Infrared divergences and the renormalization group in the theory of fully developed turbulence

L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev

V. A. Steklov Mathematics Institute, USSR Academy of Sciences, Leningrad Branch

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The three dimensional stochastic Navier-Stokes equation with a random force correlator $\sim k(k^2+m^2)^{-\varepsilon}$ is studied by the renormalization group method which is applicable beyond the ε-expansion. Real IR pumping corresponds to the region $\varepsilon \geqslant 2$. It is shown that the formulas $\Delta_\varphi = 1 - 2\varepsilon/3$, $\Delta_{\varphi'} = 2 + 2\varepsilon/3$, and $\Delta_t = -\Delta_\omega = -2 + 2\varepsilon/3$ for the critical scaling indices are only valid in the UV pumping range $0 < \varepsilon < 2$; for $\varepsilon \geqslant 2$ the indices do not depend on ε and are identical to the Kolmogorov indices. The second Kolmogorov hypothesis (independence of the viscosity coefficient) is substantiated in the real region $\varepsilon \geqslant 2$ by the renormalization group method. It is possible to prove the first Kolmogorov hypothesis (independence of m) for a simultaneous velocity correlator only for the region $0 < \varepsilon < 2$. In this region, an ε-expansion of the form $\varepsilon^{1/3}p(\varepsilon)$ is obtained for the analog of the Kolmogorov constant C_1. Here, $p(\varepsilon)$ is a series in ε; the first term is $C_1(\varepsilon) = (40\varepsilon/3)^{1/3}$ which results in $C_1 \cong 3$ for real $\varepsilon = 2$. For comparison, the experimental value is $C_1 = 1.4$–2.7.

1. INTRODUCTION

The basic problem in the theory of fully developed (homogeneous, isotropic) turbulence of an incompressible liquid is that of justifying Kolmogorov's postulates from first principles, i.e., in the framework of a microscopic model. The latter is usually taken to be stochastic hydrodynamics[1] described by the Navier-Stokes equation

$$\nabla_t \varphi_i = \nu_0 \Delta \varphi_i - \partial_i p + F_i, \quad \nabla_t \equiv \partial_t + (\varphi\partial), \tag{1}$$

where φ_i is the transverse (due to incompressibility) velocity field, and p and F are the pressure and transverse random external force per unit mass, respectively [all these quantities are functions of $x \equiv (t, \mathbf{x})$]; ν_0 is the viscosity. We take for F a gaussian distribution with $\langle F \rangle = 0$ and correlator

$$\langle F_s(x) F_l(x') \rangle = (2\pi)^{-4} \iint d\omega\, d\mathbf{k}\, P_{sl} d(k) \exp i[\mathbf{k}(\mathbf{x}-\mathbf{x}') - \omega(t-t')], \tag{2}$$

where $P_{sl} = \delta_{sl} - k_s k_l / k^2$ is the perpendicular projector and $d(k)$ is a given function of $k \equiv |\mathbf{k}|$ and parameters. The random force modulates the stochasticity and simultaneously acts as an energy pump balancing the dissipation in the system. The pumping power E is connected with the function $d(k)$ in Eq. (2) by the familiar relation

$$E = (2\pi)^{-3} \int d\mathbf{k}\, d(k). \tag{3}$$

The pumping in a realistic model should be in the infrared, i.e., the basic contribution in Eq. (3) should come from the region of $k \lesssim m$, where m is the inverse characteristic size of the energy-pumping large-scale eddies. The function $d(k)$ in Eq. (2) should contain m and at least one parameter for guaranteeing the arbitrariness of E for fixed m. The inverse dissipative length $\Lambda \equiv E^{1/4} \nu_0^{-3/4}$ plays the role of a maximum momentum in the problem, i.e., an ultraviolet cutoff, where the Reynolds number $\mathrm{Re} = (\Lambda/m)^{4/3}$ for fully developed turbulence is of order 10^4–10^6. The inertial range, characterized by the Kolmogorov scaling, is determined by the inequality $m \ll k \ll \Lambda$ for momenta and $E^{1/3} m^{2/3} \ll \omega \ll \nu_0 \Lambda^2$ for frequencies.

It is convenient to describe the basic quantities in the model (1) in terms of the pair of fields $\Phi \equiv \varphi, \varphi'$, where $\varphi' \equiv \varphi_i'(x)$ is a transverse (as is φ) auxiliary field. The derived quantities are statistical averages (Green's functions) $\langle \Phi \ldots \Phi \rangle$, the most important of which are the velocity correlator $\langle \varphi\varphi \rangle$ and the response function $\langle \phi\phi' \rangle$. The Green's functions of the φ-field can be considered either as dynamic (different-time) or static; the static Green's functions are independent of time for all fields.

The basic propositions of the phenomenological Kolmogorov-Obokov theory[1,2] reduce to the following two hypotheses: 1) in the region $k \gg m$, $\omega \gg \nu_0 m^2$, the Green's functions depend only on the total pumping power E and do not depend on its detailed structure, in particular on m; 2) in the region $k \ll \Lambda$, $\omega \ll \nu_0 \Lambda^2$, the Green's functions do not depend on the viscosity ν_0. Therefore, in the inertial range where both conditions are fulfilled, the only dependence is on the parameter E. Hypothesis 1 has various interpretations: in the standard treatments (Ref. 1, p. 391; Ref. 2, p. 189), it is formulated without any limitations (we call this variant 1A), while in some recent work, e.g., Ref. 3, it is assumed that m is absent in static objects, but may occur in dynamic ones (variant 1B).

It follows from the Kolmogorov hypotheses and dimensional considerations that in the inertial range

$$\langle \varphi(x_1) \varphi(x_2) \rangle = (rE)^{2/3} f_{\varphi\varphi}, \quad \langle \varphi(x_1) \varphi'(x_2) \rangle = r^{-3} f_{\varphi\varphi'},$$

where $r = |\mathbf{x}_1 - \mathbf{x}_2|$, $t = t_1 - t_2$, and f is an unknown scaling function of the dimensionless arguments $Et^3 r^{-2}$ and mr. Hypothesis 1A allows a dependence only on the first argument, but hypothesis 1B allows a dependence on both arguments that necessarily vanishes at $t = 0$. In any case, the equations given express scaling invariance with respect to consistent dilation of the fields, coordinates, times, and m for fixed E and ν_0 with definite Kolmogorov indices

$$\Delta_\varphi = -1/3, \quad \Delta_{\varphi'} = 10/3, \quad \Delta_t = -\Delta_\omega = -2/3, \quad \Delta_m = 1, \tag{4}$$

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where the momentum scaling was taken as unity, $\Delta_k = -\Delta_r = 1$.

Many studies (see, e.g., Refs. 3–9) have been devoted to the problem of justifying the Kolmogorov hypotheses. The conventional approach is to consider schematic self-consistency equations with dressed lines and to pose the problem of proving the existence of Kolmogorov-type solutions to these equations. The basic problem in this approach is the existence of infrared (IR) divergences in the schematic graphs with Kolmogorov lines. In spite of some success, a generally accepted solution does not exist at this time. Another comparatively new approach, the use of quantum-field techniques of the renormalization group (RG) has been used successfully in solving the critical scaling problem in the theory of critical phenomena.[10–12] The RG method explains the origin of the scaling and provides a method of calculating the critical indices analogous to Eq. (4) in the form of a series in the parameter $\varepsilon \equiv 4-d$, the deviation of the dimensionality of space $\mathbf{x}$ from four, whose actual value is $\varepsilon = 1$. This technique was generalized to the theory of turbulence in Ref. 14, where model 1 with the specific function

$$d(k) = g_0 v_0^3 k (k^2+m^2)^{-\varepsilon} \qquad (5)$$

in the correlator (2) was considered. The constant g_0 in Eq. (5) guarantees the arbitrariness of E, and m is the infrared mass (the indices ≪0≫ indicate parameters which will be renormalized; m will not). The parameter ε in the exponent of Eq. (5) is the analog of $4-d$ in the Wilson scheme. Real IR pumping corresponds to the region $\varepsilon \geqslant 2$; for $0<\varepsilon<2$ pumping in Eq. (5) is in the ultraviolet (UV) and a cutoff at Λ in Eq. (3) is understood. The RG method in Ref. 14 showed the existence of a critical scaling with indices

$$\Delta_\varphi = 1-2\varepsilon/3,\ \Delta_{\varphi'} = 2+2\varepsilon/3,\ \Delta_t = -\Delta_\omega = -2+2\varepsilon/3,\ \Delta_m = 1 \qquad (6)$$

(without corrections of order ε^2, ε^3, etc.), coinciding with Eq. (4) for $\varepsilon = 2$. Results of Ref. 14 were subsequently augmented and generalized (the dimensionality of the component operators was obtained[15,16], and magnetohydrodynamics[17] and turbulent mixing of passive impurities[18] were studied).

In all these treatments only the ε-expansion of critical dimensionalities was considered, in actuality always a massless model with $m = 0$ in Eq. (5), because the dimensionality does not depend on m and the coefficients of the ε-expansion diagrams are finite for $m \to 0$ in agreement with hypothesis 1A. However, this proves nothing for finite ε because in the diagrams a UV-divergence appears for $m \to 0$ (the function $m^{1-\varepsilon}$ is a simple example; its coefficients in the ε-expansion are finite for $m \to 0$, but it itself diverges for $\varepsilon > 1$). Questions concerning hypothesis 1 were not discussed in Refs. 14–18 and they generally are unrelated to the validity of RG because there the asymptotic case $m \to 0$ for finite ε was considered; this is a case for which the scaling functions are undetermined by the RG method.

The second previously undiscussed question is the connection to the real parameter E and the problem of how the critical indices (6) in all regions $\varepsilon \geqslant 2$ are frozen at their Kolmogorov values for $\varepsilon = 2$. In the massless model $\varepsilon > 2$ is not allowed due to the IR divergence (3) and thus $\varepsilon = 2$ is taken as the actual value. However, for $m \neq 0$ any $\varepsilon \geqslant 2$ corresponds to some model of IR-pumping and the indices should not depend on ε.

In this paper we consider the question taking into account m in Eq. (5) outside of the framework of the ε-expansion. The basic results are the following: 1) The RG method is used to justify hypothesis 2 (independence of ν_0) for $\varepsilon \geqslant 2$; 2) The same method is used to justify the critical scaling for fixed E and ν_0 with indices (6) for $0<\varepsilon<2$ and indices (4) for $\varepsilon \geqslant 2$ [The freezing of the indices (6) at their values for $\varepsilon = 2$ is due to the dependence of g_0 in Eq. (5) on m for fixed E]; 3) The development analogous to Ref. 3 of infrared perturbation theory leads to a proof of hypothesis 1B in the interval $0<\varepsilon<2$; for the Kolmogorov constant in the spectrum of the static correlator[1] in the interval $0<\varepsilon<2$, we obtained an ε-expansion of the form $\varepsilon^{1/3}p(\varepsilon)$, where $p(\varepsilon)$ is a series in ε whose first term $(40\varepsilon/3)^{1/3}$ is $(80/3)^{1/3} \cong 3$ for $\varepsilon = 2$ (the experimental values[1,19] are 1.4–2.7). The basic unsolved problem (unrelated to the validity of the RG method) is justification for $\varepsilon > 2$ of hypothesis 1B (1A is necessarily false).

For greater clarity and to establish analogies with the theory of critical phenomena, we consider simultaneously model 1 and the standard φ^4 model of critical statics, elucidating in detail the ideas and techniques of RG.

2. INFRARED SINGULARITIES OF PERTURBATION THEORY DIAGRAMS

We consider the problem in the quantum-field formulation, having in view a model of a classical random field Φ whose correlation functions (Green's functions) are determined by functional averages with weighting $\exp S(\Phi)$, where $S(\Phi)$ is a given action functional. It is known that any stochastic dynamics of the form (1) is completely equivalent to some quantum-field model (for a short proof, see Ref. 15). In particular, our problem is the equivalent of the theory of two transverse vector fields $\Phi = \varphi, \varphi'$ with action

$$S(\Phi) = \varphi' D_F \varphi'/2 + \varphi'[-\partial_t \varphi + \nu_0 \Delta\varphi - (\varphi\partial)\varphi], \qquad (7)$$

where D_F is the correlator (2) with the function (5). Necessary integrations over the arguments x and summations over indices are understood. The Green's functions of any field model have standard Feynman diagrams. For Eq. (7) they coincide with the diagram technique of Wyld.[1,4] The lines in the diagrams are associated with the bare propagators $\langle\Phi\Phi\rangle_0$; for the model (7) in the ω, p-representation we have $\langle\varphi', \varphi'\rangle_0 = 0$ and

$$\langle\varphi\varphi'\rangle_0 = \langle\varphi'\varphi\rangle_0^* = (-i\omega + \nu_0 p^2)^{-1},\quad \langle\varphi\varphi\rangle_0 = \langle\varphi\varphi'\rangle_0 d(p) \langle\varphi'\varphi\rangle_0, \qquad (8)$$

where $d(p)$ is given by Eq. (5). From the vector indices, which are omitted in Eq. (8), all lines are multiples of the transverse projector. The interaction (7) corresponds to the three-line vertex

$$-\varphi'(\varphi\partial)\varphi = \varphi_s' v_{skl} \varphi_k \varphi_l / 2$$

with vertex multiplier

$$v_{skl} = -(\partial_k \delta_{sl} + \partial_l \delta_{sk}),$$

where, in the momentum representation $\partial \to -ip$, for momentum p flowing into the vertex through the field φ'.

To each object F in Eq. (7) one can assign definite momentum d_F^p, frequency d_F^ω, and total $d_F = d_F^p + 2d_F^\omega$ canonical dimensionalities[15], which are determined from the

TABLE I.

F	$\varphi(x)$	$\varphi'(x)$	Λ, m, M	ν, ν_0	E	g_0	g
d_F^p	-1	4	1	-2	-2	2ε	0
d_F^ω	1	-1	0	1	3	0	0
d_F	1	2	1	0	4	2ε	0

requirement of nondimensionality (momentum and frequency separately) of each term of Eq. (7). The dimensionalities are given in Table I along with the renormalized parameters g, ν, and M, which will appear below.

We use for comparison the simplest static (time-independent) standard model with one scalar field $\varphi = \varphi(\mathbf{x})$ and action

$$S(\varphi)=-\varphi(-\Delta+m_0^2)\varphi/2-g_0\varphi^4/24, \tag{9}$$

where integration over $\mathbf{x}$ is understood, g_0 is the coupling constant (charge), and $m_0^2 = T - T_c$ is the deviation of the temperature T from the critical value. The model (9) is considered in the d-dimensional space $\mathbf{x}$ with UV-cutoff $\Lambda = r_{\min}^{-1}$, where $r_{\min}$ is the interatomic distance. Each value of F in Eq. (9) corresponds to one (momentum) canonical dimensionality d_F (see Table II, which includes the renormalized parameters). We note these differences: the dimensionalities of values in Eq. (9) are completely determined by the dimensionality of the space d while in Eq. (7) they do not depend on d, except for φ', and for g_0 are completely determined by the parameter ε in Eq. (5).

In the theory of critical phenomena one seeks the asymptotic form of Green's functions in the region $p, m_0 \ll \Lambda$ (temperatures close to T_c, distances large in comparison with $r_{\min}$) for which one considers $g_0 = \text{const}\,\Lambda^{4-d}$ with const $\lesssim 1$. In order to be specific we consider the correlator $D = \langle\varphi\varphi\rangle$ in the momentum representation. It is found from the Dyson equation (p is the external momentum)

$$D^{-1}=p^2+m_0^2-\Sigma(p), \tag{10}$$

where $\Sigma(p)$ is an infinite sum of all 1-irreducible diagrams (see Fig. 1) whose vertices correspond to the multiplier g_0 and whose lines correspond to the bare propagator $(k^2 + m_0^2)^{-1}$. We clarify the essence of the IR problem for dimensionality $d = 4-2\varepsilon$ (the actual value is $2\varepsilon = 1$). For $0 < 2\varepsilon < 1$ there are, in the first graphs of (10), independent of p and m_0, algebraic UV-diverging terms $g_0^n\Lambda^{2-2n\varepsilon}$ (where n is the order of perturbation theory) which correspond to the simple shift T_c in $m_0^2 = T - T_c$. If we consider the value of T_c in m_0^2 to be known exactly, it is necessary to discard all such terms. This is implemented by subtracting their values for $p = m_0 = 0$ from all graphs of the form (10). After these subtractions the integrals for $0 < 2\varepsilon < 1$ become UV-convergent, the cutoff Λ can be eliminated (taken as ∞), and the series (10) takes the form

$$D^{-1}=(p^2+m_0^2)\Big[1+\sum_{n=1}^{\infty}(g_0p^{-2\varepsilon})^n c_n(m_0/p,\varepsilon)\Big]. \tag{11}$$

For $p \sim m_0 \ll \Lambda$ and $\varepsilon > 0$ the dimensionless parameter of the expansion $g_0 p^{-2\varepsilon} \sim (\Lambda p^{-1})^{2\varepsilon}$ in Eq. (11) is not small and it is necessary to sum the series. This is the first IR-problem which is solved by the RG method. The second IR-problem, occurring in the region $m_0 \ll p$, is connected with singularities of the coefficient c_n in Eq. (11) for $m_0/p \to 0$ and cannot be handled by RG.

In a clearer formulation the first problem reduces to a determination of the asymptotic value of $D_\lambda = D(\lambda p, \lambda m_0)$ for $\lambda \to 0$ (everything is fixed except for λ). This procedure is nontrivial for $\varepsilon > 0$ due to the presence in the c_n of poles in ε connected with the appearance of new UV-divergences in the graphs (10) for $\varepsilon = 0$. We call the task of removing poles in ε the UV-problem. It is solved by the UV-renormalization procedure whose arbitrariness leads to the equations of RG (Sec. 3). The connection of the IR- and UV-problems noted above (in the absence of the latter, the format is also absent for $\varepsilon \to 0$) shows how the UV-renormalization and the techniques of RG have a definite relation to the IR-problem for small ε.

All these things can be generalized immediately to the model (7); the role of the charge is played by the parameter g_0 in Eq. (5) and the parameter ε in Eq. (6), now totally unrelated to the dimensionality of space, plays the role of 4-d in Eq. (9). For the exact correlator $D = \langle\varphi\varphi\rangle$ in the ω, p-representation, an analogous of the series in the brackets of Eq. (11) is

$$1+\sum_{n=1}^{\infty} g_0^n p^{-2n\varepsilon} c_n(\omega/\nu_0 p^2, m/p, \varepsilon), \tag{12}$$

and the first IR-problem reduces to determining the asymptotic value $D_\lambda = D(\lambda p, \lambda m, \lambda^2\omega)$ for $\lambda \to 0$. We emphasize that this IR-problem is nontrivial also in the region of UV-pumping $0 < \varepsilon < 2$. We also note that in this region, on the one hand, $E = \Lambda^4 \nu_0^3$ from the definition of Λ, but, on the other hand, $E \sim g_0 \nu_0^3 \Lambda^{4-2\varepsilon}$ from calculating Eq. (3) (more exact equations are given in Sec. 5); thus, $g_0 \sim \Lambda^{2\varepsilon}$ as in Eq. (9). The first IR-problem is solved by the RG method for any fixed (in the limit $\lambda \to 0$) ratio m/p; thus the second IR-problem (for which asymptotically $m/p \to 0$) can be dis-

TABLE II.

F	$\varphi(x)$	m_0, m, Λ, M	g_0	g
d_F	$d/2-1$	1	$4-d$	0

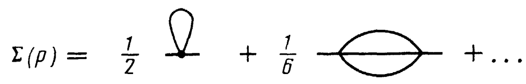

FIG. 1. The functions $\Sigma(p)$ in the φ^4 model.

cussed in the framework of a general solution of the first problem.

3. RENORMALIZATION AND THE RG EQUATIONS

The UV-divergences (in this case poles in ε) of the models considered are removed by the multiplicative renormalization procedure. It amounts to the following: the initial; action $S(\Phi)$ is referred to as unrenormalized, its parameters e_0 (the letter e designates the whole set of parameters) are referred to as bare; these are considered as some (remaining to be determined) functions of new renormalized parameters e, with the new renormalized action considered as a functional $S_{\rm ren}(\Phi) = S(Z_\Phi \Phi)$ with some (remaining to be determined) constants of the renormalized field Z_Φ (one for each independent component of Φ). In the unrenormalized Green's functions $W_n = \langle \Phi ... \Phi \rangle$, the averaging is carried out with weight exp $S(\phi)$ and in the renormalized ones $W_n^{\rm ren}$, averaging is carried out with weighting exp $S_{\rm ren}(\Phi)$; it follows from the connection between S and $S_{\rm ren}$ that $W_n^{\rm ren} = Z_\Phi^{-n} W_n$, where $W_n = W_n(e_0, \varepsilon, ...)$ where the dots indicate other arguments of coordinates for momenta, and $W_n^{\rm ren}$ and Z_Φ are expressed correspondingly through the variables e. The correspondence $e_0 \leftrightarrow e$ is assumed to be one-to-one; thus, the independent variables can be chosen to be either of the two sets e_0 or e. It is considered in the renormalization equations that

$$W_n^{\rm ren}(e, \varepsilon, \dots) = Z_\Phi^{-n}(e, \varepsilon) W_n(e_0(e, \varepsilon), \varepsilon, \dots).$$

The functions $e_0(e, \varepsilon)$ and $Z_\Phi(e, \varepsilon)$ can be chosen arbitrarily corresponding to the arbitrary choice of normalization of the field and parameters e for given e_0. The basic assertion of renormalization theory is that these functions can be chosen such that they guarantee the UV-finiteness (in this case finiteness in the limit $\varepsilon \to 0$) of the functions $W_n^{\rm ren}(e, \varepsilon, ...)$ for fixed e. For such a choice all UV-divergences (poles in ε) appear in the functions $e_0(e, \varepsilon)$ and $Z_\Phi(e, \varepsilon)$ and are absent in $W_n^{\rm ren}(e, \varepsilon, ...)$.

The RG equations are written for the functions $W_n^{\rm ren}$, which differ from the initial W_n only by normalization and, thus, can be used equally validly for a critical scaling analysis. We present a brief derivation of the RG equations. The requirement of eliminating singularities does not determine the functions $e_0(e, \varepsilon)$ uniquely because an arbitrariness remains allowing introduction in the functions (and through them in $W_n^{\rm ren}$) of an additional dimensional parameter, the renormalized mass M:

$$W_n^{\rm ren}(e, M, \varepsilon, \dots) = Z_\Phi^{-n}(e, \varepsilon) W_n(e_0(e, M, \varepsilon), \varepsilon, \dots).$$

Variation of M for fixed e_0 leads to variations of e, Z_Φ, and $W_n^{\rm ren}$ for fixed $W_n(e_0, \varepsilon, ...)$. We denote by $\tilde{\mathscr{D}}_M$ the differential operator $M\partial_M$ for fixed e_0. Applying it on both sides of the relation $Z_\Phi^n W_n^{\rm ren} = W_n$ leads to the basic RG equation

$$[\mathscr{D}_{\rm RG} + n\gamma_\Phi] W_n^{\rm ren}(e, M, \varepsilon, \dots) = 0, \quad \gamma_\Phi \equiv \tilde{\mathscr{D}}_M \ln Z_\Phi, \tag{13}$$

where $\mathscr{D}_{\rm RG}$ is the operator $\tilde{\mathscr{D}}_M$, expressed in terms of e and M.

We turn now to our specific models. General rules exist (for an analysis of the dimensionalities of 1-irreducible Green's functions. see Ref. 20, Chap. 5) which allow one to determine which renormalization constants are necessary for eliminating the IR-divergences. Three of them are necessary for model (9): the field renormalization constant Z_φ and two parameters $m_0 = mZ_m$, and $g_0 = gM^{2\varepsilon} Z_g$, where M is the renormalized mass, and the renormalized charge g and all constants Z are dimensionless. Their actual form depends on the choice of subtraction scheme. In the theory of critical phenomena and in Refs. 15–18 on turbulence, the most convenient one in practice is the scheme of minimal subtraction,[21] in which all Z have the form

$$Z = Z(g, \varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk} \varepsilon^{-k}, \tag{14}$$

where a_{nk} are numerical coefficients independent of any parameters. It follows from the definition of $\mathscr{D}_{\rm RG}$ and the renormalization parameters according to the above equations that in this model

$$\mathscr{D}_{\rm RG} = \tilde{\mathscr{D}}_M + \beta\partial_g - \gamma_m \mathscr{D}_m.$$

Here and in the following, we use $\mathscr{D}_x \equiv x\partial_x$ for any parameter of the renormalization theory; for any Z_i

$$\gamma_i \equiv \tilde{\mathscr{D}}_M \ln Z_i, \quad \beta \equiv \tilde{\mathscr{D}}_M g = g(-2\varepsilon - \gamma_g). \tag{15}$$

These identities determine the β-function and the anomalous dimensionalities γ_i. We call these in general RG-functions. One calculates the renormalization constants Z_i from the diagrams of perturbation theory and the RG-functions follow from Eq. (15). In more detail, for any Z_i of the form (14) we have from Eq. (15)

$$\gamma_i = \tilde{\mathscr{D}}_M \ln Z_i = \beta\partial_g \ln Z_i,$$

where for $i = g$ it follows from Eq. (15) that $\beta = g(-2\varepsilon - \beta\partial_g \ln Z_g)$; thus, the β-function is expressed algebraically through $\partial_g \ln Z_g$, and the remaining γ_i for $i \neq g$ are calculated using $\gamma_i = \beta\partial_g \ln Z_i$ for the corresponding Z_i. All RG-functions are constructed as a series in g and for constants of form (14) the functions $\gamma_i(g)$ do not depend on ε.

This scheme is also applicable to the model (7). Conventional analysis shows[14,15] that it is even easier than the model (9) from the point of view of renormalization because only one independent renormalization constant Z_ν is necessary for the complete removal of the IR-divergences (poles in ε):

$$\nu_0 = \nu Z_\nu, \quad g_0 = gM^{2\varepsilon} Z_g, \quad Z_g = Z_\nu^{-3}. \tag{16}$$

Renormalizations of the fields in the parameter m are not necessary, i.e., $Z_\Phi = Z_m = 1$; thus, in Eq. (13) $\gamma_\Phi = 0$, and the contribution of $\mathscr{D}_m$ is absent in $\mathscr{D}_{\rm RG}$:

$$\mathscr{D}_{\rm RG} W_n^{\rm ren}(g, \nu, m, M, \dots) = 0, \quad \mathscr{D}_{\rm RG} \equiv \tilde{\mathscr{D}}_M + \beta\partial_g - \gamma_\nu \mathscr{D}_\nu. \tag{17}$$

It follows form Eqs. (15) in the last equality of Eq. (16) that $\gamma_g = -3\gamma_\nu$ and

$$\beta(g) = g(-2\varepsilon + 3\gamma_\nu(g)), \tag{18}$$

i.e., only the one RG-function $\gamma_\nu(g)$ is independent. It is calculated via Z_ν as a series in g. Only the first coefficient is well known[15]; $\gamma_\nu(g) = ag + ...$; in the three-dimensional problem $a = 1/20\pi^2$. It follows from the positivity of a that the β-function (18) for small ε must have a fixed point $g_* 2\varepsilon/$

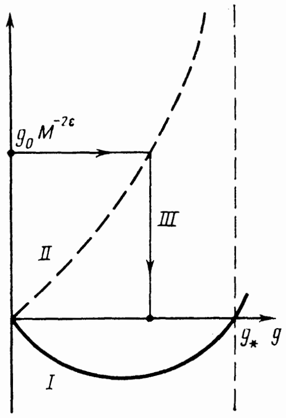

FIG. 2. Trace I is a graph of the β-function; trace II is a graph of g^{Z_g}; trace III is a graph of the solution of Eq. (27).

$3a + O(\varepsilon^2)$, at which $\beta(g_*) = 0$, and $\beta'(g_*) > 0$. The behavior of the β-function (calculated for small ε and guessed for larger) is shown as a solid line in Fig. 2.

4. SOLUTION OF THE RG EQUATIONS: INVARIANT CHARGE

The fields in model (7) are unrenormalized; thus, $W_n^{\text{ren}} = W_n$, and there is a difference only in the choice of variables and the form of perturbation theory (in g or in g_0). We consider as a qualitative example the correlator $D = \langle \varphi\varphi \rangle$ in the ω, p-representation. In renormalized variables $D = D(p,m,\omega,\nu,g,M)$, where the perpendicular projector in the vector indices is omitted and the function D is calculated as a series in g. In lowest order it coincides with the bare propagator (8), with the changes $\nu_0 \to \nu$ and $g_0 \to gM^{2\varepsilon}$ [see Eq. (16)]. Hence,

$$D = \frac{g\nu^3 M^{2\varepsilon}(p^2+m^2)^{-\varepsilon} p}{|-i\omega+\nu p^2|^2} + \ldots = g\nu M^{2\varepsilon} p^{-3}(p^2+m^2)^{-\varepsilon} R, \quad (19)$$

where the dots indicate contributions of higher order in g and R is some function of dimensionless arguments:

$$R = R(s, g, z, u), \quad s \equiv p/M, \quad z \equiv \omega/\nu M^2, \quad u \equiv m/p. \quad (20)$$

In agreement with Eq. (17), the function D satisfies $\mathscr{D}_{\text{RG}} D = 0$. Substituting Eq. (19) into this equation and taking into account Eq. (18), we obtain for R

$$LR(s, g, z, u) = 0, \quad L \equiv -\mathscr{D}_s + \beta\partial_g - (2-\gamma_\nu)\mathscr{D}_z + 2\gamma_\nu \quad (21)$$

(recall that $\mathscr{D}_x \equiv x\partial_x$ for any variable). The solution of Eq. (21) for known RG-functions can be written in the form

$$R(s, g, z, u) = R(1, \bar{g}, \bar{z}, u)\psi^2, \quad (22)$$

where $\bar{g} = \bar{g}(s,g)$ is the invariant charge, implicitly determined by the relation

$$\ln s = \int_g^{\bar{g}} \frac{dx}{\beta(x)}, \quad \bar{g}(1,g) = g, \quad (23)$$

and

$$\psi = \exp\left\{\int_g^{\bar{g}} dx \frac{\gamma_\nu(x)}{\beta(x)}\right\} = \left(\frac{s^{2\varepsilon}\bar{g}}{g}\right)^{1/3}, \quad \bar{z} = zs^{-2}\psi, \quad (24)$$

where the second equality for Ψ follows from Eqs. (18) and (23) (in models of the type (9) all RG-functions are usually independent). We note that the values $\bar{g}$, $\bar{z}$, and $\bar{u} = u$ are the first integrals of Eq. (21), normalized with respect to g, z, and u for $s = 1$. We show for illustration an explicit expression for $\bar{g}$ obtained from Eqs. (18) and (23) in the one-loop approximation ($\gamma_\nu(g) = ag$, $a = 1/20\pi^2$):

$$\bar{g}(s, g) = 2\varepsilon g/[2\varepsilon s^{2\varepsilon} + 3ag(1-s^{2\varepsilon})]. \quad (25)$$

Analogous equations for the static (simultaneous) correlator $D_{\text{ST}} = (2\pi)^{-1}\int d\omega D$ can be obtained by integrating Eq. (22) over ω, but it is simpler to derive it from first principles: in place of Eq. (19) we have (in the same notation)

$$D_{\text{ST}}(p, m, g, \nu, M) = g\nu^2 M^{2\varepsilon} p^{-1}(p^2+m^2)^{-\varepsilon} R(s, g, u), \quad (26)$$

where the RG equation for the new function R (in lowest order $R = 1/2$) has the form $LR = 0$ with $L = -\mathscr{D}_s + \beta\partial_g + \gamma_\nu$. Its solution is $R(s,g,u) = R(1,\bar{g},u)\Psi$.

When one calculates the function R in Eqs. (19) and (26) by perturbation theory, one introduces the parameters $s^{-2\varepsilon} = (Mp^{-1})^{2\varepsilon}$ lwhich grow as the order in g increases, as illustrated by the series for g in Eq. (25). The RG equations show that these powers $s^{-2\varepsilon}$ form combinations of $\bar{g}$ and Ψ, having, as we shall see, simple asymptotic forms for $s \to 0$.

For future purposes it is important to estimate the value of the renormalized charge g. Initially in the model the bare parameters are given; the renormalized ones are expressed in terms of them through Eqs. (16) with arbitrary choice of M (the natural choice is $M \sim \Lambda$). For a given M and g_0, the value of g is found from

$$g_0 M^{-2\varepsilon} = gZ_g(g).$$

The behavior of the right hand side as a function of g can be established, linking it with the β-function [see the discussion following Eq. (15)]:

$$\partial_g \ln Z_g = -1/g - 2\varepsilon/\beta(g),$$

i.e.,

$$\partial_g \ln(gZ_g) = -2\varepsilon/\beta(g) \geqslant 0$$

and

$$g_0 M^{-2\varepsilon} = gZ_g(g) = g\exp\left\{-\int_0^g dx\left[\frac{2\varepsilon}{\beta(x)} + \frac{1}{x}\right]\right\}, \quad (27)$$

where the term $1/x$ guarantees convergence for $x = 0$. For the β-function shown in Fig. 2 the behavior of the right hand side of Eq. (27) is shown in the same plot by the broken trace; for $g \to g_*$ it diverges. The graphical solution of Eq. (27) is shown in the same figure: one can see that some value of g in the interval $(0, g_*)$ corresponds uniquely to any arbigrary value of $g_0 M^{-2\varepsilon}$, i.e., $g < g_* \sim \varepsilon$ is a small value for small ε for any g_0. It follows from Eq. (23) that $\bar{g}(s,g)$ for any $s = p/M$ is also located within the interval $(0, g_*)$, that it is a monotonic function of s, and that

$$\bar{g}(s, g) \to g_* \quad \text{for} \quad s = p/M \to 0. \quad (28)$$

In the approximation (25), $\bar{g} \to g_* = 2\varepsilon/3a$ for $s \to 0$.

The property (28) supplies a solution using Eqs. (19)–(24) of the first IR-problem, the determination of the asymptotic value for $\lambda \to 0$ of D_λ obtained from D in Eq. (19)

by the λ-substitution

$$p,\ m,\ \omega\to\lambda p,\ \lambda m,\ \lambda^2\omega;$$

where M, g, and ν are fixed. For this substitution

$$\bar{g}_\lambda=\bar{g}(\lambda s,\ g)\to g_*$$

for $\lambda\to 0$ and from Eq. (24) we have

$$\bar{z}_\lambda\sim\psi_\lambda\sim\lambda^{2\varepsilon/3},$$

which in Eq. (22) leads to

$$R(1,\ \bar{g}_\lambda,\ \bar{z}_\lambda,\ u)\to R(1,\ g_*,\ 0,\ u)$$

for $\lambda\to 0$ (it is known from the diagrams that the limits exist) so that we find from Eqs. (19) and (22) the desired asymptotic value

$$D_\lambda=c\lambda^{-3-2\varepsilon/3}$$

where the coefficient c is independent of λ and ω. The dependence on λ in the asymptotic value has disappeared because the λ-substitution in Sec. 2 was chosen simply according to canonical dimensionalities. To obtain a more complete picture of the asymptotic value, the critical scaling, it is necessary to change the form of the λ-substitution.

5. CRITICAL SCALING

We consider a λ-substitution of the form

$$p,\ m,\ \omega\to\lambda p,\ \lambda^{\Delta_m}m,\ \lambda^{\Delta_\omega}\omega$$

with some critical dimensionalities $\Delta_{m,\omega}$ ($\Delta_p=1$ is the normalization). When the function F with the λ-substitution has a power-law asymptotic form $F_\lambda\sim\lambda^{\Delta_F}$, for $\lambda\to 0$, one says a critical scaling occurs for F, and the index $\Delta_F=\Delta[F]$ (the second designation is for complicated F) is called the critical dimensionality of F. Property (28) guarantees the existence of a critical scaling (see below) for the model as a whole with definite critical dimensionalities of the fields and parameters. The dimensionality of the arbitrary Green's function in the x-representation is the sum of the dimensionalities of the fields that go into it while that in the momentum representation is obtained from the Fourier transformation relations, in particular

$$\Delta[D(p,\ \omega,\dots)]=2\Delta_\varphi-3-\Delta_\omega,\ \Delta[D_{\mathrm{ST}}(p,\dots)]=2\Delta_\varphi-3,$$
$$\Delta_\varphi+\Delta_{\varphi'}=3. \tag{29}$$

The last inequality follows from the normalization to $\delta(\mathbf{x}_1-\mathbf{x}_2)$ of the response function $\langle\varphi(x_1)\varphi'(x_2)\rangle$ for $t_1=t_2+0$.

We show that a scaling with definite critical dimensionalities actually follows from the equations of Sec. 4. It is clear from Eq. (28) that the asymptotic value for $\lambda\to 0$ does not change if we make the substitution $\bar{g}\to g_*$ in the equations, which gives

$$D=\psi_*^2 g\nu M^{2\varepsilon}p^{-3-2\varepsilon}\xi(u)f(\bar{z},\ u),\ \bar{z}=\psi_*\omega\nu^{-1}p^{-2}$$

for the dynamic correlator and

$$D_{\mathrm{ST}}=\psi_* g\nu^2 M^{2\varepsilon}p^{-1-2\varepsilon}\xi(u)f(u)$$

for the static one, where $\Psi_*=(s^{2\varepsilon}g_*g^{-1})^{1/3}$, and ξ and f are scaling functions:

$$\xi(u)=(1+u^2)^{-\varepsilon},\ f(u)\equiv R(1,\ g_*,\ u),$$
$$f(\bar{z},\ u)\equiv R(1,\ g_*,\ \bar{z},\ u). \tag{30}$$

It is easy to verify that with the substitution $s=p/M$, the values of g, ν, and M are always grouped so that they do not depend on M in the combination $g\nu^3M^{2\varepsilon}=g_0\nu_0^3$ (the relation follows from Eq. (16)), which leads to

$$D=a_0^{1/3}g_*^{2/3}p^{-3-2\varepsilon/3}\xi(u)f(\bar{z},u),$$
$$D_{\mathrm{ST}}=a_0^{2/3}g_*^{1/3}p^{-1-4\varepsilon/3}\xi(u)f(u), \tag{31}$$
$$a_0\equiv g_0\nu_0^3,\ \bar{z}\equiv a_0^{-1/3}g_*^{1/3}\omega p^{-2+2\varepsilon/3}.$$

These equations for fixed a_0 and arbitrary functions f describe the scaling with the dimensionalities (6); one can easily be convinced of the consistency of the indices using Eq. (29). Justification of the critical scaling in models of the type (9) can be carried out in the same way; the critical dimensionalities of the variables e_i are determined in general by $\Delta_i=d_i+\gamma_i(g_*)$, where d_i is the canonical dimensionality and $\gamma_i(g)$ is the corresponding RG function (15). Usually all RG-functions β and γ_i are independent; thus, $\gamma_i(g_*)$ can be expressed as a series in ε. In our model there is only one independent RG-function $\gamma_\nu(g)$; it follows from the connection (18) that the value $\gamma_\nu(g_*)=2\varepsilon/3$ is determined exactly without corrections ε^2, ε^3 etc., this is a unique feature of the model. It should be emphasized that the scaling is due to property (28), and this property in turn is due to the behavior of the β-function shown in Fig. 2. The smallness of ε is needed only to guarantee this behavior; it is assumed that when ε grows the picture deforms, but does not qualitatively change. Thus, g_* continuously shifts, but does not disappear; this is the basis for belief that there is a critical scaling and that the indices are continuous in ε for large ε.

In models of the form (9) the parameter analogous to a_0 in the λ-substitution is always fixed; thus, equations of type (31) are final answers for the critical scaling. In the theory of Kolmogorov-Obukov, one speaks about the scaling for fixed E and ν_0; thus, it is necessary to express g_0 in Eq. (31) through the pumping power E to obtain final results.

Performing the integration in Eq. (3) with the function (5) and cutoff Λ gives

$$E=\frac{g_0\nu_0^3}{4\pi^2(2-\varepsilon)}\Bigg[(\Lambda^2+m^2)^{2-\varepsilon}+\frac{m^{4-2\varepsilon}}{(1-\varepsilon)}-\frac{(2-\varepsilon)m^2(\Lambda^2+m^2)^{1-\varepsilon}}{(1-\varepsilon)}\Bigg]. \tag{32}$$

The Reynolds number $\mathrm{Re}=(\lambda/m)^{4/3}$ is of order 10^4–10^6, so we have from Eq. (32)

$$a_0\equiv g_0\nu_0^3=EB(\Lambda,\ m,\ \varepsilon)\approx\begin{cases}c_1E\Lambda^{2\varepsilon-4} & \text{for}\quad 2>\varepsilon>0,\\ c_2Em^{2\varepsilon-4} & \text{for}\quad \varepsilon>2,\end{cases} \tag{33}$$

where $c_1\equiv 4\pi^2(2-\varepsilon)$, $c_2\equiv c_1(1-\varepsilon)$, and the determination of the exact function $B(\lambda,m,\varepsilon)$ in Eq. (33) is clear from comparison with Eq. (32); in particular $B(\lambda,m,2)\approx 2\pi^2/\ln(\Lambda/m)$ for $\varepsilon=2$. The simple approximations (33) are useful outside the transition region close to $\varepsilon=2$, which has a small width $\sim 1/\ln\mathrm{Re}$; hence, we consider below that approximations (33) are valid everywhere right up to $\varepsilon=2$. It follows from Eq. (33) that for the λ-substitution with fixed E and ν_0 (and $\Lambda=E^{1/4}\nu_0^{3/4}$), the parameter a_0 in Eq. (33)

in the region of UV-pumping $0<\varepsilon\leqslant 2$ is fixed. Thus, from Eq. (31), one has a scaling with dimensionalities (6). A dependence on ν_0 remains in this region, entering through $\lambda^{2\varepsilon-4}$ in a_0 and disappearing for $\varepsilon=2$. In the actual region of IR-pumping $\varepsilon\geqslant 2$, the dependence on ν_0 in a_0 disappears, but $a_0\sim m^{2\varepsilon-4}$ for the λ-substitution acquires the dimensionality $\Delta_a=2\varepsilon-4$. Thus, a scaling emerges from Eqs. (29) and (31) with different dimensionalities $\Delta_\varphi'=\Delta_\varphi+\Delta_a/3$ and $\Delta_\omega'=\Delta_\omega+\Delta_a/3$ which coincide with Eq. (4), as is easily seen. In this way we have proven the basic assertions 1 and 2 made at the end of Sec. 1.

6. SCALING FUNCTIONS AND THE FIRST KOLMOGOROV HYPOTHESIS

Equation (31) for $D_{\rm ST}$ describes the critical asymptotic value in the region $p,m\ll\Lambda$ for arbitrary $u=m/p$. The inertial range corresponds to the additional condition $u\ll 1$ so that $\xi(u)\simeq 1$ [see Eq. (30)]. Hypothesis 1B (se Sec. 1) implies elimination of m in the term $a_0^{2/3}f(u)$ for $u\to 0$, i.e., taking into account Eq. (33)

$$f(u)\approx\begin{cases}\text{const}=f(0) & \text{for}\quad 2\geqslant\varepsilon>0,\\ \text{const}\,u^{4(2-\varepsilon)/3} & \text{for}\quad \varepsilon\geqslant 2.\end{cases}\tag{34}$$

Strictly speaking, the Kolmogorov theory is relevant only for the case of real IR-pumping with $\varepsilon\geqslant 2$; Eq. (34) is a generalization to arbitrary $\varepsilon>0$.

The scaling functions (30) are in no way fixed by the RG equations. They are systematically calculated by an ε-expansion in the theory of critical phenomena: the initial functions R in Eqs. (19) and (26) are calculated as a series $\Sigma_n g^n R_n$ of renormalized perturbation theory, where R_n is the sum of contributions of diagrams of order n. For the substitution $R=\Sigma g^n R_n$ in Eq. (30), $g\to g_*$, and expanding g_* and R_n in ε, we obtain the desired ε-expansion of the scaling function; for example

$$f(u,\ \varepsilon)=\sum\varepsilon^n f_n(u)$$

for $D_{\rm ST}$. It is important for the calculation of a finite order in ε that there be a finite number of diagrams, since $g_*\sim\varepsilon$ and R_n does not contains poles in ε.

We are interested in the asymptotic value as $u\to 0$. It is known from an analysis of diagrams that the coefficients $f_n(u)$ of the ε-expansion have only weak singularities of the type $u\ln u$, i.e., they are finite for $u=0$; thus, one can postulate hypothesis 1 in the framework of the ε-expansion. However, this does not prove it for finite ε because for any $\varepsilon>0$, however small, there are diagrams diverging for $m\to 0$. This is all true for models of type (9); in the general case the argument of the scaling function is the invariant quantity

$$u\sim mp^{-1}(Mp^{-1})^{\gamma_m^*},$$

where $\gamma_m{}^*\equiv\gamma_m(g_*)$ is the anomalous dimensionality of the parameter m (for us $\gamma_m=0$ due to the absence of a renormalized m). Hence, the problem which interests us has already occurred in the theory of critical phenomena where the method of operator expansion was employed, which allows the asymptotic value as $u\to 0$ to be found for finite ε (Refs. 22, 23). We clarify this procedure for the model (9). The role of a_0 in equations of type (31) is played by g_0; thus, all dependence on m is contained in the function $f(u)$, and its singularity in u consists of singularities in m of the renormalized correlator D. These are studied with the help of the following fact, proved inthe Wilson theory of the renormalized operator expansion (see Ref. 20, Chap. 20). For $x\equiv(x_1+x_2)/2=\text{const}$ and $r\equiv|x_1-x_2|\to 0$,

$$\varphi(x_1)\varphi(x_2)\approx\sum c_i(r)F_i(x),\tag{35}$$

where the c_i are analytic numerical coefficients for $m\to 0$, and F_i are all possible combinations with the symmetry of the problem of renormalized localized component operators, i.e., combinations like $\varphi^2(x)$, $\varphi(x)\Delta\varphi(x)$, $\varphi^4(x)$ of φ and its derivatives. The coincidence of field arguments in these terms leads to new IR-divergences in the Green's functions . A generalized renormalization theory of component operators exist that is quite complicated partly due to their mixing in renormalization. For our purposes it is only important that to each F_i in Eq. (35) one can assign a definite critical dimensionality $\Delta_i=d_i+\gamma_i{}^*$, where d_i is its canonical dimensionality (the sum of the canonical dimensionalities of the fields and their derivatives comprising F_i), and $\gamma_i{}^*$ is the anomalous dimensionality, calculated by the RG method in the form of an ε-expansion.

The correlator D is obtained by averaging Eq. (35); on the right hand side the values $\langle F_i\rangle=m^{d_i}a_i(g,m/M)$ appear whose asymptotic values for $m\to 0$ are determined (with justification from RG) by the critical dimensionalities Δ_i and $\Delta_m:\langle F_i\rangle\sim m^{\Delta_i/\Delta_m}$ (in hydrodynamics $\Delta_m=1$). In view of the fact that $m\sim u$, the operators F_i in Eq. (35) give rise to nonanalytic contributions u^{Δ_i/Δ_m} in $f(u)$; one must add to them all possible analytic contributions (of power u^2).

The most important contributions for $u\to 0$ are those with the smallest Δ_i which are those with the smallest $d_i(\varepsilon=0)$ in the framework of the ε-expansion. We will call operators with $\Delta_i<0$ "dangerous" because their contributions are divergent for $u\to 0$. Dangerous operators do not occur in the ε-expansion because $\Delta_i=n_i+O(\varepsilon)$, where $n_i>0$ is the total number of multipliers φ and ∂ in F_i (see the table of dimensionalities in Sec. 2). For the model (9) the operator $F=\varphi^2(x)$ has the smallest n_i in Eq. (35) ($F=1$ in Eq. (35) yields a contribution which is analytic in m^2; $F=\varphi(x)$ is forbidden by the symmetry $\varphi\to-\varphi$), giving rise to the contribution $u^{2+O(\varepsilon)}$ in $f(u)$; to this one must add $u^{4+O(\varepsilon)}$ corrections from the operators $\varphi\Delta\varphi$, φ^4, etc. The positivity of all Δ_i is guaranteed by the finiteness of $f(u=0,\varepsilon)$ in spite of the divergence of separate diagrams for $m\to 0$ with finite ε. Strictly speaking, the positivity of Δ_i is guaranteed only within the ε-expansion, i.e., for asymptotically small ε because we actually know only the finite remainders of the ε-expansion of Δ_i. However, in the theory of critical phenomena, there is a simple argument for the finiteness of $f(u=0,\varepsilon)$, namely the very existence of a critical (massless) system; thus, the positivity of Δ_i is not called into question. We note that the form of the massless correlator for the real value $2\varepsilon=1$ is well known from experiments: $D\sim p^{-2+\eta}$, where $\eta\approx 0.05$ is the Fisher index (see Ref. 12, p. 62). Insertion of this correlator in the schematic graph for Eq. (10) leads to IR-divergences that are the exact analog of the problem attacked in Ref. 3. The problem was not solved in Ref. 3 and does not occur in the theory of critical phenomena; the alternative RG technique in Eq. (35) gives a solution of the problem without direct solution of the problem of Ref. 3.

One can imagine (though there is no proof) that the operator expansion (35) is correct for the static correlator in our problem; the total canonical dimensionality F_i plays the role of the d_i. The relation $\Delta_i = n_i + O(\varepsilon)$ remains valid (see the dimensionalities in Sec. 2); thus, in terms of the ε-expansion, the most important operators in Eq. (35) are, as before, those with the smallest number of fields and their derivatives, the most important of which is φ^2. The renormalization and critical dimensionalities of the first component operators with $d(\varepsilon=0) \leqslant 4$ were studied in Refs. 15 and 16. In comparison with model (9), it is possible in model (7) to calculate exactly some critical dimensionalities due to the termination of the series in ε, which gives

$$\Delta[\varphi^n] = n\Delta_\varphi = n(1-2\varepsilon/3). \tag{36}$$

This rule was proven in Ref. 16 for $n \leqslant 4$. We have generalized it to any n; the proof is based on Ward identities which express Galilean invariance and is omitted for lack of space.

One can see from Eq. (36) that all operators φ^n become dangerous ($\Delta < 0$) for $\varepsilon > 3/2$, i.e., even before reaching the border $\varepsilon = 2$ of the region of actual IR-pumping. We assume below that only the operators φ^n can be dangerous in the region $0 < \varepsilon < 2$. This was proven for operators with $d(\varepsilon=0) \leqslant 4$ in Ref. 16 and is supported indirectly in the general case by the results of Sec. 7. In the region $0 < \varepsilon < 3/2$ there are no dangerous operators, i.e., there are no diverging contributions to $f(u)$ for $u \to 0$ and $f(u=0)$ is finite; thus, the hypothesis (34) in this region is proven. In the region $3/2 < \varepsilon < 2$ all operators φ^n are dangerous, but there are none of the most dangerous with the smallest value of Δ (the larger n, the more dangerous); thus, one must sum all their contributions, which is not done in the theory of critical phenomena (and in fact is impossible) due to a lack of knowledge of exact dimensionalities. The necessary summation is carried out in Sec. 7 where it is shown, as a result of the summation for $m \to 0$, one obtains a finite expression for D_{ST}, i.e., the hypothesis (34) holds in the whole interval $0 < \varepsilon < 2$ and, thus, from Eqs. (31), (33), and (34)

$$D_{\mathrm{ST}} = (c_1 E \Lambda^{2\varepsilon-4})^{2/3} g_*^{1/3} p^{-1-4\varepsilon/3} f(u=0). \tag{37}$$

This equation is the generalization in the region $0 < \varepsilon < 2$ of the phenomenological Kolmogorov spectrum

$$D_{\mathrm{CT}} = 2\pi^2 A E^{2/3} p^{-11/3},$$

where the analog of the Kolmogorov constant A ($A = C_1$ in the rotation of Ref. 1, p. 183) in Eq. (37) is

$$A(\varepsilon) = c_1^{2/3} g_*^{1/3} f(u=0)/2\pi^2, \tag{38}$$

where $c_1 = 4\pi^2(2-\varepsilon)$ (see below following Eq. (33)), and g_* and $f(u=0)$ are series in ε. Actually, we only know their first terms:

$$g_* = 40\pi^2\varepsilon/3 + \ldots$$

(see the end of Sec. 3), and $f(u=0) = 1/2 + \ldots$ [see after Eq. (26)] from which $A(\varepsilon) \approx 2(5\varepsilon/3)^{1/3}$. The spectrum (37) coincides with that of Kolmogorov at the boundary $\varepsilon = 2$; in lowest order $A = A(\varepsilon = 2) = (80/3)^{1/3} \approx 3$, which is not too bad [from experiments, $A = 1.4$–2.7 (Refs. 1, 19)].

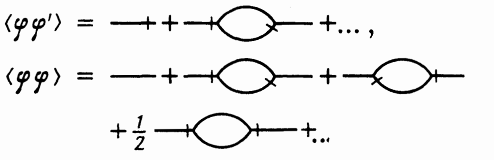

FIG. 3. The one-loop approximation for pair correlators in the model (7).

7. INFRARED PERTURBATION THEORY

We analyze the leading singularities in m of diagrams of perturbation theory in g_0 in the model (7) for the example of the pair of correlators $\langle\varphi\varphi\rangle$ and $\langle\varphi\varphi'\rangle$. They are given in the single loop approximation in Fig. 3 with lines corresponding to bare propagators (8): an uncrossed line designates the field φ, a singly crossed line designates the field φ', and there are no doubly crossed lines because $\langle\varphi'\varphi'\rangle_0 = 0$. Let $\tilde{p}$ be the four-dimensional external momentum in the diagrams ($\tilde{p}_0 = \omega$, $\tilde{p}_i = \mathbf{p}_i$ for $i = 1,2,3$), and $\tilde{k}$ be the total momentum flowing in the line $\varphi\varphi$ in the loops; thus, $\tilde{p} - \tilde{k}$ flows in the second line. We assume that $\tilde{p}$ is located in the inertial range so that $\tilde{p} \gg m$ (this notation always signifies $p \gg m$ and $\tilde{p}_0 = \omega \gg \nu_0 m^2$). We introduce some fixed limiting mass μ with $\tilde{p} \gg \mu \gg m$, dividing the region of weak ($\tilde{k} \leqslant \mu$) and strong ($\tilde{k} \geqslant \mu$) momenta, and label the lines of the diagram accordingly. The integration over the circulating momentum $\tilde{k}$ in the loops is divided into two regions: 1) both lines of the loop are strong and 2) one of the lines is weak and the other is strong (two weak lines are forbidden by $\tilde{p} \gg \mu$). The singularity in m is evidently produced by the weak line $\varphi\varphi$. Its contribution, neglecting the dependence of vertices and the strong line on weak $\tilde{k}$, is identified as the multiplier

$$\langle\varphi^2(x)\rangle_0 = (2\pi)^{-4} \int d\tilde{k} \langle\varphi\varphi\rangle_0(\tilde{k}) \sim \int d\mathbf{k}\, d(k) k^{-2} \sim g_0 m^{2-2\varepsilon}, \tag{39}$$

which diverges for $m \to 0$ for $\varepsilon > 1$. The corresponding approximation for the one-loop diagrams is given in Fig. 4. The shortened loop signifies $\langle\varphi^2(x)\rangle_0$ and the index $\ll 0 \gg$ is the zeroth approximation in g_0. The coefficient 1/2 in one of the diagrams of Fig. 3 is lost because there are two lines $\varphi\varphi$ in its loop and the weak one can be either of them.

The procedure of extracting the leading singularities in m corresponds to an operator expansion of the form (35): to first order in g_0 only $\langle\varphi^2\rangle_0$ appears, in the next two-loop approximation $\langle\varphi^4\rangle_0$ and a correction of first order in $\langle\varphi^2\rangle_0$ appears, and finally all completely dressed multipliers $\langle\varphi^{2n}\rangle$ occur. Taking into account the dependence of the strong lines and the vertices occurring on them on weak momenta (i.e., momenta of the φ field of the multipliers $\langle\varphi^{2n}\rangle$) would correspond to the calculation in Eq. (35) of operators with their derivatives which we do not consider dangerous for

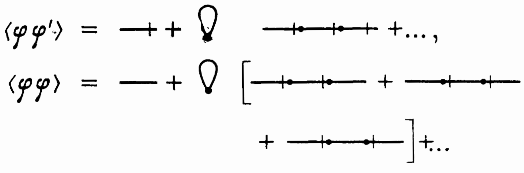

FIG. 4. The one-loop diagrams neglecting the dependence of the vertices and stong lines on weak momenta.

$\varepsilon < 2$. It should be noted that from the point of view of a direct count of orders of magnitude it is always more advantageous to use the next weak lines to dress the already existing weak block $\langle \varphi^{2n} \rangle$ and not to construct the block $\langle \varphi^{2n+2} \rangle$. In fact, the first process is characterized by the dimensionless parameter $g_0 m^{-2\varepsilon}$ and the second one by $g_0 m^{2-2\varepsilon} p^{-2}$ because the new weak line only produces the singularity $\sim g_0 m^{2-2\varepsilon}$ (see above) which is nondimensionalized with the strong momentum p, and the weak block g_0 is completely nondimensionalized by m. If we limit ourselves to perturbation theory, it would be possible to conclude from what was said above that the main contribution is produced only by the completely dressed operator φ^2 and all other φ^{2n} operators are not important, which corresponds to the approximation of Fig. 4 with an exact loop. However, we intend to estimate $\langle \varphi^{2n} \rangle$ by the RG method which allows calculation not only of canonical, but also anomalous dimensionalities. We know from Eq. (36) that for $\varepsilon = 3/2$ all operators φ^n are equally important; for $\varepsilon > 3/2$ the operator φ^n is more important than the larger n and, all φ^n are necessary.

We introduce a simple method for their approximation. Representing the field $\Phi = \varphi, \varphi'$ in Eq. (7) in the form of a sum $\Phi = \Phi_< + \Phi_>$ of weak $\Phi_< = \varphi_<, \varphi_<'$ and weak $\Phi_> = \varphi_>, \varphi_>'$ components leads to

$$\int D\Phi ... = \int D\Phi_< \int D\Phi_> ... .$$

What is of interest is the average of strong fields $D_> = \langle \Phi_> \Phi_> \rangle$ (arguments x are dropped) which is represented by the functional integral

$$D_> = \int D\Phi \Phi_> \Phi_> \exp S(\Phi)$$

accurate to within an unimportant normalizing multiplier [we note that in models of type (7) the vacuum loops are absent[15] and, thus, one can consider $\int D\Phi \exp S(\Phi) = 1$]. The definition of $D_>$ can be rewritten as

$$D_> = \int D\Phi_< D_>(\Phi_<) \int D\Phi_> \exp S(\Phi), \qquad (40)$$

where

$$D_>(\Phi_<) \equiv \left\{ \int D\Phi_> \exp S(\Phi) \right\}^{-1} \int D\Phi_> \Phi_> \Phi_> \exp S(\Phi). \qquad (41)$$

The idea of Eq. (41) is that a strong propagator in a fixed [similar to frozen-in impurities (see Ref. 11, Chap. 10)] weak external field and Eq. (40) represents the weak-field statistical average, and hence the average over the exact statistics, i.e., that determined by the whole $S(\Phi)$. When $\Phi = \phi_< + \phi_>$ is substituted in the action of Eq. (41), its interaction $\varphi'(\varphi\partial)\varphi$ produces several terms. In the zeroth approximation only the $\varphi_>'(\varphi_< \partial)\varphi_>$ term is retained, and Eq. (41) has the sense of bare propagators with the substitution $\partial_t \to \partial_t + \varphi_< \partial$. The weak field $\varphi_<(x)$ is not strictly homogeneous, but its inhomogeneity is weak and in the zeroth approximation can be neglected, which means that replacing the values $\langle \varphi_<(x_1) ... \varphi_<(x_n) \rangle$ appearing in Eq. (40) with localized averages $\langle \varphi_<(x) ... \varphi_<(x) \rangle$ that are independent of x (this is equivalent in the language of diagrams to neglecting the dependence of strong lines and vertices on weak momenta). This operation corresponds to the substitution $\varphi_<(x) \to v \equiv \varphi_<(0)$ in Eq. (41); the latter is just a simple random value and not a random field. Then the quantities (41) have the sense of bare propagators after the substitution $\partial_t \to \partial_t + v\partial$, i.e., $\omega \to \omega - vp$ in the momentum-frequency representation; the transition to this representation is possible due to the homogeneity of v. The functional average over $\Phi_<$ in Eq. (40) becomes equivalent to the average $\langle\!\langle ... \rangle\!\rangle$ over a distribution of the random variable v:

$$\langle\!\langle v^n \rangle\!\rangle \equiv \langle \varphi_<^n(x) \rangle = \int D\Phi_< \varphi_<^n(x) \int D\Phi_> \exp S(\Phi),$$

so that it follows from Eqs. (40) and (41) that

$$\langle \varphi\varphi' \rangle = \langle \varphi'\varphi \rangle^* = \left\langle\!\!\left\langle \frac{1}{-i(\omega - vp) + \nu_0 p^2} \right\rangle\!\!\right\rangle,$$
$$\langle \varphi\varphi \rangle = \left\langle\!\!\left\langle \frac{1}{-i(\omega - vp) + \nu_0 p^2} d(p) \frac{1}{i(\omega - vp) + \nu_0 p^2} \right\rangle\!\!\right\rangle. \qquad (42)$$

These equations were obtained previously in Ref. 3 by another method. Use of the functional integral simplifies their derivation. More importantly, it allows one to consider Eq. (42) as the zeroth approximation of some new infrared perturbation theory since the prescription for constructing a series of corrections is clear.

The first of Eqs. (42) was given in Ref. 3, but the second was not because in Ref. 3 the pumping function $d(p)$ was considered equal to zero in the inertial range; in our case it is a power of the momentum. This difference is important and deserves some comment. We feel that it is necessary to regard the model of pumping (5) in exactly the same way as fluctuation models of the type (9) in the theory of critical phenomena. These are not exact, but phenomenological models in the spirit of the Landau theory are intended only for a description of IR asymptotic values.

Justification of the model of power-law pumping (5) in the theory of turbulence is, evidently, the same type of problem as that of justifying the fluctuation models in the theory of critical phenomena. The finite pumping $d(p)$, disappearing outside a region of width of order m, plays the role of an exact micro-model. Its spreading in the region $p \gg m$ requires multiple bifurcations, or, in diagram language, higher order perturbation theory. Of course, the corresponding theory (Ref. 2, § 32) predicts a scaling that can be considered as an indirect justification of the power-law pumping model (5) for $p \gg m$. However, even in the absence of a strict justification, such models, as shown by experience with the theory of critical phenomena, can be used qualitatively successfully as a basis for the RG model.

Equations (42) are the result of summation of all contributions φ^n to Eq. (35) to lowest order in g_0 for c_i. When there are no other dangerous operators (which we consider true in the region $0 < \varepsilon < 2$), the corrections in Eq. (42) contain only m^Δ with positive indices and, thus, disappear for $m \to 0$. In the region $\varepsilon > 2$ it is impossible to trust Eqs. (42) because it is known from the RG analysis[16] that for $\varepsilon \geqslant 2$ the operator of local energy dissipation becomes dangerous

$$F = -\nu_0 (\partial_i \varphi_k + \partial_k \varphi_i)^2 / 2,$$

and, consequently, all its powers; thus, as ε increases new dangerous operators will appear.

The quantities $\langle\!\langle v^{2n} \rangle\!\rangle$ occur in the expansion in v of Eq. (42) that coincide with the exact (because the total action

$S(\Phi)$ enters in Eq. (40)) local averages $\langle \varphi^{2n}(x) \rangle$ (averages of odd powers equal zero). Figure 4 corresponds to the first order in v^2 in Eq. (42). The behavior of $\langle\langle v^{2n} \rangle\rangle = \langle \varphi^{2n} \rangle$ for $m \to 0$ is determined by the critical scalings of the operators φ^{2n} (see Sec. 6) known from Eq. (36):

$$\langle\langle v^{2n} \rangle\rangle \sim m^{2n\Delta_\varphi},$$

for $\varepsilon > 3/2$ they diverge for $m \to 0$. In dynamic objects (42) there is no obvious reason for the disappearance of the dependence on m, i.e., hypothesis 1A is not fulfilled even in the UV-pumping region $0 < \varepsilon < 2$. The transition to static objects corresponds to integration over ω in Eq. (42); thus, the dependence on v disappears because vp is a simple frequency shift, i.e., inthe region $0 < \varepsilon < 2$ [where we trust Eqs. (42)] hypothesis 1B is fulfilled and D_{ST} does not depend on m. We note that integration over ω leads to a disappearance of the dependence on m in Eqs. (42) in each order in v^2; in particular, the integral over ω goes to zero for the coefficient for $\langle\langle v^2 \rangle\rangle$ in Fig. 4.

Hence, we have shown that the coefficients for the dangerous averages $\langle \varphi^{2n} \rangle$ in static objects become zero; thus, hypothesis 1B is satisfied (as ε increases) until operators with derivatives become dangerous. On the basis of the results of Ref. 16 we say that this occurs for $\varepsilon \geqslant 2$; to prove hypothesis 1B for $\varepsilon > 2$ it is necessary to show that the contributions of these new dangerous operators sum up to the known power of Eq. (34). This is clearly not a simple problem and requires refinements of the present technique.

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[1]A. S. Monin and A. M. Yaglom, *Statisticheskaya gidromekhanika*, Vol. 2, Nauka, Moscow, 1969 (*Statistical Fluid Mechanics II*, MIT Press, Cambridge, 1971).
[2]L. D. Landau and E. M. Lifshitz, *Gidrodinamika*, Nauka, Moscow, 1986 (*Fluid Mechanics*, Pergamon, Oxford, 1959).
[3]V. S. L'vov, Preprint IAE Siberian Branch No. 53, 1977; V. I. Belinicher and V. S. L'vov, Preprint IAE Siberian Branch No. 333, 1986; Zh. Eksp. Teor. Fiz. **93**, 533 (1987) [Sov. Phys. JETP **66**, 303 (1987)].
[4]H. W. Wyld, Ann. Phys. **14**, 143 (1961).
[5]R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959); Phys. Fluids **7**, 575 (1964); Phys. Fluids **8**, 2219 (1965).
[6]G. A. Kuz'min and A. Z. Patashinskii, Zh. Eksp. Teor. Fiz. **62**, 1175 (1972) [Sov. Phys. JETP **35**, 620 (1972)]. (reference missing to PMTF which has no vol. no.) Preprint IYaF Siberian Branch 76/20, 1976.
[7]V. E. Zakharov and V. S. L'vov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **18**, 1470 (1975).
[8]S. S. Moiseev, A. V. Tur, and V. U. Yanovskii, Trudi mezhdunarodnoi rabochei gruppy no fizike nelineinykh yavlenii i turbulentnosti, Nauk. dimak, Kiev, 1983, p. 1693; A. V. Tur and V. V. Yanovskii, Preprint IKI 12-03, 1986; R. Z. Sagdeev, S. S. Moiseev, A. V. Tur, and V. V. Yanovskii, *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, ed. R. Z. Sagdeev, Mir, Moscow, 1986, p. 138.
[9]B. B. Kadomstev, *Voprosy teorii plazmy*, Vol. 4, Atomizdat, Moscow, 1964; *Plasma Turbulence*, Academic, New York, 1965.
[10]K. Wilson and J. Kogut, *Renormalizatsionnaya gruppi i ε-razlozhenie*, Mir, Moscow, 1975.
[11]Sh. Ma, *Modern Theory of Critical Phenomena*, Benjamin, Reading, Ma., 1976 (Russ. Transl., Mir, Moscow, 1980).
[12]A. Z. Patashinskii and V. L. Pokrovskii, *Fluktuatsionnaya teoriya fazovykh perekhodov*, Nauka, Moscow, 1982 (Fluctuation Theory of Phase Transitions, Pergamon, Oxford, 1979).
[13]A. N. Kolmogorov, Dok. Akad. Mauk SSSR **30**, 299 (1941).
[14]C. De Dominicis and P. C. Martin, Phys. Rev. A **29**, 419 (1979).
[15]L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **57**, 268 (1983) (in Russian).
[16]L. Ts. Adzhemyan, A. N. Vasil'ev, and M. Gnatich, Teor. Mat. Fiz. **74**, 180 (1988) (in Russian).
[17]J. D. Fournier, P. L. Sulem, and A. Pouquet, J. Phys. A **15**, 1393 (1982). L. Ts. Adzhemyan, A. N. Vasil'ev, and M. Gnatich, Teor. Mat. Fiz. **64**, 196 (1985) (in Russian).
[18]L. Ts. Adzhemyan, A. N. Vasil'ev, and M. Gnatich, Teor. Mat. Fiz **68**, 72 (1984) (in Russian).
[19]T. Nakano, Ann. Phys. **73**, 326 (1972).
[20]John C. Collins, *Renormalization*, Cambridge U. P., New York, 1984 (Russ. Transl., Mir, Moscow, 1988).
[21]G. 'tHooft, Nucl. Phys. B **61**, 455 (1973).
[22]E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. Lett. **32**, 473 (1974).
[23]D. R. Nelson, Phys. Rev. B **14**, 1123 (1976)

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