eigenmodes can have growth exponents larger than 0.235. At $l=1$ the solution of (4.7) which is regular at the caviton center, and the solution of (4.6) which decreases as $\rho\to\infty$, are given by

$$E\approx A_1(\alpha)\left(\frac{\rho_*}{\rho}\right)^{1/2}\sin\left[\frac{1}{2\sqrt{3}}\ln\frac{\rho}{\rho_*}+\varphi_{1*}(\alpha)\right],$$

$$-\pi<\operatorname{Re}\varphi_{1*}(\alpha)\leqslant\pi, \tag{4.13}$$

$$\tilde{E}\approx B_1\rho^{-1/2}\sin\left(\frac{1}{2\sqrt{3}}\ln\rho+\varphi_1\right),\quad 0<\varphi_1<\pi,$$

which coincide with the zeros of the analytic function

$$\chi_1(\alpha)=A_1(\alpha)\sin\left[\varphi_{1*}(\alpha)-\frac{1}{2\sqrt{3}}\ln\rho_*-\varphi_1\right].$$

With the aid of the relation

$$\varphi-\varphi_*+\frac{5}{2\sqrt{3}}\ln\rho_*=\pi N$$

[see (4.3) and (4.4)] we can explicitly distinguish the dependence of the function $\chi_1(\alpha)$ on the number N:

$$\chi_1(\alpha)=A_1(\alpha)\sin[\varphi_{1*}(\alpha)-\varphi_1-{}^1/_5(\varphi_*-\varphi+\pi N)]. \tag{4.14}$$

All five types of the asymptotic ($N\gg 1$) dipole spectrum in the region $\operatorname{Re}\alpha<0$ (except for the eigenvalue $\alpha=-2/3$ which is common to all cases) are listed in Table III.

5. CONCLUSIONS AND GENERALIZATIONS

In the cases considered above we have considered only one of the infinite sets of bound states present in a self-similar caviton, i.e., "single-mode" regimes of a scalar collapse. The family of single-mode centrosymmetric self-similar regimes was parametrized with the aid of two integers: the number $N=0, 1, 2,\ldots$ of the populated bound state, and the number $M=1, 2, 3,\ldots$ of the term of the sequence of self-similar solutions existing for a given N and arranged in increasing order of $E^2(0)$. In this family only two solutions, ($N=0$, $M=1$) and ($N=1$, $M=1$), were stable against arbitrary small perturbations.

The deduction that self-similar solutions with $M\geqslant 2$, i.e., with

$$E^2(0)>{}^{44}/_9 \tag{5.1}$$

are unstable extends also to multimode centrosymmetric self-similar regimes of a scalar collapse, and in the case $E^2(0)\gg 1$ or $N\gg 1$ also to asymmetric self-similar scalar-collapse regimes. To all appearances, the instability criterion (5.1) is perfectly general, although it has so far not been fully proved. It is remarkable that the maximum instability exponent of all the investigated self-similar solutions that satisfy criterion (5.1) exceeds the maximum quasimode growth exponent:

$$\max(-\operatorname{Re}\alpha)>-\alpha_0^-=[2E^2(0)+{}^1/_4]^{1/2}-{}^1/_2-{}^4/_3. \tag{5.2}$$

A more variegated picture is revealed by examination of self-similar solutions with

$${}^{14}/_9<E^2(0)<{}^{44}/_9, \tag{5.3}$$

which include both stable and unstable ones. It is not clear at present whether the instability of the centrosymmetric solutions with $N\geqslant 2$ and $M=1$ leads to asymmetric self-similar solutions with the same N or to decay of the collapsing caviton into several new cavitons with fewer populated states. It is useful to note in this connection that the instability of the self-similar solutions that satisfy condition (5.3), with large populated-state numbers, become stabilized even if the ground state is insignificantly populated. Indeed, the shape of the caviton is determined completely by the values of E^2 in

TABLE III.

N (mod 5)	Z_1	α
0	2	−0.085
1	4	−0.286, −0.215±1.2i
2	5	−1.176, −0.10±1.03, −0.056
3	3	−2.10, −0.065
4	3	−5.33, −0.072

that region where this number is estimated to have its maximum. The size of the indicated region (of the same order as the size of the localization region of the ground state) is exponentially small compared with the size of the localization region of the Nth excited states. Therefore, even starting with exponentially small (relative to the parameter N) energy ratios of the waves trapped in the ground and nth excited states, the maximum energy density E^2, meaning also the collapse dynamics, is determined entirely by the waves that are in the ground state.

We conclude by repeating the main results of the present article. They comprise a proof of the existence of self-similar collapse regimes that are stable against infinitesimally small perturbations, and a determination of the necessary stability condition (5.3) that restricts radically the class of solutions that need be investigated. Particularly important is the conclusion that the self-similar solution obtained in Ref. 3 ($N = 0, M = 1$), which, to all appearances, determines the dynamics of the scalar collapse under a wide range of initial conditions. This conclusion seems to point to stability of an analogous self-similar vector-collapse regime, which has not been found so far, but which has been proposed in a number of existing and stable models of strong Langmuir turbulence.

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