Charged-particle dynamics in the field of two electromagnetic waves

V. A. Balakirev, V. A. Buts, A. P. Tolstoluzhskii, and Yu. A. Turkin

Kharkov Institute of Physics and Engineering, Ukrainian Academy of Science
(Submitted 19 May 1988)
Zh. Eksp. Teor. Fiz. **95**, 1231–1245 (April 1989)

We examine charged-particle dynamics in a constant external magnetic field and the field due to two arbitrarily polarized electromagnetic waves. We derive the general condition for the onset of local instability of particle motion, valid for all particle-field resonances—namely, Cherenkov, cyclotron, normal- and anomalous-Doppler cyclotron, and parametric Cherenkov resonances. We go on to show how all presently known conditions for the onset of local instability follow as special cases.

The interaction dynamics of charged particles with regular fields due to electromagnetic waves is a topic of considerable interest in the solution of problems dealing with charged-particle acceleration and with plasma heating by external fields for the purposes of controlled thermonuclear fusion.

Charged particles are of course known to interact efficiently with electromagnetic waves under resonant conditions. The most familiar elementary resonances between charged particles and electromagnetic waves are Cherenkov, normal- and anomalous-Doppler cyclotron, and parametric resonances. To this list we must also add the resonance that occurs when a charged particle interacts with an electromagnetic wave inside a periodic medium.[1] This reduces formally to a Cherenkov resonance between the particle and the virtual wave that arises in the periodic medium.

The natue of the ensuing particle motion depends heavily on both the separation between resonances and the width of each of the linear resonances.[2–4] If the wave is of low amplitude, so that the width of a nonlinear resonance is less than the separation between adjacent resonances, then the particle will interact efficiently only with a single, isolated resonance. In that event, particle motion will be regular over the entire phase plane, except for narrow stochastic layers near the separatrices where these layers are formed.[2,3] This picture changes qualitatively at wave amplitudes high enough that the sum of the halfwidths of neighboring resonances exceeds their separation—i.e., when nonlinear resonances overlap. Particle motion then becomes chaotic.

The chaotic dynamics of particle motion and the conditions under which it arises are of considerable interest, particularly in the development of charged-particle acceleration techniques. Indeed, on the one hand, the onset of stochastic instability can lead to the disruption of acceleration in resonant accelerators, but on the other, it opens new avenues for the implementation of stochastic acceleration methods.[2,5,6]

There is a sizable literature on chaotic motion of charged particles. Relativistic and nonrelativistic charged-particle motion in a constant magnetic field and in the field of a longitudinal plasma wave was examined in Refs. 5–7. Chaotic particle motion in the field of a longitudinal-wave packet was studied in Ref. 8. In Ref. 9, a solution was found for the chaotic motion of an oscillator in either a prescribed or a self-consistent field of a transverse electromagnetic wave propagating perpendicular to an external magnetic field. References 2, 11, and 13, which deal with chaotic particle dynamics in the field of two waves but with no external magnetic field, are also to be noted.

In the present paper, we examine the interaction of an arbitrarily polarized electromagnetic wave with charged particles in an external magnetic field, all in the presence of a medium—in particular, a plasma. With overlap of nonlinear resonances as our criterion, we identify a general condition under which charged-particle motion in an external electromagnetic field becomes stochastic. We derive a condition for the validity of all known resonant interactions between particles and a field; it subsumes the presently known conditions for the onset of local instability of charged-particle motion as special cases.

1. CHARGED-PARTICLE DYNAMICS IN THE FIELD OF AN ELECTROMAGNETIC PLANE WAVE

We consider the motion of a charged particle in a constant, externally applied magnetic field H_0 directed along the z-axis, and in the field of an electromagnetic plane wave, which in the general case has components

$$\vec{\mathscr{E}} = \mathrm{Re}[\mathbf{E}\exp(i\mathbf{kr}-i\omega t)],$$

$$\vec{\mathscr{H}} = \mathrm{Re}\left\{\frac{c}{\omega}[\mathbf{kE}]\exp(i\mathbf{kr}-i\omega t)\right\},$$

$$\mathbf{E}=\mathbf{E}_0(\alpha_x,\ i\alpha_y,\ \alpha_z), \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_x i\alpha_y, \alpha_z)$ is the polarization vector of the wave.

With no loss of generality, we may assume that the wave vector $\mathbf{k}$ has only two nonvanishing components, k_x and k_z. In terms of the dimensionless variables

$$t \to \omega t, \quad \mathbf{r} \to \frac{\omega}{c}\mathbf{r}, \quad \mathbf{p} \to \frac{\mathbf{p}}{mc}$$

the equations of particle motion can be reduced to the form

$$\frac{dp}{dt} = \left(1-\frac{\mathbf{kp}}{\gamma}\right)\mathrm{Re}(\vec{\mathscr{E}}e^{i\Psi}) + \frac{\omega_H}{\gamma}[\mathbf{ph}] + \frac{\mathbf{k}}{\gamma}\mathrm{Re}[(\mathbf{p}\vec{\mathscr{E}})e^{i\Psi}],$$

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\gamma}, \quad \frac{d\Psi}{dt} = \frac{\mathbf{kp}}{\gamma} - 1, \tag{2}$$

where $h = H/H_0$, $\omega_H = eH_0/mc\omega$, $\mathscr{E} = eE/mc\omega$, $\Psi = kr - t$, $\mathbf{k}$ is a unit vector in the direction of wave propagation, $\gamma = (1+p^2)^{1/2}$ is the particle energy, and $\mathbf{p}$ is its momentum.

Multiplying the first of Eqs. (2) by $\mathbf{p}$, we obtain an equation for the rate of change of particle energy:

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$$\frac{d\gamma}{dt}=\mathrm{Re}(\mathbf{v}\vec{\mathscr{E}}e^{i\psi}). \qquad (3)$$

Making use of Eq. (3), we find the following constant of the motion from the system of equations (2):

$$\mathbf{p}-\mathrm{Re}(i\mathscr{E}e^{i\psi})+\omega_H[\mathbf{rh}]-\mathbf{k}\gamma=\mathrm{const}. \qquad (4)$$

For subsequent analysis, it is convenient to transform to new variables $p_\perp$, $p_\parallel$, θ, ξ, and η defined by

$$p_x=p_\perp\cos\theta,\quad p_y=p_\perp\sin\theta,\quad p_z=p_\parallel,$$

$$x=\xi-\frac{p_\perp}{\omega}\sin\theta,\quad y=\eta+\frac{p_\perp}{\omega_H}\cos\theta. \qquad (5)$$

in terms of these variables, Eqs. (2) and (3) take the form

$$\frac{dp_\perp}{dt}=(1-k_zV_\parallel)\sum_{n=-\infty}^{\infty}\mathscr{E}_0\left(\alpha_x\frac{n}{\mu}J_n-\alpha_yJ_n'\right)\cos\theta_n$$

$$+\mathscr{E}_0k_xV_\parallel\sum_{n=-\infty}^{\infty}\frac{n}{\mu}J_n\cos\theta_n,$$

$$\frac{dp_\parallel}{dt}=\mathscr{E}_0k_zV_\perp\sum_{n=-\infty}^{\infty}\left(\alpha_x\frac{n}{\mu}J_n-\alpha_yJ_n'\right)\cos\theta_n$$

$$+\mathscr{E}_0\alpha_z\sum_{n=-\infty}^{\infty}\left(1-\frac{n\omega_H}{\gamma}\right)J_n\cos\theta_n, \qquad (6)$$

$$\frac{d\gamma}{dt}=\mathscr{E}_0\sum_{n=-\infty}^{\infty}\left(\alpha_xV_\perp\frac{n}{\mu}J_n-\alpha_yV_\perp J_n'+\alpha_zV_\parallel J_n\right)\cos\theta_n,$$

$$\frac{d\theta}{dt}=-\frac{\omega_H}{\gamma}+\mathscr{E}_0\frac{1-k_zV_\parallel}{p_\perp}\sum_{n=-\infty}^{\infty}\left(\alpha_xJ_n'-\alpha_y\frac{n}{\mu}J_n\right)\sin\theta_n$$

$$+\mathscr{E}_0\frac{k_x}{p_\perp}\sum_{n=-\infty}^{\infty}(V_\perp\alpha_yJ_n+V_\parallel\alpha_zJ_n')\sin\theta_n,$$

$$\frac{d\xi}{dt}=-\mathscr{E}_0\frac{1}{\omega_H}\alpha_y(1-k_zV_\parallel)\sum_{n=-\infty}^{\infty}J_n\sin\theta_n$$

$$-\mathscr{E}_0\frac{k_zV_\perp}{\omega_H}\alpha_y\sum_{n=-\infty}^{\infty}\frac{n}{\mu}J_n\sin\theta_n,$$

$$\frac{dz}{dt}=V_\parallel,$$

where

$$\mu=\frac{p_\perp}{\omega_H},\quad J_n=J_n(\mu),\quad J_n'=\frac{d}{d\mu}J_n(\mu),\quad \theta_n=k_zz+k_x\xi-n\theta-t.$$

Let us consider the case of a small amplitude electromagnetic wave, $\mathscr{E}_0\ll1$. The particle will then interact efficiently with the wave if it fulfills one of the resonance conditions

$$k_zV_{\parallel0}+s\frac{\omega_H}{\gamma_0}-1=0,\qquad s=0,\ \pm1,\pm2,\ldots \qquad (7)$$

Assuming that one such condition is in fact satisfied and introducing the resonant phase angle θ_s, we obtain the following equations from (6) after averaging, which describe particle motion in the case of an isolated resonance

$$\frac{dp_\perp}{dt}=\frac{1}{p_\perp}(1-k_zV_\parallel)W_s\mathscr{E}_0\cos\theta_s,$$

$$\frac{dp_\parallel}{dt}=\frac{1}{\gamma}k_zW_s\cos\theta_s,\quad \frac{d\gamma}{dt}=\frac{\mathscr{E}_0}{\gamma}W_s\cos\theta_s,$$

$$\frac{d\theta_s}{dt}=\Delta_s\equiv k_zV_\parallel+s\frac{\omega_H}{\gamma}-1. \qquad (8)$$

Here

$$W_s=\alpha_xp_\perp\frac{s}{\mu}J_s-\alpha_yp_\perp J_s'+\alpha_zp_zJ_s.$$

The third equation in this system is a consequence of the first two. Note that we have neglected terms proportional to $\Delta_s\mathscr{E}_0$ in the first two equations, and terms of order $\mathscr{E}_0$ in the last.

We shall assume that the change in particle energy resulting from its interaction with the electromagnetic field is small [($\gamma=\gamma_0+\tilde{\gamma}_s$, $|\tilde{\gamma}_s|\ll\gamma_0$, where γ_0 is the energy at which the resonance condition (7) holds]. Retaining terms to first order in $\tilde{\gamma}_s$, the last two equations in the system (8), in conjunction with the approximate integral of the motion

$$p_z-k_z\gamma=\mathrm{const}, \qquad (9)$$

yield a closed set of two equations that determine $\tilde{\gamma}_s$ and θ_s:

$$\frac{d\tilde{\gamma}_s}{dt}=\mathscr{E}_0\frac{W_s}{\gamma_0}\cos\theta_s,\qquad \frac{d\theta_s}{dt}=\frac{k_z^2-1}{\gamma_0}\tilde{\gamma}_s. \qquad (10)$$

These are the equations of a mathematical pendulum. It is then straightforward to find the width of the isolated nonlinear resonance.

$$\Delta\theta_s=4\left[\frac{(k_z^2-1)\mathscr{E}_0W_s}{\gamma_0^2}\right]^{1/2}. \qquad (11a)$$

The width of the nonlinear resonance may conveniently be expressed in energy units:

$$\Delta\tilde{\gamma}_s=4\left(\frac{\mathscr{E}_0W_s}{k_z^2-1}\right)^{1/2}. \qquad (11b)$$

The resonance conditions (7) and the approximate integral of the motion (9) give the separation between adjacent resonances

$$\delta\gamma_s=\gamma_{0,s+1}-\gamma_{0,s}=\frac{\omega_H}{1-k_z^2}. \qquad (12)$$

Equations (11) and (12) then imply that when

$$\mathscr{E}_0\geqslant\omega_H^2/16W_s(1-k_z^2),$$

the width $\Delta\tilde{\gamma}_s$ of the nonlinear resonance is greater than the separation $\delta\gamma_s$ between resonances; in other words, resonances overlap. Under such conditions, particle motion becomes quite complicated, and in fact chaotic. It should be noted that condition (13), which relates to the onset of stochastic instability of particle motion, is quite general, and holds in all the most important instances of resonant interaction between particles and electromagnetic waves. Thus, Eq. (11a) or (11b) gives the width of the nonlinear resonance that occurs for Cherenkov interaction ($s=0$) between a particle and field, and for the resonances associated with the normal ($s>0$) and anomalous ($s<0$) Doppler effects. Accordingly, Eq. (13) is the condition for the onset of stochastic instability of motion due to overlap of these resonances.

We now investigate the expression for the overlap of resonances in a number of specific instances of particle interaction with electromagnetic fields.

1.1. Consider the case of particle interaction with a longitudinal wave under conditions of Cherenkov resonance ($s=0$). For such a wave, we have $\alpha_x = k_x$, $\alpha_z = k_z$, $\alpha_y = 0$. From (13), we obtain the following criterion for the overlap of a Cherenkov resonance with its neighboring normal and anomalous Doppler resonances:

$$\mathscr{E}_0 > \frac{1}{16}\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_H \mu^{1/2}}{\gamma_0(1-k_z^2)}. \qquad (14)$$

Apart from a multiplicative factor, the condition (14) for the onset of stochastic behavior is the same as the analogous expression obtained by Shklyar.[6] The reason for the factor is that the width of the nonlinear resonance was estimated only to order of magnitude in Ref. 6.

1.2. Next, consider a transverse electromagnetic wave propagating perpendicular to an external magnetic field. In this situation, the overlap of resonances stems from a change in particle energy γ. Two cases are possible here, corresponding to different wave polarizations. For a TE-wave [polarization $\alpha = (0,1,0)$], the resonance-overlap criterion takes the form

$$\mathscr{E}_0 \gtrsim \omega_H^2/16 p_{\perp 0} J_s'(\mu) \qquad (15)$$

independent of the longitudinal velocity of the particle.

For a TM-wave (polarization $\alpha = (0,0,1)$), the resonance-overlap criterion takes the form

$$\mathscr{E}_0 \gtrsim \omega_H^2/16 p_{z0} J_s(\mu). \qquad (16)$$

In contrast to (15), the amplitude at which resonance overlap sets in depends strongly on the longitudinal momentum.

The condition (16) for the onset of stochastic behavior is approximate, inasmuch as it was derived with no regard for the interaction of nonlinear resonances. Since there exist no good analytic methods for describing stochastic components of the motion, it is natural to turn to a numerical integration of Eqs. (2). One important aspect of the numerical analysis is of course the accuracy of the results that are obtained. A check on the accuracy of solutions is available if one makes use of the constant of the motion (4), and exercises local adaptive control over the integration step size. The integral (4) can then be computed to an accuracy of better than 10^{-4}.

To determine quantitatively the degree of chaos inherent in the particle motion under the influence of the external fields (that is, to obtain a measure of the divergence of initially neighboring trajectories), we have computed the K-entropy (see Refs. 3 and 8) over the entire phase plane of initial values. As might be expected, for small-amplitude external fields the particle motion is regular and takes place inside isolated resonances, apart from certain regions near the separatrices where chaotic domains arise for any nonzero amplitude of an external wave. In fact, as can be seen from Fig. 1, the value of the K-entropy is small (close to zero) in the vicinity of both Cherenkov and Doppler resonances. At the boundaries of these domains, near the separatrices between resonances, finite-width stochastic layers are formed. Within these layers, the K-entropy is nonzero. As the external wave amplitude increases, so does the width of the stochastic layer, and at amplitudes that exceed the resonance-overlap amplitude A_*, global stochastic instability sets in.[3] Under these circumstances, the motion becomes chaotic over the entire phase domain of the Doppler resonance ($s = -1$), and the K-entropy becomes nonzero (see Fig. 2).

For motion in the vicinity of the Cherenkov resonance ($s=0$), the domain of initial phase values giving regular motion becomes progressively more restricted as the wave amplitude rises. Even when the wave amplitude exceeds the critical value by a factor of ten ($A \sim 10 A_*$), or if we use a

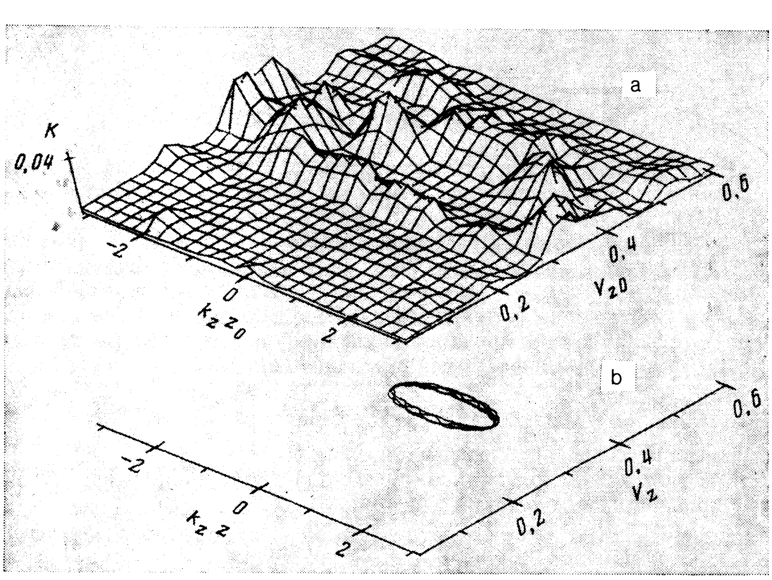

FIG. 1. (a) Distribution of K-entropy in the ($k_z z_0, V_{z0}$) phase plane of initial conditions in the case of isolated resonances; b) particle trajectory in the ($k_z z, V_z$) phase plane under the same conditions.

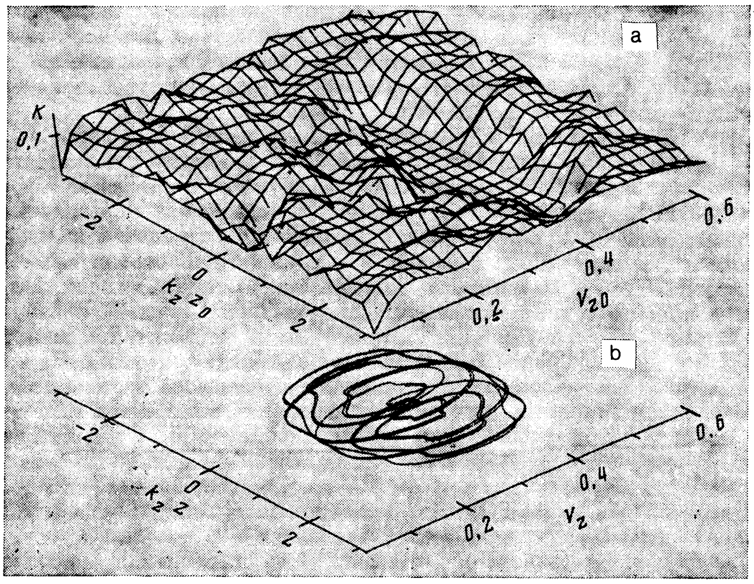

FIG. 2. a) Distribution of K-entropy in the $(k_z z_0, V_{z0})$ phase plane of initial conditions in the case of global stochastic behavior; b) particle trajectory originating in a regular region of the $(k_z z, V_z)$ phase plane under the same conditions.

more accurate value[9] of A_*, giving $A \sim 3A_*$, a domain of regular motion is retained, which confirms our previous results.[10] Regular trajectories are then not localized within the original phase domains, and over the course of time they migrate into regions of phase space from which chaotic trajectories emerge.

1.3. Let us consider a plane-polarized electromagnetic wave propagating at some angle to an external magnetic field in a medium with refractive index $n > 1$. Let the particle be described by Cherenkov resonance conditions ($s = 0$).

For a (TM-polarized wave with $\boldsymbol{\alpha} = (\cos\varphi, 0, \sin\varphi)$, where φ is the angle between the wave vector and the z-axis, the overlap criterion for the Cherenkov and adjacent cyclotron resonances may be written in the form

$$\mathscr{E}_0 \geqslant \frac{\omega_H V_{z0}}{16 J_0(\mu)\gamma_0(1-V_{z0}^2)\sin\varphi}. \tag{17}$$

We then find that as the longitudinal velocity of the particle increases, the wave amplitude required for the onset of stochastic instability of the particle motion rises. For a TE wave with $\boldsymbol{\alpha} = (0,1,0)$, the resonance overlap criterion takes the form

$$\mathscr{E}_0 \geqslant \frac{\omega_H^2 V_{z0}^2}{16(1-V_{z0}^2)p_{\perp 0}J_1(\mu)}. \tag{18}$$

In this case, the wave amplitude at which the particle motion becomes chaotic decreases with increasing transverse momentum.

2. OSCILLATOR DYNAMICS IN THE FIELD OF TWO ELECTROMAGNETIC WAVES

In the present section, we derive the criterion for the onset of stochastic instability of charged-particle motion in a constant magnetic field and in the field of two arbitrarily polarized electromagnetic plane waves. We show that conditions exist for which stochastic instability can develop for particle interactions with waves of arbitrarily small amplitude. We go on to treat examples of the use of the criterion that is derived to analyze particle motion in the field of a wave packet and in a periodic medium.

Consider the motion of a charged particle in a constant magnetic field and in a field due to two arbitrarily polarized electromagnetic waves.

$$\mathbf{E}_{1,2} = \mathrm{Re}\{A_{1,2}\boldsymbol{\alpha}_{1,2}\exp i\Psi_{1,2}\},$$

$$\mathbf{H}_{1,2} = \mathrm{Re}\left\{\frac{c}{\omega_{1,2}}A_{1,2}[\mathbf{k}_{1,2}\boldsymbol{\alpha}_{1,2}]\exp i\Psi_{1,2}\right\},$$

$$\boldsymbol{\alpha}_{1,2} = (\alpha_{x1,2}, \alpha_{y1,2}, \alpha_{z1,2}), \quad \Psi_{1,2} = \mathbf{k}_{1,2}\mathbf{r} - \omega_{1,2}t; \tag{19}$$

where the subscripts 1 and 2 refer to the first and second wave respectively, $\boldsymbol{\alpha}$ is the unit polarization vector, the z-axis points along the external magnetic field $\mathbf{H}_0$, $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector, and ω and A are the wave frequency and amplitude.

The basic assumption that we make in deriving the criterion for stochastic motion of a particle in the field of two electromagnetic waves is that at small wave amplitudes, the particle interacts resonantly with one of the waves. This assumption then enables us to make use of certain of the results of Sec. 1.

As in the case of a single wave (Sec. 1), it is a straightforward matter to derive the equations describing the change in oscillator energy upon interaction with one of the two waves:

$$\frac{d\gamma_{1,2}}{dt} = \mathscr{E}_{1,2}\,\mathrm{Re}[W_s^{(1,2)}\exp(i\theta_s^{(1,2)})],$$

$$\frac{d\theta_s^{(1,2)}}{dt} = \Delta_s^{(1,2)}, \tag{20}$$

where

$$\Delta_s^{(1,2)} = k_{z\,1,2}V_z + s\omega_{H\,1,2}/\gamma - 1, \quad \omega_{H\,1,2} = eH_0/mc\omega_{1,2},$$

and $W_s^{(1,2)}$ takes the form

$$W_s^{(1,2)} = V_\perp\cos\beta_{1,2}\left[\alpha_{x\,1,2}\frac{s}{\mu_{1,2}}J_s(\mu_{1,2}) - \alpha_{y\,1,2}J'(\mu_{1,2})\right]$$

$$+ iV_\perp\sin\beta_{1,2}\left[\alpha_{y\,1,2}\frac{s}{\mu_{1,2}}J_s(\mu_{1,2}) - \alpha_{x\,1,2}J_s'(\mu_{1,2})\right]$$

$$+ \alpha_{z\,1,2}V_zJ_s(\mu_{1,2}). \tag{21}$$

Here $\mu_{1,2} = k_{\perp 1,2} p_\perp / \omega_{H1,2}$, $k_{\perp 1,2}$ is the projection of the wave vector on the x-y plane, $\beta_{1,2}$ is the angle between $\mathbf{k}_{1,2}$ and the x-axis, and $J_s(x)$ is the Bessel function of order s.

Notice that Eq. (21) for $W_s^{(1,2)}$ is a generalization of the analogous expression (8) to the case of a wave propagating in an arbitrary direction. Subscripts 1 and 2 in Eq. (20) refer to two different cases: 1) the particle lies within an isolated nonlinear resonance $\gamma = k_{z1} p_z + s\omega_{H1}$ with the first wave, and in accordance with the assumption we have made, its interaction with this wave is independent of the presence of the second wave; 2) the particle, situated at a resonance $\gamma = k_{z2} p_z + s\omega_{H2}$, interacts only with the second wave.

Making use of the approximate integral of the motion

$$k_{z1,2}\gamma - p_z = C_{1,2}, \tag{22}$$

which is valid within any isolated resonance (as in Sec. 1), we obtain an expression for the halfwidth of the nonlinear resonance:

$$\Delta\gamma_{1,2} = 2\left(\mathcal{E}_{1,2}\gamma_{1,2}\left|\frac{W_s^{(1,2)}}{k_{z1,2}^2 - 1}\right|\right)^{1/2}. \tag{23}$$

Let us now find the energy difference between resonances. For definiteness, we assume that a particle with initial energy γ_1 and longitudinal momentum p_{z1} is in resonance with the first wave; that is, it lies at the point A on the straight line 1 in Fig. 3, the latter being defined by the resonance condition

$$\gamma = k_{z1} p_z + s\omega_{H1}. \tag{24}$$

Particle motion is then consistent with the constant of the motion,

$$k_{z1}\gamma - p_z = k_{z1}\gamma_1 - p_{z1}. \tag{25}$$

An increase in the wave amplitudes results in an increase in the resonance widths (the shaded areas in Fig. 3) and, for certain wave amplitudes, an overlap between these regions results in the particle crossing over into the resonance region of the second particle.

$$\gamma = k_{z2} p_z + n\omega_{H2} \tag{26}$$

(line 2 in Fig. 3), and then moving along line 4—that is, it conforms to the constant of the motion

$$k_{z2}\gamma - p_z = C_2, \tag{27}$$

which holds in the resonance region (26). This momentum will also correspond to over-lap of the resonances (24) and (26).

The problem may be stated as follows: knowing γ_1 and p_{z1} (the initial state of the particle), find γ_2 and p_{z2}—that is, the coordinates of point B (Fig. 3). To do so, it is necessary to determine the constant C_2 in the integral (27); then, using the resonance condition (26) and the constant of the motion (27), it is straightforward to find γ_2, and consequently the energy difference $\delta\gamma = |\gamma_1 - \gamma_2|$ between resonances.

From (25) and (27), we find that at point C (Fig. 3),

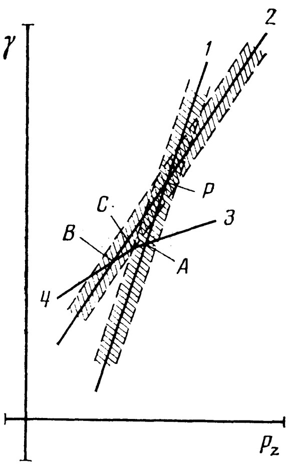

FIG. 3. Resonance lines for particle motion in the field of two electromagnetic waves.

$$(k_{z2} - k_{z1})\gamma_C = C_2 - k_{z1}\gamma_1 - p_{z1}.$$

Noting that $\gamma_C = \gamma_1 - \Delta\gamma_1$ [where $\Delta\gamma_1$ is the halfwidth of the resonances (24)], we find the constant C_2:

$$C_2 = k_{z2}\gamma_1 - p_{z1} - (k_{z2} - k_{z1})\Delta\gamma_1. \tag{28}$$

Making use of (26) and (27) with the constant (28), we may determine γ_2:

$$\gamma_2 = \frac{k_{z2} C_2 - n\omega_{H2}}{k_{z2}^2 - 1}. \tag{29}$$

The final separation in energy between resonances takes the form

$$\delta\gamma = \gamma_1 - \frac{k_{z2}^2\gamma_1 - k_{z2} p_{z1} - k_{z2} - \Delta\gamma_1 (k_{z2} - k_{z1}) - n\omega_{H2}}{k_{z2}^2 - 1}. \tag{30}$$

Clearly, resonances will overlap if the sum of the halfwidths of nonlinear resonances exceeds the distance between them:

$$\Delta\gamma_1 + \Delta\gamma_2 \geqslant \delta\gamma. \tag{31}$$

Some simple manipulation gives the following criterion for overlap of resonances:

$$\left|k_{z1}\Delta\gamma_1 + k_{z2}\Delta\gamma_2 - \frac{\Delta\gamma_1 + \Delta\gamma_2}{k_{z2}}\right|$$
$$\geqslant \left|\frac{n\omega_{H2}}{k_{z2}} - \frac{s\omega_{H1}}{k_{z1}} + \gamma_1\frac{k_{z2} - k_{z1}}{k_{z1}k_{z2}}\right|. \tag{32}$$

Note that the prerequisite for stochastic instability of particle motion in the field of two electromagnetic waves takes the form (32) not just for the example depicted in Fig. 3 (two slow waves), but for other combinations of waves (two fast waves, a fast and a slow wave) and positions of point A relative to the intersection of the resonance lines as well.

Condition (32) is quite universally applicable. Since for $\mathbf{k}_1 = \mathbf{k}_2$, $\omega_{H1} = \omega_{H2}$, $n = s + 1$, the criterion (13) for stochastic motion of an oscillator in the field of a single wave follows directly.

We see from (32) that when

$$\frac{n\omega_{H2}}{k_{z2}}-\frac{s\omega_{H1}}{k_{z1}}+\frac{k_{z2}-k_{z1}}{k_{z1}k_{z2}}\gamma_1=0 \tag{33}$$

overlap sets in at arbitrarily small wave amplitudes. Equation (33) corresponds to the point at which the resonance lines in Fig. 3 intersect.

As examples of the use of (32), let us examine the motion of a charged particle in the field of a wave packet and in a periodic medium.

2.1. Charged-particle motion in the field of a wave packet

Consider a charged particle with $V_\perp = 0$, moving in the field of a wave packet

$$E_z = \mathrm{Re}\sum_k E_k \exp(ikz-i\omega t) \tag{34}$$

with a characteristic separation Δk between wave vectors ($\Delta k \ll k$). The particle will then interact with the wave field under resonance conditions,

$$\omega(k)=kV, \tag{35}$$

and the criterion (32) for stochastic instability will take the form

$$2|(k_1^2-1)\Delta\gamma| \gtrsim \gamma_1 \Delta k/(k_1k_2). \tag{36}$$

Taking $k_{1,2} = k_{1,2}c/\omega_{1,2}$, $k_2 - k_1 = \Delta k$ and $\Delta\omega = \Delta k(\partial\omega/\partial k)$, we obtain the following condition for the onset of stochastic instability:

$$\left(\frac{eE_k}{m}k\right)^{1/2} \gtrsim \frac{\gamma^{2/3}}{4}\left|\frac{\partial\omega}{\partial k}-V\right|\Delta k, \tag{37}$$

which at low energies is the same as the criterion derived in Ref. 2. Note that as the particle energy rises, so does the field amplitude at which stochastic instability sets in.

22. Criterion for stochastic motion of a charged particle in a periodic medium

We may represent the field of a wave in a spatially periodic medium as an expansion in plane waves.

$$E=\sum_n E_n \exp[i\omega t+i(k+n\varkappa_d)z], \tag{38}$$

where $\varkappa_d = 2\pi/d$, d is the spatial period of the inhomogeneity, and the E_h are the amplitudes of the spatial harmonics, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

For a weakly inhomogeneous medium with dielectric constant

$$\varepsilon=\varepsilon_0(1+a\cos\varkappa_d z),$$

where $a \ll 1$, the amplitudes of harmonics quickly drop off with increasing n; that is,

$$E_n \sim E_0 a^n \ll E_0, \quad n \neq 0.$$

If we then consider the motion of a charged particle with $V_\perp = 0$, which is initially at the resonance $\omega = kV$ corresponding to the fundamental frequency of oscillation ($n = 0$), we obtain the following criterion for stochastic particle motion in a periodic medium:

$$\left(\frac{eE_0}{m}k\right)^{1/2} = \omega_B \gtrsim \frac{\gamma^{3/2}}{2}\frac{\varkappa_d}{k}\omega. \tag{39}$$

The nature of the motion of a charged particle in a wave field excited by a beam of relativistic charged particles moving in a periodic medium is of practical interest. This is the way in which electromagnetic waves are excited in certain kinds of free-electron lasers. The basic radiation mechanism consists of parametric Cherenkov radiation resulting from charged-particle motion in a periodic medium; the wavelength excited by the charged particles is then

$$\lambda=2\pi/k=d/2\gamma^2.$$

The wave amplitude E_0 may be estimated by assuming that the wave field captures particles from the beam, i.e., that $\mathrm{Im}\,\omega \sim \omega_B$, where $\mathrm{Im}\,\omega$ is the differential amount by which oscillations excited by the beam grow. The criterion (39) for the onset of stochastic particle motion then becomes

$$\frac{\mathrm{Im}\,\omega}{\omega} \gtrsim \frac{1}{4\gamma^{1/2}}. \tag{40}$$

Thus, for growth rates satisfying (40), particle motion in the field of a wave excited by these particles becomes stochastic. In actual fact, however, this does not happen, as the growth rates of a wave excited by a beam in a spatially periodic medium turn out to be rather small.

$$\frac{\mathrm{Im}\,\omega}{\omega} \sim \frac{\alpha}{\gamma^\nu},$$

where $\upsilon \geqslant 1/2$, and α is proportional to the beam density to a certain extent. As a rule, $\alpha \ll 1$, and stochastic instability therefore fails to develop when there is particle motion in a periodic medium.

3. CHARGED-PARTICLE DYNAMICS IN THE FIELD OF TWO LONGITUDINAL WAVES

In the previous section, we demonstrated that a particle moving in the field of two longitudinal waves may be captured by one of them. Such a situation is possible when the phase velocities of the wave differ significantly. In that event, a nonresonant wave may affect particle motion in such a way that a stochastic layer arises in the phase plane in the vicinity of the separatrix that distinguishes capture trajectories from pass-through trajectories.[2–4,11] As the wave amplitude increases (and the phase velocities approach one another), the nonlinear resonances corresponding to each of the waves also approach, and ultimately start to overlap. Under these circumstances, particle motion becomes chaotic.

Below we point out that during particle motion in the field of two longitudinal waves, there appear—in addition to Cherenkov resonances—a set of parametric resonances, whose overlap also leads to the onset of chaotic particle motion. The wave amplitudes required for the overlap of the parametric resonances may then be substantially lower than for the Cherenkov resonances.

The equation describing particle motion in the field of two longitudinal waves is of the form

$$\ddot{z}=-\frac{e}{m}E_1\sin(k_1z-\omega_1t)-\frac{e}{m}E_2\sin(k_2z-\omega_2t), \quad (41)$$

where e and m are the charge and mass of the particle, $k_{1,2}$ and $\omega_{1,2}$ are the longitudinal wave numbers and frequencies of the longitudinal waves, and $E_{1,2}$ represents the wave amplitude.

It proves convenient to transform (4) using the new variables

$$\zeta=k_1z-\omega_1t, \quad \tau=k_2z-\omega_2t. \quad (42)$$

Equation (41) then takes the form

$$\left(1-q\frac{d\zeta}{d\tau}\right)^{-3}\frac{d^2\zeta}{d\tau^2}=-\mathscr{E}_1\sin\zeta-\mathscr{E}_2\sin\tau, \quad (43)$$

where

$$q=k_2/k_1, \quad \mathscr{E}_{1,2}=\frac{ek_1}{mk_2^2\Delta^2}E_{1,2}, \quad \Delta=\frac{\omega_1}{k_1}-\frac{\omega_2}{k_2}.$$

To further simplify the derivation, we assume that

$$\left|q\frac{d\zeta}{d\tau}\right|\ll 1. \quad (44)$$

Estimating a typical value of $d\zeta/d\tau$ to be the width of the separatrix $2\mathscr{E}_1^{1/2}$ corresponding to the nonlinear resonance of the particle in the field of wave number 1, we may rewrite (44) in the form

$$2q\mathscr{E}_1^{1/2}\ll 1. \quad (45)$$

Over the domain of parameters for which (45) is satisfied, the particle motion is described by the pendulum equation, with an external harmonic driving force at unit frequency:

$$\frac{d^2\zeta}{d\tau^2}+\mathscr{E}_1\sin\zeta=-\mathscr{E}_2\sin\tau. \quad (46)$$

With the replacement

$$\zeta=\tilde{\zeta}+\mathscr{E}_2\sin\tau, \quad (47)$$

Eq. (46) in turn can be reduced to the form

$$\frac{d^2\tilde{\zeta}}{d\tau^2}+\mathscr{E}_1\sum_{s=-\infty}^{\infty}J_s(\mathscr{E}_2)\sin(\tilde{\zeta}+s\tau)=0. \quad (48)$$

Each term of the sum in (48) corresponds to a nonlinear resonance

$$d\tilde{\zeta}/d\tau=-s. \quad (49)$$

When the overlap condition

$$2\mathscr{E}_1^{1/2}\{[J_s(\mathscr{E}_2)]^{1/2}+[J_{s+1}(\mathscr{E}_2)]^{1/2}\}\geqslant 1 \quad (50)$$

for parametric resonances is satisfied, the particle motion becomes stochastic. A numerical solution of Eq. (46) confirms the validity of the criterion for a transition to stochastic particle motion. The resonance condition (49) may be rewritten as

$$\omega_1-k_1V=s(\omega_2-k_2V). \quad (51)$$

The set of resonances (51) corresponds to plasma-wave scattering by particles. It is clear from a comparison of the condition (50) for overlap of parametric resonances with the analogous condition for Cherenkov resonances.

$$2q[\mathscr{E}_1^{1/2}+(\mathscr{E}_2/q)^{1/2}]>1,$$

that when $\mathscr{E}_1-\mathscr{E}_2$ in the first case, overlap sets in at smaller values of the wave amplitude.

4. MOTION OF AN ENSEMBLE OF OSCILLATORS IN THE FIELD OF AN ELECTROMAGNETIC WAVE

As stochastic instability develops, particle trajectories become exceedingly complicated, and are only amenable to study by numerical methods. Sometimes, however, this very complexity makes it possible to simplify the problem considerably by utilizing the methods of statistical physics. To illustrate this point, consider the evolution of the distribution function for an ensemble of oscillators in a constant magnetic field H_0 that points in the z-direction, and in the field of an arbitrarily polarized external electromagnetic wave. Mutual interactions of particles and wave excitation by particles will be neglected (low-density beam approximation). Under these conditions, the problem of the motion of the ensemble reduces to a one-particle problem. The criterion for the onset of stochastic instability is given for each particle by (13); we shall assume that the electric field strength of the external field satisfies this condition. In order to study the diffusion in energy of the particles belonging to the ensemble, we use Eqs. (8), which may conveniently be rewritten as

$$\dot{\gamma}=\mathscr{E}_0\sum W_n\cos\theta_n/\gamma, \quad \theta_n=k_zV_z+n\omega_H/\gamma-1. \quad (52)$$

Since stochastic instability will have set in, we may assume that the phases of the resonances are random and independent. Bearing in mind that the terms on the right-hand side of the first of Eqs. (52) are of small amplitude, we may substitute the unperturbed values of the variables. Then the first equation in (52) yields the following expression for the correlation function:

$$\mathscr{K}(\tau)\equiv\langle\dot{\gamma}(t+\tau)\dot{\gamma}(t)\rangle=\frac{1}{2}(\mathscr{E}_0/\gamma)^2$$
$$\times\mathrm{Re}\left\{\exp[i\tau(k_zV_0-1)]\sum W_s^2\exp(is\omega_H\tau/\gamma)\right\}, \quad (53)$$

where

$$\langle\gamma\rangle=\int_0^{2\pi}\frac{dk_z}{2\pi}z_0\frac{1}{2\pi}\int_0^{2\pi}d\varphi_0\,\gamma.$$

Using the addition theorem for cylindrical functions,[14] we can expand the sum on the right-hand side of (53) to yield

$$\mathscr{K}(\tau)=\frac{1}{2}\left(\frac{\mathscr{E}_0}{\gamma}\right)^2\mathrm{Re}\left\{e^{i(k_zV_0-1)\tau}\left[(\alpha_+^2e^{-i\omega_H\tau/\gamma}+\alpha_-^2e^{i\omega_H\tau/\gamma})\frac{p_\perp^2}{4}J_0(\Pi)+\alpha_z^2p_\parallel^2J_0(\Pi)+\frac{1}{2}p_\perp^2\alpha_+\alpha_-\varkappa e^{i\omega_H\tau/\gamma}J_0(\Pi)+\frac{1}{2}p_\perp p_\parallel\alpha_zJ_1(\Pi)\left(\frac{\alpha_+\varkappa+\alpha_-}{\varkappa^{1/2}}\right)\right]\right\}, \quad (54)$$

where

$$\alpha_\pm \equiv \alpha_x \pm \alpha_y, \quad \varkappa \equiv (1-e^{-i\omega_H\tau/\gamma})/(1-e^{i\omega_H\tau/\gamma}),$$
$$\Pi \equiv k_x p_\perp \sin(\omega_H\tau/2\gamma)/2^{1/2}\omega_H.$$

Making use of (54), we can easily find the variance $\sigma^2 \equiv \langle (\Delta\gamma)^2 \rangle$. Indeed Eq. (52) gives

$$\Delta\gamma = \int_0^t \dot{\gamma}(t')\,dt', \tag{55}$$

so for the variance we have

$$\sigma^2 = 2\int_0^t d\tau\,(t-\tau)\mathscr{K}(\tau). \tag{56}$$

At small times

$$t \ll t_0 \ll \min\left[\frac{\gamma}{\omega_H},\ (k_z V_0 - 1)^{-1},\ 2^{1/2}\gamma/k_x p_\perp\right]$$

the quantity $\mathscr{K}(\tau)$ can be treated as a constant, yielding a quadratic dependence for the variance:

$$\sigma^2 = \mathscr{K}(0)t^2. \tag{57}$$

At large times, the main contribution to the integral in (56) comes from values of τ less than t_0. We then obtain

$$\sigma^2 \approx 2\mathscr{K}(0)t_0 t. \tag{58}$$

If the wave propagation is strictly perpendicular to the external magnetic field ($k_2 = 0$, $p_2 = 0$), then only cyclotron resonances are possible, and their overlap is governed by relativistic effects.[12,15] Equation (54) for the correlation function then simplifies to a form that agrees with the one derived in Ref. 15. In that same paper, we demonstrated good agreement between the analytic results and numerical calculations.

Note that according to the numerical results obtained in Ref. 15, the mechanism considered here for particle interactions with a field can be an efficient means of heating and accelerating charged particles—the mean particle energy, in a time of order 100 periods, increased from $\langle\gamma\rangle = 2$ to $\langle\gamma\rangle = 5$.

CONCLUSION

To summarize, we have obtained the most general condition (32) for the onset of stochastic instability in the motion of a charged particle undergoing interaction with an electromagnetic wave. That condition is applicable to all known resonances, namely Cherenkov, cyclotron, normal and anomalous Doppler cyclotron, and parametric resonance. All existing conditions for the onset of local instability may be derived from our condition as special cases.

One important assumption used in the derivation of (32) should be pointed out; it is clear from the derivation that the presence of a second wave did not change the conditions for resonant interaction of a particle with one of the waves. That is, the existence of new resonances due to the presence of a second wave (these might be called parametric resonances) was not taken into consideration.

Condition (32) suggests the existence of a number of novel results in addition to those that are already known. We have already commented on some of these above. Here were point out only two of the most interesting. In the (γ, p_z) plane, the resonance lines give rise to a grid of resonances. Both along the resonance lines and at their intersection points, stochastic instability can develop at arbitrarily small electromagnetic wave amplitudes, and this can then result in particle diffusion. Constraints on this diffusion can be derived from the constant of the motion (22). In fact, when the constant of the motion (22) is integrated over some short time interval, it will also give rise to a straight line in the (γ, p_z) plane. In general, the straight lines due to resonances and those due to the integrals will not be the same. They will, however, coincide under conditions of self-resonance, and it is precisely near self-resonance that diffusion will be most efficient.

Self-resonance conditions are of interest for other reasons as well. As we have seen above, the distance between resonances is determined along the lines controlled by the constants of the motion. Therefore, as a system approaches self-resonance, this distance increases, and larger and larger electromagnetic wave amplitudes are required for stochastic instability to develop.

The authors thank Ya. B. Fainberg his stimulating interest in this work and for fruitful discussions of our results.

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Translated by Marc Damashek