Low-frequency conductivity of a plasma in a stochastic magnetic field

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The low-frequency longitudinal-conductivity operator for a plasma in a stochastic magnetic field is obtained in a linear approximation. It is shown that the connection between the electric field and the current is essentially nonlocal. Models of a stochastic magnetic field in unbounded space and in a bounded layer are considered. They are used to obtain expressions for the plasma conductivity in the low-collision-frequency regime. A procedure for renormalizing the conductivity is introduced. The expressions obtained for the conductivity operator are used to determine the penetration of a field **E** and of a current **j**, applied externally, into a plasma. It is shown that the $\mathbf{E}(x,t)$ and $\mathbf{j}(x,t)$ distributions in a plasma with a stochastic magnetic field can differ greatly from the corresponding distribution in the case of the classical skin effect.

1. INTRODUCTION

One characteristic feature of a plasma confined by a magnetic field is its anomalously high thermal conductivity, which suggests that the electrons may straggle randomly through the confining field. This straggling is characterized by a diffusion coefficient of the order of the electron thermal diffusivity $\varkappa$. During the time ν^{-1} of an electron-ion collision the electron is displaced a distance of order $l = (\varkappa/\nu)^{1/2}$. Clearly, at such distances the connection of the low-frequency ($\omega < \nu$) field with the current is nonlocal. Moreover, anomalous mobility decreases the magnetization of the electrons and increases thereby their contribution to the transverse components of the dielectric tensor. These effects can manifest themselves in time-dependent processes, in which strongly inhomogeneous fields can be generated, such as in the onset of MHD instabilities in a plasma, in experiments on rapid increase of the current,[1,2] etc.

It is assumed in the present paper that the cause of the random straying of electrons across the confining field is stochastic destruction of magnetic surfaces. In other words, we assume the magnetic field in which the plasma is located to have a random component $\delta\mathbf{B}$ that leads to spatial diffusion of the force lines,[10] and consider quasistationary fields in such a medium. In the linear approximation, these fields are described by the equations

$$\operatorname{rot}\mathbf{E}^{\mathrm{M}} = -\frac{1}{c}\frac{\partial\mathbf{B}^{\mathrm{M}}}{\partial t}, \quad \operatorname{rot}\mathbf{B}^{\mathrm{M}} = \frac{4\pi}{c}\mathbf{j}^{\mathrm{M}}, \qquad (1)$$
$$\operatorname{div}\mathbf{j}^{\mathrm{M}}=0, \quad \mathbf{j}^{\mathrm{M}}=\hat{\sigma}^{\mathrm{M}}\mathbf{E}^{\mathrm{M}},$$

where the superscript "M" labels microscopic (non-averaged) quantities. The operator $\hat{\sigma}^{\mathrm{M}}$ depends on $\delta\mathbf{B}$ and is consequently stochastic. We assume that the spatial inhomogeneity of the medium (transverse to the average magnetic field), which is connected with this dependence, is microscopic: our aim is to derive equations for the macroscopic fields **E** and **B**, i.e., the fields averaged over a physical infinitesimal volume. Under the usual ergodicity assumptions this averaging is equivalent to a statistical one, so that

$$\mathbf{E}=\langle\mathbf{E}^{\mathrm{M}}\rangle_B, \quad \mathbf{j}=\langle\sigma^{\mathrm{M}}\mathbf{E}^{\mathrm{M}}\rangle_B, \quad \mathbf{E}^{\mathrm{M}}=\mathbf{E}+\delta\mathbf{E}, \qquad (2)$$

where $\langle\,\rangle_B$ denotes ensemble-averaging of realizations of the random field $\delta\mathbf{B}$. The desired equations are obtained by averaging Eq. (1). To close them we must express **j** as a functional of **E**. If the microscopic Ohm's law is linear this functional is also linear: $\mathbf{j} = \hat{\sigma}\mathbf{E}$, where $\hat{\sigma}$ is some new operator. Our aim in the present paper is to calculate the renormalized (macroscopic) conductivity $\hat{\sigma}$ in the low-frequency limit $\omega < \nu$. By way of illustration, we consider the skin effect in a plasma placed in a magnetic field with weakly disturbed surfaces.

2. MODEL OF STOCHASTIC MAGNETIC FIELD

We formulate as a preliminary the basic assumptions concerning the statistical properties of the magnetic field in which the plasma is placed. In the case of an infinite medium, we represent **B** by

$$\mathbf{B}(\mathbf{r})=\mathbf{B}_0+\delta\mathbf{B}(\mathbf{r}), \qquad (3)$$

where $\mathbf{B}_0$ is a constant vector along the z axis and $\delta\mathbf{B}(\mathbf{r})$ is a random vector in the xy plane; it is assumed for the time being that **B** is independent of time. We shall assume the random field $\delta\mathbf{B}$ to be statistically uniform and isotropic; it is then characterized by two correlation lengths, transverse $\delta_\perp$ and longitudinal $\delta_\parallel$. For tokamaks it is customary to use the estimate $\delta_\parallel \sim qR_0$, where q is the safety factor and R_0 is the major radius of the tokamak, i.e., $\delta_\parallel \sim 10^3$ cm. The value assumed for $\delta_\perp$ in theoretical papers is of order $c/\omega_{pe} \sim 0.1$ cm (Refs. 3 and 4), where ω_{pe} is the plasma frequency. According to experiment,[5,6,9] $\delta_\perp \sim 0.1 - 1$ cm.

It is phenomenologically convenient to express the field lines of the random field (3) in Lagrangian variables, with the coordinates x and y of the point **r** of a certain line regarded as a random function of the length s of an arc of this line, measured from an arbitrary initial point. We shall need below the distribution functions of the coordinates $x'(s')$ and $y'(s')$ of the points of a field line passing through a fixed stationary point $r(s)$, i.e., the conditional probability density $W(r', s'/r,s)$. In accord with the assumptions made concerning the statistical properties of the magnetic field, we have

$$W(x', y', s'/x, y, s)=\prod_{\alpha=x,y} W_0(\Delta_\alpha/\Delta), \qquad (4)$$

where $\Delta_\alpha = r'_\alpha - r_\alpha$, $\Delta = s' - s$. For $|\Delta| \gg \delta$, the field lines diffuse, i.e.,

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$$W_0(\Delta_\alpha/\Delta)=[4\pi D_m|\Delta|]^{-1/2}\exp[-\Delta_\alpha^2/4D_m|\Delta|], \qquad (5)$$

where D_m is the diffusion coefficient of the field lines. (The condition $D_m \neq 0$ can serve as a definition of the stochasticity of the field). In a tokamak plasma under quiescent conditions, $D_m \sim 10^{-4}$–10^{-7} cm.[6,9] Henceforth we assume $\delta_\parallel$ to be the shortest of the characteristic longitudinal lengths, i.e., we use in fact the limit $\delta_\parallel = 0$ and correspondingly always use for W_0 its asymptotic value (5).

An important property of a stochastic field is an exponential divergence of the field lines. The corresponding length L_k depends on the intensity $|\delta B/B|^2$ of the magnetic fluctuations. In the magnetic-turbulence case which we have in mind[6] we have $L_k \gg \delta_\parallel$.

In addition to an unbounded medium we consider also a stochastic layer—a situation in which the field lines diffuse only in the region $0 \leqslant x \leqslant a$, outside of which $D_m = 0$. (Similar layers are produced, for example, when magnetic islands overlap). There is at present no information that permits the form of the function $W(x',s'/x,s)$ to be determined for this case [it is assumed that the distribution of y' is given as before by Eq. (5)]. Some general properties of this function can, however, be indicated. For small Δ, such that $D_m|\Delta| \ll a^2$, the field line should diffuse "freely," and D_m can be a function of x. In the opposite limit, $D_m|\Delta| \gg a^2$, the probability distribution should tend to a certain limiting function independent of x' and s'.

We use for $W(x',s'/x,s)$ a model function satisfying the conditions

$$W(x',s'/x,s)=\sum_{n=-\infty}^{+\infty}\{W_0[(x'-x-2na)/\Delta] + W_0[(x'+x-2na)/\Delta]\}, \qquad (6)$$

where W_0 is defined by Eq. (5), $0 \leqslant x$, and $x' \leqslant a$. If Δ is small, only the first term with $n=0$ in the first addend is significant; as $\Delta \to \infty$, W tends to the limit $1/a$ inside the layer. This means that the force lines fill the layer ergodically.

The function (6) describes the model of a layer in which the statistical properties of the field $\delta\mathbf{B}$ are the same as assumed above for an unbounded medium, but any force line that reaches the layer boundary $x'=0$ or $x'=a$ is specularly reflected from it in the xz plane.

Stochastic magnetic fields can be produced by superimposing on the regular component stationary perturbation from external sources (as in the case of a stochastic magnetic diaphragm). In this case $\delta\mathbf{B}(r)$ is independent of time. A different situation is realized when the field component $\delta\mathbf{B}$ is the result of development of low-frequency turbulence and is a random function of the time. A broad spectrum (from 10^3 to 10^6 Hz) of magnetic-field fluctuations is observed in many experiments. The role of the oscillation depends substantially on whether the plasma motion that accompanies them is frozen-in or not. In the former case the oscillations do not change the topology of the field lines, i.e., from the phenomenological viewpoint they do not alter the realization of the random magnetic field, but cause only some "wiggling" of this realization. More significant are oscillations for which the freezing-in condition is not met, since they lead to a restructuring of the magnetic field. In first order, taking only this process into account, we can assume that the characteristic frequencies Ω_B of $\delta\mathbf{B}$ are bounded by the condition

$$\Omega_B < c^2/\sigma_0\delta_\perp^2 \approx \nu c^2/\omega_{pe}^2\delta_\perp^2 \quad (\sigma_0=\omega_{pe}^2/4\pi\nu), \qquad (7)$$

i.e., $\Omega_B \lesssim \nu$. For stationary turbulence, Eqs. (4) and (5) remain in effect also if $\delta\mathbf{B}=\delta\mathbf{B}(t)$. Noted that the familiar difficulty raised by the impossibility of uniquely describing the field-line motion in time-dependent fields does not arise in this case, since we always have in mind a field line passing through a certain fixed point r.

Let us assess now the character of the electron motion in a stochastic field for large mean free paths $\lambda > \delta_\parallel$. At each instant of time the electron moves practically along a magnetic field line, but undergoes a slow displacement $\delta(t)$ in a direction perpendicular to the field $\mathbf{B}$. This displacement can be due to collisions or drift, but is assumed to be slow enough to neglect its direct contribution to the drift across the magnetic field $\mathbf{B}_0$. The time interval t_0 during which the displacement remains small compared with $\delta_\perp$ can be called the lifetime of the particle on the field line. When t_0 is determined, account must be taken of the exponential enhancement of the transverse displacement by the stochastic divergence of the field lines; t_0 can then be estimated from the relation

$$d_0\exp[L_\parallel(t_0)/L_k]=\delta_\perp,\ d_0=\langle|\delta(t_0)|\rangle_B, \qquad (8)$$

where $L_\parallel(t)$ is the characteristic value of the particle displacement along the line during a time t. For $\nu t_0 \gg 1$, the longitudinal motion has the character of longitudinal diffusion and $L_\parallel(t)=(D_\parallel t)^{1/2}$, with a longitudinal diffusion coefficient $D_\parallel = v_{Te}^2/\nu$. We have then to within a logarithmic factor

$$\nu t_0 = L_k^2/\lambda^2,\ \lambda=v_{Te}/\nu.$$

An important role is played in what follows by the statistical characteristics of the particle trajectories $R(t,r)$ that pass through a fixed point r at an instant of time $t=0$. In an approximation in which the Larmor radius is zero, it can be assumed, if $\nu t_0 \gg 1$ that over times short compared with t_0 the particle moves along a field line, i.e.,

$$R(t,r)=r'[s'(t)],\ s'=s+\Delta(t),\ \Delta(0)=0. \qquad (9)$$

If $|\delta\mathbf{B}| \ll |\mathbf{B}_0|$ the influence of the inhomogeneity of the field (3) on the longitudinal motion can be neglected. The particle displacement $\Delta(t)$ along the field line is then determined only by its longitudinal velocity $v(t)$ (with collisions taken into account), but is independent of $\delta\mathbf{B}$.

The sign of the longitudinal electron velocity is reversed many times by collisions over a time of order t_0. The trajectories (9) pass therefore many times through a region of space in which the values of $\delta\mathbf{B}$ are statistically correlated. This regime was first considered by Rosenbluth and Rechester.[7] It must be assumed in this regime that $\Omega_B t_0 < 1$, since the change of the magnetic field itself can impose a limit on t_0.

As the mean free path λ increases, the coupling of the electron to a definite field line decreases; in the limit $\lambda \gg L_k$ a move of the particle from one line to another can cause practically the entire trajectory to pass through field regions that are statistically independent of one another. Yet the correlation between the longitudinal and transverse motions is preserved here: since we have $\delta_\parallel \ll \lambda$, the transverse (to the mean field $\mathbf{B}_0$) displacement of the particle constitutes diffusion with an instantaneous diffusion coefficient $D_\perp(t) = D_m |v(t)|$, where $v(t)$ is the longitudinal velocity. The statistical trajectory characteristics that follow from such a picture can be adequately described by the following model: the electron moves strictly along a field line, but its displacement formally has the form

$$\Delta(t) = \mathrm{sign}[v(0)] \int_0^t |v(t')|\,dt'. \tag{10}$$

Since the integral contains the absolute value of the velocity, the particle does not return after stopping (after v becomes equal to zero) along its preceding path but, moving in the same direction as before, reaches the next section of the field line, an event statistically indistinguishable from a move to "another" field line in real motion. It is readily verified that the expression given above for $D_\perp(t)$ also holds in this model.

3. LOW-FREQUENCY LONGITUDINAL CONDUCTIVITY OF A PLASMA IN A STOCHASTIC MAGNETIC FIELD

We proceed now directly to calculate the plasma conductivity and consider first the part σ_e due to the electron contribution. In a sufficiently strong field it can be assumed that

$$\mathbf{j}_e{}^{\mathrm{M}} = \mathbf{b} j_\parallel{}^{\mathrm{M}}, \quad \mathbf{b} = \mathbf{B}/|\mathbf{B}|, \tag{11}$$

the subscript "e" of $j_e{}^{\mathrm{M}}$ will be omitted, since the ion contribution to this current component can be neglected. We have here

$$j_\alpha = \langle b_\alpha j_\parallel{}^{\mathrm{M}} \rangle_B, \quad \alpha = x, y, z. \tag{12}$$

At long mean free paths, $\lambda \gg \delta_\parallel$, it is tempting to represent $j_\parallel{}^{\mathrm{M}}$ by a path integral. This can be done by neglecting electron–electron collisions. In this case the electrons do not interact with one another, and the collisions with ions can be regarded as motion in specified random external field. We assume initially that the magnetic field $\delta\mathbf{B}$ is stationary and that the time dependence of $\mathbf{E}$ and $\mathbf{j}$ is given by

$$\mathbf{E} = \mathbf{E}(r)\exp(-i\omega t), \quad \mathbf{j} = \mathbf{j}(r)\exp(-i\omega t). \tag{13}$$

We can then write in the zero-Larmor-radius approximation, assuming a Maxwellian electron distribution in the initial velocities,

$$j_\parallel{}^{\mathrm{M}}(r) = \frac{\omega_{pe}{}^2}{4\pi v_{Te}{}^2} \left\langle \int_0^\infty v(0) v(t) E_\parallel{}^{\mathrm{M}}[R(t)] \exp(i\omega t)\,dt \right\rangle_p, \tag{14}$$

where $v(t) = \dot{R}(t) E_\parallel{}^{\mathrm{M}} = \mathbf{b}\mathbf{E}^{\mathrm{M}}$, and $\langle\ \rangle_p$ denotes averaging over the electron initial velocities and over the variables that determine the ion configurations.

When relation (9) is valid, Eq. (14) can be expressed as an integral along a field line:

$$j_\parallel{}^{\mathrm{M}}(s) = \int_{-\infty}^{+\infty} \sigma_\parallel(s-s') E_\parallel{}^{\mathrm{M}}(s')\,ds', \tag{15}$$

where

$$j_\parallel{}^{\mathrm{M}}(s) \equiv j_\parallel{}^{\mathrm{M}}[r(s)], \quad E_\parallel{}^{\mathrm{M}}(s') \equiv E_\parallel{}^{\mathrm{M}}[r'(s')],$$

$$\sigma_\parallel(s-s') = \frac{\omega_{pe}{}^2}{4\pi v_{Te}{}^2} \int_0^\infty \langle v(0) v(t) \delta(s-s'-\Delta(t)) \rangle_p \exp(i\omega t)\,dt, \tag{16}$$

or equivalently,

$$j_\parallel{}^{\mathrm{M}}(s) = \int_{-\infty}^{+\infty} \sigma_\parallel(q) E_\parallel{}^{\mathrm{M}}(q) \exp(iqs)\,dq, \tag{17}$$

$$\sigma_\parallel(q) = \frac{\omega_{pe}{}^2}{4\pi v_{Te}{}^2} \int_0^\infty \langle v(0) v(t) \exp[i(\omega t - q\Delta(t))] \rangle_p\,dt, \tag{18}$$

where $E_\parallel{}^{\mathrm{M}}$ is the Fourier component of the field with respect to the variable s.

We consider (17) first for $\nu t_0 \gg 1$. Clearly, $\sigma_\parallel(q)$ is here the usual longitudinal conductivity of a plasma in a uniform magnetic field with $\omega \ll \nu$, and the expression for it is well known:

$$\sigma_\parallel(q) = \frac{\omega_{pe}{}^2}{4\pi} \frac{p}{\nu p + q^2 v_{Te}{}^2},$$

$$\sigma_\parallel(s) = \frac{\omega_{pe}{}^2}{4 v_{Te}} \left(\frac{p}{\nu}\right)^{1/2} \exp\left(-\frac{(\nu p)^{1/2}}{v_{Te}}|s|\right), \tag{19}$$

where $p = -i\omega$; we have introduced this quantity with an aim of using later a Laplace transformation. To obtain this expression in the context of the proposed approach we describe, following Ref. 10, the electron–ion collisions by a Brownian-motion model. Here $v(t)$ is a normal Markov process with a correlation function[11]

$$\langle v(t_1) v(t_2) \rangle_p = v_{Te}{}^2 \exp(-\nu|t_1 - t_2|), \tag{20}$$

where ν is a constant having the meaning of the electron-collision frequency. Just as any linear function of a normal function, $\Delta(t)$ is a normal function of t, with

$$\langle \Delta^2(t) \rangle_p = 2 v_{Te}{}^2 [|t|\nu + \exp(-\nu|t|) - 1]/\nu^2. \tag{21}$$

Taking (18) into account, we express the longitudinal conductivity in the form

$$\sigma_\parallel(q) = \frac{\omega_{pe}{}^2}{4\pi v_{Te}{}^2 q^2} \int_0^\infty \frac{\partial^2}{\partial t^2} \left\langle \exp\left(-iq\int_0^t v(t')\,dt'\right) \right\rangle_p \times \exp(-pt)\,dt, \tag{22}$$

and, using the relation

$$\langle \exp(-iqz) \rangle = \exp(-q^2\langle z^2 \rangle/2), \tag{23}$$

which is valid for any normal random quantity, we obtain (19) in the limit $p \ll \nu$.

Returning to expression (12) for j_α, we recall that it is not yet possible to average in it over an ensemble of $\delta\mathbf{B}$ realizations, since $\delta\mathbf{B}$ governs not only the trajectories $r'[s'(t)]$ whose statistical properties are assumed known, but also the field $E_\parallel{}^{\mathrm{M}}$ in (15). To exclude a random electric-field component, we note that the principal mechanism that generates this component is variation of $j_\parallel{}^{\mathrm{M}}$ along a field line. Consequently div $\mathbf{j}_\parallel{}^{\mathrm{M}} \neq 0$ and the current flow is accompanied by charge separation and by the onset of a small-scale electrostatic potential. At low frequencies, the solenoidal part $\delta\mathbf{E}$ of the small-scale fluctuating field is negligible compared with the electrostatic part. Thus,

$$\mathbf{E}^{M}=\mathbf{E}-\nabla\varphi,\ E_{\parallel}^{M}(q)=E(q)-iq\varphi(q). \qquad (24)$$

To exclude the potential field component we use the plasma quasineutrality condition

$$\operatorname{div}\mathbf{j}_{\perp}^{M}(r)-\int_{-\infty}^{+\infty}\sigma_{\parallel}(s-s')\frac{\partial^2\varphi(s')}{\partial s'^2}ds'$$

$$=-\int_{-\infty}^{+\infty}\sigma_{\parallel}(s-s')\frac{\partial E_{\parallel}(s')}{\partial s'}ds'. \qquad (25)$$

The current component $j_{\perp}^{M}$, to which the electrons make a negligible contribution, depends substantially on the transverse (to $\mathbf{B}$) scale of variation of the potential. To estimate this scale, we examine in greater detail the right-hand side of (25), which serves in this equation as a source and can alternatively be written in the form $p\rho_e(E_{\parallel})$, where $\rho_e(E_{\parallel})$ is the density of the electron charge produced by the field $E_{\parallel}$. It is easily shown that

$$\langle\rho_e(E_{\parallel})\rangle_B=0,$$

i.e., $\rho_e(E_{\parallel})$ is a purely fluctuating quantity. It is evident from (25) that the value of $\rho_e(E_{\parallel})$ at this point is determined by an integral along the field line passing through it. The values of $\rho_e(E_{\parallel})$ at two nearby points on the xy plane will therefore differ greatly if the corresponding field lines are not correlated on an appreciable part of their effective length that contributes to the integral. This takes place, in view of the divergence of the field lines, when the distance between the points is still small compared with $\delta_{\perp}$. For $\omega t_0\sim 1$ the scale of variation of $\rho e(E)$ and hence also the potential is equal to d_0 in (8). Since $d_0\ll\delta_{\perp}$ and $\delta_{\perp}$, as indicated, has a value ~ 0.1 cm, i.e., close to or less than the typical ion Larmor radius r_i in a hot plasma, we shall assume that $d_0<r_i$. The ion motion across the magnetic field $\mathbf{B}$ is thus found to be magnetized, and assuming for simplicity a Maxwellian ion distribution function we can take the ion density to be a Boltzmann distribution, i.e.,

$$\rho_i=-\frac{\varphi}{4\pi d_i^2},\quad \operatorname{div}\mathbf{j}_{\perp}^{M}=-\frac{p}{4\pi d_i^2}\varphi, \qquad (26)$$

where ρ_i is the perturbed charge density and $d_i=v_{Ti}/\omega_{pi}$ is the ion Debye radius. Substituting this result in (25) we obtain an integral equation that relates the potential at the point r to its values on a field line passing through this point. This equation is solved using a Fourier expansion in the variable s. The result is

$$\varphi(q)=iq\sigma_{\parallel}(q)d_i^2(p-q^2d_i^2\sigma_{\parallel}(q))^{-1}E_{\parallel}(q). \qquad (27)$$

Using this result, we can, by virtue of (24), express $E_{\parallel}^{M}(q)$ in terms of the Fourier transform of the mean field $E(q)$. Substituting the resultant expression in (17) we get

$$j_{\parallel}^{M}(s)=\int E_{\parallel}(q)\sigma_{\parallel}^{ren}(q)\exp(iqs)dq, \qquad (28)$$

$$j_{\parallel}^{M}(s)=\int\sigma_{\parallel}^{ren}(s-s')E_{\parallel}(s')ds', \qquad (29)$$

where the renormalized quantities $\rho_{\parallel}^{ren}(q)$ and $\sigma_{\parallel}^{ren}(s)$ differ from the unrenormalized ones in (19) by the substitutions $v_{Te}^2\to v_T^2$, $v_T^2=(T_e+T_i)/m_e$. In the calculation of the conductivity tensor we confine ourselves to the component $\sigma_{zz}\equiv\sigma$, which is needed to solve the skin-effect problem. We shall assume that $\mathbf{E}(r)=\mathbf{E}(x)$. We obtain the current component $j_z(x)$ by averaging (12) in which we put $b_z=1$, and by using expression (29). The latter depends on $\delta\mathbf{B}$ only via the function $r'(s')$, so that the average $\langle\ \rangle_B$ is effected with the aid of the distribution function (4) or (8). The result is

$$j(x)=\int_{-\infty}^{+\infty}\sigma(x,x')E(x')dx', \qquad (30)$$

$$\sigma(x,x')=\int_{-\infty}^{+\infty}\sigma_{\parallel}^{ren}(s-s')W(x',s'/x,s)ds'. \qquad (31)$$

For an unbounded medium we have

$$\sigma(x,x')=\sigma_{\infty}(x-x')=\sigma_0\frac{\beta^{1/2}}{2}\exp(-\beta^{1/2}|x-x'|), \qquad (32)$$

$$\sigma_{\infty}(k)=\int_{-\infty}^{+\infty}\sigma_{\infty}(x)\exp(-ikx)dx=\sigma_0/(1+k^2/\beta), \qquad (33)$$

where

$$\beta=(\nu p)^{1/2}/D_m v_T,\ \sigma_0=\omega_{pe}^2/4\pi\nu.$$

In the case of a stochastic layer of thickness a we have

$$\sigma(x,x')=\sum_{n=-\infty}^{+\infty}\{\sigma_{\infty}(x-x'-2na)+\sigma_{\infty}(x+x'-2na)\}. \qquad (34)$$

We proceed now to the case of extremely large mean free paths $\lambda>L_k$. If the field $\delta\mathbf{B}$ is independent of time as before, the preceding expressions remain in force, but the explicit form of the function $\sigma_{\parallel}(q)$ is altered, since $\Delta(t)$ in (18) is now given by (10). For $\omega\ll\nu$ we can put $\omega=0$ in (18). Describing the collisions again by the Brownian-motion model (20) and putting $v(t)=v_{Te}u(\nu t)$, where $u(\tau)$ is a normal random function for which

$$\langle u(0)u(\tau)\rangle=\exp(-|\tau|),$$

we obtain

$$\sigma_{\parallel}(q)=\sigma_0 f(q\lambda), \qquad (35)$$

$$f(z)=\int_0^{\infty}\left\langle u(0)u(\tau)\exp[-iz\operatorname{sign}[u(0)]]\int_0^{\tau}|u(\tau')|d\tau'\right\rangle_u d\tau \qquad (36)$$

(the average in (36) is over the ensemble of realizations of the function u). It can be shown that $f(z)\propto z^{-5/3}$ for $z\gg 1$. Thus averaging the field along the electron trajectory cuts off the contribution of the Fourier components with $q>\lambda^{-1}$. We shall not analyze this case in greater detail, since in the $\lambda>L_k$ regime we can consider the more realistic situation $\delta\mathbf{B}=\delta\mathbf{B}(t)$. If the magnetic field varies with time, particles with different prior histories will pass through a fixed point r at different instants, and the current will fluctuate. The characteristic frequency Ω of these fluctuations becomes higher than the frequency Ω_B, owing to the stochastic instability of the force lines, in exactly the same way manner in which the spatial scale of the ρ_e fluctuations decreases compared with $\delta_{\perp}$: it is easy to obtain the estimate $\Omega\sim\Omega_B\exp(-\lambda/L_k)$. It is natural to assume that $\Omega \gg \omega$, and then the characteristic frequency of the purely fluctuating quantities in ρ_e and φ will equal Ω. Taking this into account, we can estimate the term in the left-hand side of the quasineutrality equation (25) in the Fourier representation:

$$(\mathrm{div}\, j_\perp^M)_q \sim \Omega\varphi(q)/4\pi d_i^2,$$

$$\left(\sigma_\parallel(s-s')\frac{\partial^2\varphi(s')}{\partial s'^2}\right)_q \sim \sigma_0 q^2\varphi(q).$$

It follows hence that the electronic term prevails over the ionic if $(q\lambda)^2 > \Omega/\nu$. Thus, if $\Omega > \nu$ the quasineutrality is maintained by ion displacement over the entire range of q, which determines the current density according to (35) and (36). Since Ω is exponentially large compared with Ω_B and $\Omega_B \sim \nu$, it can be assumed that the condition $\Omega \gg \nu$ is met for $\lambda > L_k$. This means that the contribution of the electrostatic part of the field to the current $j_\parallel^M$ can be neglected, i.e., it is possible to replace $E_\parallel^M$ by $E_\parallel$ in (15) and (17).

From the pattern of the particle motion at $\lambda > L_k$ it can be concluded that $j_z = \langle j_\parallel^M \rangle_B$ will have the same form in the case of a stationary magnetic field as when $\Omega_B \neq 0$.

In the calculation of $\sigma(x,x')$ it is convenient to carry out the average (15) in accordance with (31). In the case of an infinite medium we obtain in this case for

$$\sigma_\infty(k) = \int \sigma_\infty(x)\exp(-ikx)\,dx,$$

the values

$$\sigma_\infty(k) = \sigma_0 F(k^2 D_m \lambda), \tag{37}$$

$$F(z) = \int_0^\infty \left\langle u(0)u(\tau)\exp\left(-z\int_0^\tau |u(\tau')|\,d\tau'\right)\right\rangle_u d\tau. \tag{38}$$

In the limiting cases of large and small z, the function $F(z)$ takes the form

$$F(z) \to 1 - 4z/(2\pi)^{1/2} \quad \text{if } z \ll 1,$$
$$F(z) \to (2/\pi)^{1/2}/z \quad \text{if } z \gg 1. \tag{39}$$

It can be approximated for all z, accurate to about 20%, by the expression

$$F(z) = (1 + (\pi/2)^{1/2} z)^{-1}, \tag{40}$$

which leads to the same $\sigma_\infty(k)$ dependence as in the Rosenbluth–Rechester regime [Eq. (33)]. In the coordinate representation we obtain then for $\sigma_\infty(x-x')$ the expression (32) with β given by

$$\beta = [(\pi/2)^{1/2} D_m \lambda]^{-1}.$$

For a stochastic layer with $\lambda > L_k$ the relation (34) between $\sigma(x,x')$ and $\sigma_\infty(x-x')$ remains in force.

It must be noted that expressions (38)–(40) were obtained in a model in which a direct correlation exists between the transverse and longitudinal motion of the particle, so that the instantaneous diffusion coefficient is $D_\perp = D_m |v(t)|$. In the other limiting case when the transverse motion constitutes diffusion that is completely independent of the longitudinal velocity, we obtain the relation (33) (where $D_m v_T$ must be replaced by the test-particle diffusion coefficient, which can be set equal to the thermal diffusivity $\varkappa$, so that $\beta = \nu/\varkappa$). This circumstance indicates that the longitudinal conductivity has apparently a weak dependence on the specific anomalous electron-transport mechanism. At the same time, the specific dependence of the conductivity on ν and p in the Rosenbluth–Rechester regime is in fact a feature of the stochastic-magnetic-field model.

4. SKIN EFFECT

As an example of the application of the results, we consider (in the model formulation) the skin effect in a plasma situated in a weakly stochastic magnetic field. Assume that a homogeneous plasma occupies the half-space $x > 0$ and borders on a vacuum at $x = 0$. We seek an electric field $E(x,t)$ that satisfies the boundary condition $E(0,t) = W_0(t)$ at $t > 0$ and $E(x,t) \to 0$ as $x \to \infty$, and the initial conditions $E(x,0) = 0$ and $f(x,0) = 0$.

It is convenient to solve the problem by Laplace transformation with respect to time. For the Laplace transform of the electric field $E(x,p)$ we obtain the equation

$$\frac{\partial^2 E(x,p)}{\partial x^2} = \frac{4\pi p}{c^2} j(x,p). \tag{41}$$

We consider first the case when the magnetic field is stochastic in the entire plasma volume. Expression (30) for $f(x,p)$ cannot be used directly, since it was derived for an unbounded medium. Assume that the fact that the plasma is bounded in space can be taken into account by simply integrating in (30) not between infinite limits but over the region occupied by the plasma. (This assumption is analogous to the model of diffusive reflection of the particles from the surface in the theory of the anomalous skin effect in a solid[8]; as applied to a plasma, this can be interpreted as the ideal-recycling hypothesis.) Equation (4) then takes the form

$$\frac{\partial^2 E(x,p)}{\partial x^2} = \frac{b_1\beta^{1/2}}{2}\int_0^\infty \exp(-\beta^{1/2}|x-x'|)E(x',p)\,dx', \tag{42}$$

where

$$b_1 = 4\pi p\sigma_0/c^2 \equiv p/D_s.$$

This equation has a solution that satisfies the boundary conditions $E(x,p) \to 0$ as $x \to \infty$ and $E(0,p) = E_0(p)$, where $E_0(p)$ is the Laplace transform of $E_0(t)$, in the form

$$E(x,p) = [C_1\exp(k_1 x) + C_2\exp(k_2 x)]E_0(p); \tag{43}$$

here $k_{1,2}$ are the roots of the dispersive equation

$$k^4 - \beta k^2 + b_1\beta = 0, \tag{44}$$

that satisfies the condition $\mathrm{Re}\, k < 0$, and

$$C_2 = (\beta^{1/2} + k_2)/(k_2 - k_1), \quad C_1 = 1 - C_2.$$

For the current density we have from (41)

$$j(x,p) = \frac{1}{p}[C_1 k_1^2\exp(k_1 x) + C_2 k_2^2\exp(k_2 x)]j_0(p), \tag{45}$$

where

$$j_0(p) = c^2 E_0(p)/4\pi D_s$$

is the current density in the field $E_0(p)$ for classical conductivity. Note that for $p = -i\omega$ expressions (43) and (45) describe the form of the field and of the current for the harmonic function $E_0(t) = E_0 \exp(-i\omega t)$.

Let us consider the solution (43) and (45) in the Rosenbluth–Rechester regime, when $\beta = (\nu p)^{1/2}/D_m v_T$. We introduce in (43) and (45), and in the Laplace integrals that determine $E(x,t)$ and $j(x,t)$ from the transforms $E(x,p)$ and $f(x,p)$, the dimensionless variables

$$\xi = x/L, \quad \tilde{p} = pT, \quad \tau = t/T, \tag{46}$$

where

$$L = D/(D_s\nu)^{1/4}, \quad T = D^2/D_s^2\nu = L^2/D_s, \quad D = D_m v_T.$$

It is easy to verify that in terms of these variables Eq. (42) and the functions $E(\xi,\tilde{p})$ and $E(\xi,\tau)$ contain no parameters whatever indicative of the properties of the medium. The transition to the classical conductivity with $D_m \to 0$ is therefore effected by taking the limits $\xi \to \infty$ and $\tau \to \infty$; this means that even for finite D_m the distributions of j and E for $\tau \gg 1$ and $\xi \gg 1$ will have classical forms, and will, in particular, be functions of a single variable $\xi^2/\tau = x^2/D_s t$ that does not contain the parameters D_m. Figure 1 shows plots of $E(\xi,\tau)/E_0$ and $j(\xi,\tau)/j_0$ for several fixed values of τ, obtained numerically using inverse Laplace transformation for $E_0(\tau)$ in the form of a step function ($E_0(\tau) = 0$ for $\tau < 0$ and $E_0(\tau)$ for $\tau > 0$). It can be seen that E and j are negative in some regions of space; this singularity is attributed to an inductive effect when **J** and **E** are related nonlocally. For small τ the current distribution is flatter than classical, so that during the initial stage of current penetration into the plasma is accelerated. The total current flowing in the plasma, however, is found to be even smaller than classical, as can be seen from Fig. 2, which shows the function

$$I(\tau) = \int_0^\infty j(\xi,\tau)\,d\xi.$$

As $\tau \to$ the function $I(\tau)$ takes the asymptotic form

$$I(\tau) = E_0 \frac{c^2 L}{4\pi D_s}\left(\frac{2}{\pi^{1/2}}\tau^{1/2} - \frac{1}{2\Gamma(5/4)}\tau^{1/4} + \frac{1}{2} + \dots\right). \tag{47}$$

It can be seen that the limiting value corresponding to the normal skin effect (which is described by the first term) is reached very slowly. Note also that for $\xi = 0$ and as $\tau \to \infty$ the current density tends to the limiting value $E_0\sigma_0/2$. The factor 1/2 is the result of the influence of the boundary under conditions when it is reached by any field line coming from

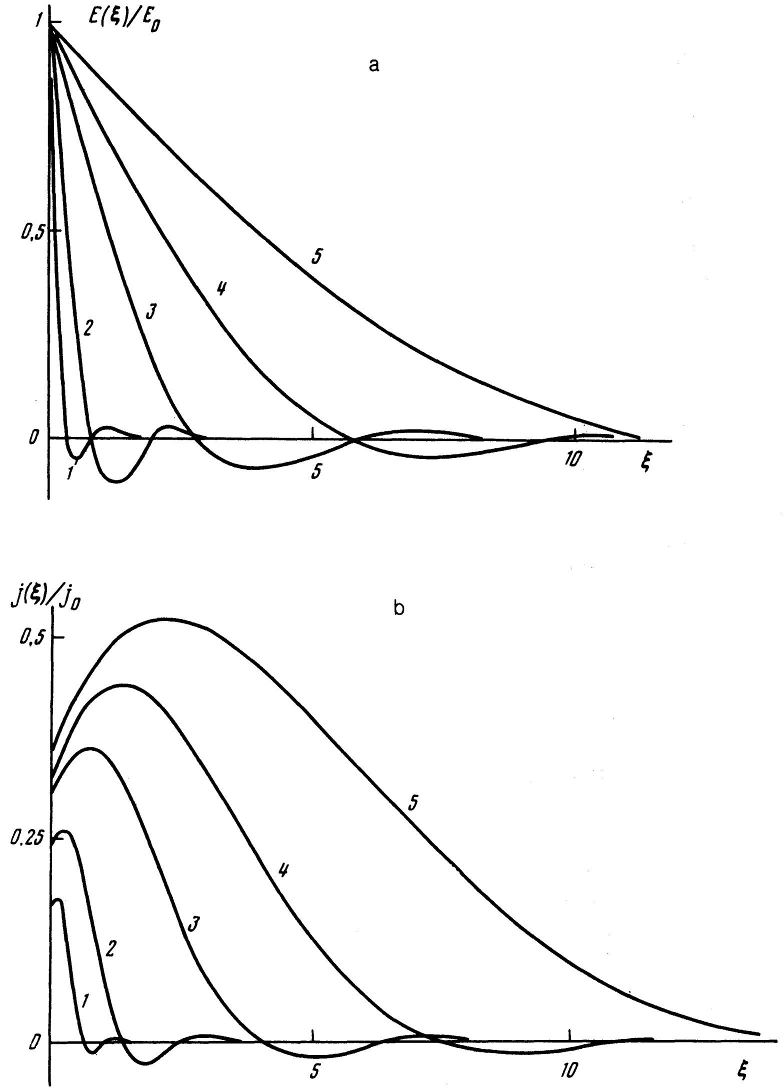

FIG. 1. Spatial distribution of the field $E(\xi)/E_0$ (a) and of the current density $j(\xi)/j_0$ (b) in the $\lambda < L_k$ regime for several fixed values of τ: 1—$\tau = 0.01$; 2—0.1; 3—1; 4—4; 5—16.

the interior of the plasma. It must, of course, be remembered that all the foregoing results are valid only for times not exceeding the electron lifetime t_0 on the line. If, however, the distributions of the fields and currents become "classical" within the time t_0, i.e., $T < t_0$, it can be assumed that they describe the entire process, and then $L^2 \lesssim D_m L_k$.

In the case of extemely long mean free paths $\lambda > L_k$ the solution of Eq. (42) can also be recast in universal form (independent of the parameters of the medium) by the variable change (46), but with different values of the characteristic scales: $L^2 = (\pi/2)^{1/2} D_m \lambda$, $T = L^2/D_s$. The values of these scales for $D_m = 10^{-5}$ cm and a plasma density 10^{13} cm^{-3} are equal to $L = 5.2$ cm and $T = 0.78$ s for a temperature 1 keV and $L = 10.4$ cm and $T = 8.9$ s for a temperature 5 keV. Note also that in the discussed case $\lambda > L_k$ there is no upper limit on the times of these processes.

We consider now an example in which the stochastic magnetic field exists only in a layer $0 < x < a$, but for $x > a$ the plasma conductivity is classical. We describe the stochastic layer by the model (6). We have then in Eq. (41)

$$j(x,p) = \int_0^a \sigma(x,x') E(x',p)\, dx', \tag{48}$$

$$j(x,p) = \sigma_0 E(x,p), \tag{49}$$

where $\sigma(x,x')$ is determined by Eqs. (34) and (32). The function $E(x,p)$ should satisfy the boundary conditions $E(x,p) \to 0$ as $x \to \infty$ and $E(0,p) = E_0(p)$, and must be continuous together with its derivative at $x = a$. Outside the stochastic layer, $x > a$, Eq. (41) has a solution that decreases as $x \to \infty$, in the form

$$E(x,p) = C(p) \exp(-(p/D_s)^{1/2}(x-a)), \tag{50}$$

where $C(p)$ is determined from the condition that the solution be continuous on going through the boundary $x = a$.

The problem for the layer $0 \leqslant x \leqslant a$ can be reduced to a problem for infinite space by noting that

$$\int_0^a \sigma(x,x') E(x',p)\, dx' = \int_{-\infty}^{+\infty} \sigma_\infty(x-x') \varepsilon(x',p)\, dx', \tag{51}$$

where the function $\varepsilon(x',p)$ was obtained by continuing $E(x,p)$ from the segment $0 \leqslant x \leqslant a$ on the entire x axis in accordance with the rule

$$\varepsilon(x,p) = E(x,p),\ 0 \leqslant x \leqslant a, \qquad \varepsilon(-x,p) = \varepsilon(x,p),\ -a \leqslant x \leqslant 0, \tag{52}$$

$\varepsilon(x + 2na, p) = \varepsilon(x,p)$, n is any integer. The derivative of $\varepsilon(x,p)$ has finite discontinuities at $x = na$ whose magnitudes are determined from the boundary conditions. Using this and the periodicity of the function $\varepsilon(x,p)$, we write for this function the equation

$$\frac{\partial^2 \varepsilon(x,p)}{\partial x^2} - \frac{4\pi p}{c^2} \int_{-\infty}^{+\infty} \sigma_\infty(x-x') \varepsilon(x',p)\, dx' = A(p)\tilde{\delta}(x) + B(p)\tilde{\delta}(x-a), \tag{53}$$

where

$$\tilde{\delta}(x) = \sum_{n=-\infty}^{+\infty} \cos(k_0 n x) \tag{54}$$

is the periodic analog of the δ function, $k_0 = \pi/a$, and $A(p)$ and $B(p)$ are amplitudes determined by the boundary conditions. It is seen from (52) that $\varepsilon(x,p)$ can be represented by the Fourier series

$$\varepsilon(x,p) = \sum_{n=-\infty}^{+\infty} \varepsilon_n(p) \cos(k_0 n x). \tag{55}$$

Substituting this expression in (52) and taking (53) into account, we obtain

$$\varepsilon(x,p) = A(p)\Phi(x,p) + B(p)\Phi(x-a,p), \tag{56}$$

$$\Phi(x,p) = \sum_{n=-\infty}^{+\infty} \frac{\cos(k_0 n x)}{k_0^2 n^2 + 4\pi p \sigma_\infty(k_0 n)/c^2}. \tag{57}$$

Taking into account the explicit form (33) of the conductivity $\sigma_\infty(k)$, we can sum the series and obtain

$$\Phi(x,p) = \sum_{i=1,2} \frac{\beta - k_0^2 \varkappa_i^2}{\beta \varkappa_i \operatorname{sh}(\pi \varkappa_i)} \operatorname{ch}[\varkappa_i(\pi - k_0 x)]; \tag{58}$$

here $\varkappa_i = k_i/k_0$, where the k_i are solutions of Eq. (44).

To calculate $E(x,p)$ in the entire range $0 \leqslant x \leqslant \infty$ we use the boundary conditions on the surfaces $x = 0$ and $x = a$ to obtain the amplitudes $A(p)$ and $B(p)$ from (56), and $C(p)$ from (50). The function $j(x,p)$ is determined from Eqs. (41) and (49). The current density $j(x,t)$ and the field $E(x,t)$ are obtained by inverse Laplace transformations using the transforms $j(x,p)$ and $E(x,p)$.

By way of example we consider a case with the field on the boundary assumed to be the step function $E_0(\tau)$. Figures 3a and 3b show respectively the numerically calculated distributions of the field and of the current density in the Rosenbluth–Rechester regime with $\xi > 0$ for several fixed instants of time. The width of the stochastic layer is assumed to be L ($\xi_0 = a/L = 1$).

It is seen from Fig. 3a that in addition to the skin-effect (i.e., the field attenuation along the ξ axis), there exist regions where $E(\xi,\tau)/E_0$ becomes negative. These spaces exist also in the Spitzer-conductivity region $\xi > \xi_0$. The current density distribution inside the stochastic layer (Fig. 3b) differs noticeably from the field distribution $E(\xi,\tau)$. Equalization of $j(\xi)$ sets in after a time $\tau \sim 0.1$ and is followed by increase of the overall level to a value j_0. A characteristic feature of the pattern is the presence of a discontinuity of j on

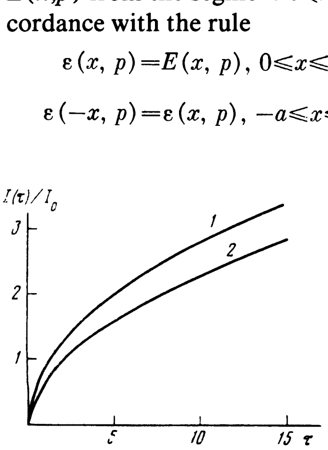

FIG. 2. Time dependence of the excited current $I(\tau) = \int_0^\infty j(\xi)\, d\xi$; 1—conductivity in a stochastic magnetic field, 2—classical conductivity.

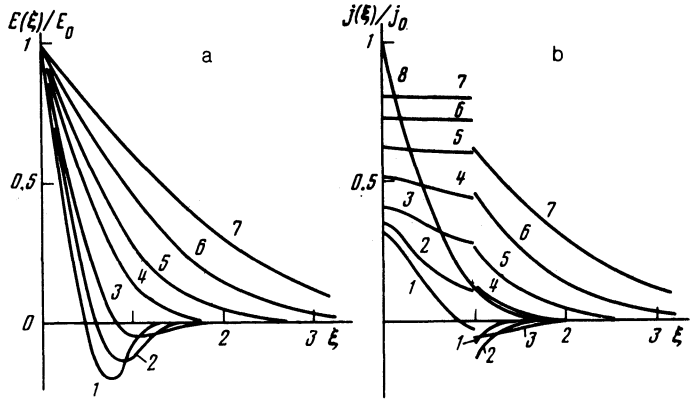

FIG. 3. Spatial distribution of the field $E(\xi)/E_0$ (a) and of the current density $j(\xi)/j_0$ (b) in the $\lambda \ll L_k$ regime in the presence of a stochastic layer, for several fixed values of τ:1—$\tau = 0.03125$; 2—0.0325; 3—0.125; 4—0.25; 5—0.5; 6—1; 7—2; 8—distribution of $j(\xi)/j_0$ for $\tau = 0.25$ in the absence of a stochastic layer.

the surface $\xi = \xi_0$. Figure 3b (curve 8) also shows for comparison the $j(\xi)$ distribution in a plasma with Spitzer conductivity in the absence of a stochastic layer.

Turning to the $\lambda > L_k$ regime, we note that the dependences of E and j on the variables ξ and τ are qualitatively quite similar to those of the case $\lambda \ll L_k$. The numerical values of the nondimensionalizing parameters can differ greatly in these two regimes.

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