Functional method for quantum ferromagnets, and nonmagnon dynamics at low temperatures

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A representation for the generating functional of the Green functions of a quantum Heisenberg ferromagnet in the nonsymmetric phase is obtained in the form of an integral over two number fields—a neutral field and a charged field. Simple criteria for the correctness of this representation are given and checked. By means of this representation the dynamics of the fluctuations of the longitudinal component of the spin at low temperatures is studied and the asymptotic form of the correlator of these fluctuations at large times is calculated.

1. INTRODUCTION

To represent the partition function and the generating functionals of different averages in the form of an integral over number fields is natural for the use of the method of steepest descent and convenient for the analysis of perturbative and nonperturbative effects. Attempts to obtain a functional representation for the quantum Heisenberg ferromagnet have been repeatedly undertaken,[1,2] but the final expressions either cannot be expressed in an explicit closed form[1] or, as in Ref. 2, do not pass the test of fulfilling trivial identities (see Sec. 2). In a paper by one of the authors[3] a method was proposed that makes it possible to write an expression for the partition function of a magnet in the nonsymmetric phase in the form of an integral over two nmber fields—a neutral field and a charged field. The elementary excitations corresponding to the charged field behaved as ordinary bosons (magnons), and the expansion in a perturbation-theory series of the functional integral of Ref. 3 reproduced the results of the operator diagram techniques of Refs. 4 and 5. We note that among these results were some which provided evidence of the defective nature of the perturbation theory developed, e.g., the "frozen character" of the fluctuations of the longitudinal component of the spin (for more detail, see Sec. 3).

In the present paper we show that the functional representation of Ref. 3. is erroneous because certain global effects were ignored in its derivation. Using, as before, the method of Ref. 3, we have obtained a correct expression together with several simple criteria for the consistency of the construction. In Sec. 3 we use this expression to consider the longitudinal fluctuations, and convince ourselves that they are "revived."

The integration fields in our integral (2.12) have remained the same as in Ref. 3; however, the charged field, despite its numerical nature, now describes not bosons, but something else. This is consistent with the obvious limitation of the idea that the excitations in a magnet have a Bose character.

Our method is not limited with respect to the spin magnitude S, although the concrete calculations pertain to the case $S=\frac{1}{2}$. In a recent paper[6] a functional integral for quantum magnets with spin $S=\frac{1}{2}$ and $S=1$ that differs substantially from ours was derived. This difference is discussed in more detail at the end of Sec. 2.

The functional approach has also been used in the study of the high-temperature dynamics.[7] It should be noted that in the constructions of Ref. 7 the above-mentioned global effects are unimportant, since the representations of Refs. 7 and 3 are organized entirely differently.

2. THE FUNCTIONAL REPRESENTATION

1. We recall how the functional integral is obtained for the simplest case of the partition function of the Ising model[8]:

$$H_I=-{}^1/_2 J_{ij}\sigma_i\sigma_j,\qquad \sigma_i=\pm1, \tag{2.1}$$

$$Z_I=\operatorname{Tr}(e^{-\beta H_I})=\int\prod_i d\varphi_i \exp(-{}^1/_2\beta\varphi_i\varphi_j J_{ij}^{-1})\operatorname{Tr}\exp(\beta\varphi_i\sigma_i)$$
$$=\int\prod_i d\varphi_i \exp\Bigl(-{}^1/_2\beta\varphi_i\varphi_j J_{ij}^{-1}+\sum_i \ln\operatorname{ch}\beta\varphi_i\Bigr). \quad 2.2)$$

Here J_{ij} is the matrix of the exchange interaction, J_{ij}^{-1} is the inverse of the matrix J_{ij}, and summation over repeated indices in implied. The meaning of the Gaussian trick used in (2.2) consists in the reduction of the trace over the entire set of states of the magnet to a product of traces over the states of each spin separately. For the quantum Heisenberg ferromagnet the direct generalization of (2.2) does not work because of the noncommutativity of the spin operators. However, for the operator $\exp(-\beta\varepsilon H_{ex})$, where

$$H_{ex}=-{}^1/_2 J_{ij}\mathbf{S}_i\mathbf{S}_j \tag{2.3}$$

and $\varepsilon\to0$, a Gaussian transformation of the type (2.2) can be performed to within terms $\sim\varepsilon^2$. Thus, writing $e^{-\beta H_{ex}}=(e^{-\varepsilon\beta H_{ex}})^{1/\varepsilon}$, with $\varepsilon\to0$, we arrive at an expression for the generating functional of the temperature Green functions of the spin operators:

$$Z(\mathbf{h})=\operatorname{Tr} T\exp\Bigl(-\beta H_{ex}+\int_0^\beta \mathbf{h}_i(t)\mathbf{S}_i\,dt\Bigr) \tag{2.4}$$

in the form[1,9]

$$Z(\mathbf{h})=\int\prod_i D\varphi_i(t)\exp\Bigl(-\frac{1}{2}\int_0^\beta dt\,\varphi_i(t)J_{ij}^{-1}\varphi_j(t)\Bigr)$$
$$\times\operatorname{Tr}\Bigl[T\exp\Bigl(\int_0^\beta dt[\varphi_i(t)+\mathbf{h}_i(t)]\mathbf{S}_i\Bigr)\Bigr]. \tag{2.5}$$

The symbol T denotes time ordering and $\mathbf{h}_i(t)$ is the external field at lattice site r_i.

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The functional integral (2.5) is understood as the limits of finite-dimensional approximations:

$$D\varphi_i(t)=\prod_{\alpha=x,y,z}\prod_{n=1}^{N}d\varphi_i^{\alpha}(\beta n/N),\quad N\to\infty. \qquad (2.6)$$

We shall rewrite (2.5) in a more convenient form, by shifting the integration fields by $-\mathbf{h}_i(t)$:

$$Z(\mathbf{h})=\int\prod_i D\varphi_i(t)\exp\Bigl(-\frac{1}{2}\int_0^\beta dt\,\varphi_i J_{ij}^{-1}\varphi_j+\int_0^\beta \mathbf{h}_i J_{ij}^{-1}\varphi_j\,dt-\frac{1}{2}\int_0^\beta dt\,\mathbf{h}_i J_{ij}^{-1}\mathbf{h}_j\Bigr)\prod_i \mathrm{Tr}\Bigl[T\exp\Bigl(\int_0^\beta dt\,\varphi_i(t)\mathbf{S}_i\Bigr)\Bigr]. \qquad (2.5')$$

The ordered operator exponential

$$A(t)=T\exp\Bigl(\int_0^t dt'\,\varphi(t')\mathbf{S}\Bigr) \qquad (2.7)$$

satisfies the equation

$$\dot A(t)=(\varphi(t)\mathbf{S})A(t) \qquad (2.8)$$

with the initial conditin $A(0)=1$. The operator $A(t)$ cannot be calculated explicitly as a functional of $\varphi(t)$. However, there exists a substitution that transforms a T-ordered exponential into a product of ordinary exponentials (see also Ref. 3). In fact, we shall consider the explicitly specified operator

$$B(t)=\exp(S^+\psi^-(t))\exp\Bigl(S^z\int_0^t\rho(t')dt'\Bigr)\times\exp\Bigl(S^-\int_0^t\psi^+(t')\exp\Bigl(\int_0^{t'}\rho(t'')dt''\Bigr)dt'\Bigr)\exp(-S^+\psi^-(0)), \qquad (2.9)$$

where $S^{\pm}=S^x\pm iS^y$, and $\psi^{\pm}(t)$ and $\rho(t)$ are certain functions of argument t. Using the commutation relations for the spin operators, we can convince ourselves that the operator $B(t)$ satisfies the equation

$$\dot B(t)=\{S^+(\dot\psi^- -\rho\psi^- -\psi^+(\psi^-)^2)+S^-\psi^+ +S^z(\rho+2\psi^+\psi^-\}B(t). \qquad (2.10)$$

The last factor in (2.9) ensures the equality $B(0)=1$. This means that the substitution

$$\varphi^z=\rho+2\psi^+\psi^-,\quad \varphi^+=\psi^+,\quad \varphi^-=\dot\psi^- -\rho\psi^- -\psi^+(\psi^-)^2, \qquad (2.11)$$

where $\varphi^{\pm}=\frac{1}{2}(\varphi^x\pm i\varphi^y)$, brings the operator $A(t)$ to the form (2.9).

Thus, regarding ρ and $\psi^{\pm}$ as new integration variables, we can calculate the trace of the T-exponential explicitly and obtain a closed functional representation for $Z(\mathbf{h})$. But, since the substitution (2.11) contains $\dot\psi^-$ in the right-hand side, some boundary or initial condition with respect to the argument t should be imposed on the filed ψ^-. The apparently natural periodic boundary conditions used in Ref. 3 make the mapping (2.11) irreversible (see below). Instead of these, in the present paper we use the Cauchy-type condition

$$\psi^-(0)=0. \qquad (2.12)$$

2. In the functional measure $D\varphi^z D\varphi^+ D\varphi^-$ we can regard the fields φ^z and $\varphi^{\pm}$ as independent complex variables; the conditions $\mathrm{Im}\varphi^z=0$ and $\varphi^+=(\varphi^-)^*$ will determine the surface along which the integration is performed. Analogously, the variables ρ and $\psi^{\pm}$ in the calculation of the Jacobian $J[\rho,\psi^+,\psi^-]$:

$$D\varphi^z D\varphi^+ D\varphi^- = J[\rho,\psi^+,\psi^-]D\rho D\psi^+ D\psi^-, \qquad (2.13)$$

are also treated as independent.

The Jacobian $J=\det\hat J$ depends on the regularization of the differential $\hat J$ of the transformation (2.11). The expressions obtained for J with the use of different regularizations will differ by a factor $\exp(\alpha\int_0^\beta\rho dt)$, where α is a certain real number (see, e.g., Ref. 10, in which the determinant of an operator similar to $\hat J$ is considered). In our case, the arbitrariness is removed by the obvious requirement that with an exchange $J_{ij}=c\delta_{ij}$ the partition function $Z(\mathbf{h}=0)$ calculated with the aid of the functional integral coincides with the following expression, which follows trivially from the kinematic identity $S^2=S(S+1)$, where S is the magnitude (maximum projection) of the spin:

$$Z(\mathbf{h}=0)=\exp(\beta(c\cdot{}^1/_2S(S+1)+\mathrm{const}))\quad\text{for}\quad J_{ij}=c\delta_{ij}. \qquad (2.14)$$

From what follows below it will be clear that to the condition (2.14) there corresponds the Jacobian (we temporarily omit the site index for each set of single-site variables)

$$J[\rho,\psi^+,\psi^-]=\mathrm{const}\exp\Bigl(-\frac{1}{2}\int_0^\beta\rho\,dt\Bigr). \qquad (2.15)$$

This value of J results from the following discretization of the transformation (2.11) ($\rho_n\equiv\rho(t_n),\ldots,t_n=n\beta/N$, $\Delta\equiv\beta/N, N\to\infty$):

$$\varphi_n^z=\rho_n+\psi_n^+(\psi_n^-+\psi_{n-1}^-),\quad \varphi_n^+=\psi_n^+,$$

$$\varphi_n^-=\frac{1}{\Delta}(\psi_n^- -\psi_{n-1}^-)-\frac{1}{2}\rho_n(\psi_n^-+\psi_{n-1}^-)-\frac{1}{4}\psi_n^+(\psi_n^-+\psi_{n-1}^-)^2. \qquad (2.11')$$

In fact, it is easy to see that if after (2.11) we make one more change of variables $\rho=\tilde\rho'-2\psi^+\psi^-$, with ψ^+ and ψ^- unchanged, the Jacobian of the transformation from the original variables $\varphi^{\pm}$, φ^z to $\tilde\rho$, $\psi^{\pm}$ will be simply $\det(\partial_t-\tilde\rho+2\psi^+\psi^-)=\det(\partial_t-\rho)$. But since the mapping $\rho\to\tilde\rho$, $\psi^{\pm}\to\psi^{\pm}$ obviously has a unity Jacobian, we conclude that

$$\det\hat J=\det(\partial_t-\rho). \qquad (2.16)$$

In the discretization (2.11) and with the condition (2.12) the right-hand side of (2.16) is the Jacobian of the transformation $\varphi_n^-=(1/\Delta)(\psi_n^- -\psi_{n-1}^-)-(1/2)\rho_n\times(\psi_n^-+\psi_{n-1}^-)$, where $n=1,\ldots,N$, and $\psi_0^-\equiv 0$; thus,

$$\det \hat{J} = \det \begin{pmatrix} \frac{1}{\Delta} - \frac{1}{2}\rho_1, 0, \ldots & & \\ -\frac{1}{\Delta} - \frac{1}{2}\rho_2, \frac{1}{\Delta} - \frac{1}{2}\rho_2, 0, \ldots & & \\ \vdots & & \\ 0, \quad . \quad . \quad . & & , \frac{1}{\Delta} - \frac{1}{2}\rho_N \end{pmatrix}$$

$$= \prod_{n=1}^{N} \left(\frac{1}{\Delta} - \frac{1}{2}\rho_n \right) \qquad (2.17)$$

as the determinant of a triangular matrix. In the limit $\Delta \to 0$ we arrive at the expression (2.15), where const $= 1/\Delta^N$.

We note that the regularization (2.11′) ensures that the operators $A(t)$ and $B(t)$ also coincide in the case when, in the discrete variants of Eqs. (2.5) and (2.7), terms of order Δ are taken into account.

3. The calculation of the trace of the operator $B(\beta)$ does not present any difficulty even for an arbitrary spin magnitude S; however, in order to avoid unnecessarily cumbersome expressions, we shall give the answer only for the case $S = \frac{1}{2}$ (the entire subsequent analysis is also performed for $S = \frac{1}{2}$). Hence follows a functional representation for $Z(\mathbf{h})$ in the form

$$Z(\mathbf{h}) = \int D\rho D\psi^+ D\psi^- e^{-\Gamma} \prod_i \Big[1 + \psi_i^-(\beta) \times \int_0^\beta \psi_i^+(t) \exp\Big(\int_0^t \rho_i \, dt' \Big) dt + \exp\Big(\int_0^\beta \rho_i \, dt \Big) \Big],$$

$$\Gamma = \int_0^\beta dt \Big(\frac{1}{2} \rho_i J_{ij}^{-1} \rho_j + 2\psi_i^+ J_{ij}^{-1} \dot{\psi}_j^- - 2\rho_i J_{ij}^{-1} (\psi_i^- \psi_j^+ - \psi_j^- \psi_j^+) + 2\psi_i^+ \psi_i^- J_{ij}^{-1} \psi_j^+ \psi_j^- - 2\psi_i^+ J_{ij}^{-1} (\psi_j^-)^2 \psi_j^+ \Big)$$

$$+ \sum_i \int_0^\beta \rho_i \, dt - \int_0^\beta \mathbf{h}_i J_{ij}^{-1} \boldsymbol{\varphi}_j \, dt. \qquad (2.18)$$

Here we have omitted the term quadratic in $\mathbf{h}$, since it does not make a contribution to the unequal-time correlators of interest to us, and the quantity φ_i in the term with the source implies the expression in terms of ρ_i and $\psi_i^\pm$ given by (2.11).

We can deform the original surface of integration into the standard surface

$$\operatorname{Im} \rho = 0, \quad \psi^+ = (\psi^-)^*, \qquad (2.19)$$

if our integral converges on all intermediate surfaces. For ferromagnetic exchange the convergence of the integral over $\psi^\pm$ is determined (and ensured) by the kinetic term $\psi_i^+ J_{ij}^{-1} \dot{\psi}_j^-$ in the Lagrangian, and such a deformation is possible.

4. Of fundamental importance is the fact that we have imposed on the integration fields ψ not boundary conditions but the initial conditions

$$\psi^-(0) = 0. \qquad (2.20)$$

In particular, this implies that the magnons described by these fields do not obey Bose-Einstein statistics in the strict sense. This, incidentally, is rather obvious consequence of the bounded character of the spin operators.

The fulfillment of the equality (2.14) is a necessary (but, of course, not sufficient) condition for the correctness of our representation (2.18). For $J_{ij} = c\delta_{ij}$

$$\Gamma = c^{-1} \sum_i \int_0^\beta dt \Big(\frac{1}{2} \rho_i^2 + 2\psi_i^+ \dot{\psi}_i^- + \rho_i \Big) \qquad (2.21)$$

and $Z(\mathbf{h} = 0)$ can be represented in the form of a product of single-site Gaussian integrals, which are easily calculated and indeed lead to (2.14). Also by explicit calculation one can convince oneself that a change of regularization, equivalent to adding to term

$$\alpha \sum_i \int_0^\beta dt \, \rho_i$$

to Γ, violates the equality (2.14) (in particular, instead of being a constant, the free energy will be a nontrivial function of β).

5. In Ref. 3, the usual (for bosons) boundary conditions $\psi^-(0) = \psi^-(\beta)$ was used instead of the initial condition $\psi^-(0) = 0$. The Jacobian of the change of variables (2.11) with boundary conditions periodic in t and in the regularization (2.11′) is equal to (compare with (2.17); $\psi_0^- = \psi_N^-$)

$$\det \hat{J}_{\text{reg}} = \det \begin{pmatrix} \frac{1}{\Delta} - \frac{1}{2}\rho_1, 0, \ldots, 0, -\frac{1}{\Delta} - \frac{1}{2}\rho_1 & \\ -\frac{1}{\Delta} - \frac{1}{2}\rho_2, \frac{1}{\Delta} - \frac{1}{2}\rho_2, 0, \ldots, 0 & \\ \vdots & \\ 0, \quad . \quad . \quad . & , -\frac{1}{\Delta} - \frac{1}{2}\rho_N, \frac{1}{\Delta} - \frac{1}{2}\rho_N \end{pmatrix}$$

$$= \prod_{n=1}^{N} \left(\frac{1}{\Delta} - \frac{1}{2}\rho_n \right) + (-1)^{N-1} \prod_{n=1}^{N} \left(-\frac{1}{\Delta} - \frac{1}{2}\rho_n \right) \xrightarrow[\Delta \to 0]{} \to \text{const sh} \left(\frac{1}{2} \int_0^\beta \rho \, dt \right) \qquad (2.22)$$

and, unlike (2.15), vanishes on the hypersurface

$$\frac{1}{2}\int_0^\beta \rho\, dt = i\pi m$$

(m is an arbitrary integer). This implies that the holomorphic mapping $(\rho_n, \psi_n^+, \psi_n^-) \to (\varphi_n^z, \varphi_n^+, \varphi_n^-)$ of complex space is not one-to-one: Several configurations of the fields $(\psi^\pm, \rho)$ correspond to one configuration of $(\varphi^\pm, \varphi^z)$ (see, e.g., Ref. 11).

The existence of configurations of the fields $(\psi^\pm, \rho)$ that make the Jacobian of the transformation (2.11) vanish upon compactification of the segment $(0, \beta)$ into a circle is preserved under modifications of this transformation that do not change the number of derivatives in the right-hand side. It is easily verified that for periodic boundary conditions there does not exist a regularization ensuring fulfillment of the equality (2.14). The change from boundary conditions to initial conditions as a way of getting rid of the zero modes was proposed by Vergeles in Ref. 12, devoted to the $SU(2)$ anomaly.

The not inelegant method proposed in Ref. 2 for obtaining a functional representation leads to a result that does not satisfy the relation (2.14). This is evidently due to the impossibility, in the approach of the authors of Ref. 2, of fixing the gauge uniquely (the gauge of Ref. 2 has a Faddeev-Popov determinant of alternting sign, i.e., is not free of double counting).

6. There exists one further way of checking the correctness of the representation (2.18)—a way which also illustrates the decisive role of the boundary conditions in the functional integral.

We shall consider the vacccum expectation value of the operator $e^{-\beta H_{ex}}$:

$$Z_0 = \langle 0 | e^{-\beta H_{ex}} | 0 \rangle = \exp\left(-\frac{M}{2} S^2 \beta J(0)\right), \qquad (2.23)$$

where

$$J(0) = \sum_j J_{ij}$$

is the zeroth spatial Fourier component of the function

$$J_{ij} = J(\mathbf{r}_i - \mathbf{r}_j), \quad M = \sum_i 1.$$

On the other hand, a functional representation for Z_0 is obtained from (2.18) by replacing $\mathrm{Tr} B(\beta)$ by

$$\langle 0 | B(\beta) | 0 \rangle = \exp\left(-S\int_0^\beta \rho\, dt\right):$$

$$Z_0 = \int D\rho D\psi^+ D\psi^- \exp\left(-\Gamma + (-S + 1/2) \sum_i \int_0^\beta \rho_i\, dt\right). \qquad (2.24)$$

The functional integral (2.24) with the condition $\psi^-(0) = 0$ can be calculated exactly, despite the nonlinear interaction of the fields $\psi^\pm$ (which prevents an exact calculation in the case of periodic boundary conditions). In fact, the bare propagator of the field ψ

$$\langle \psi_i^-(t_1) \psi_j^+(t_2) \rangle_0 = 1/2 J_{ij} \theta(t_1 - t_2) \qquad (2.25)$$

is such that the integration over $\psi^\pm$ all the contributions to the effective action [a functional of $\rho_i(t)$] that contain more than one vertex corresponding to interaction of the fields $\psi^\pm$ with each other and with the field ρ are equal to zero. As a result, the effective action W_0 will be a linear functional of $\rho_i(t)$ (Γ_ψ is that part of the action which contains the fields $\psi^\pm$):

$$\exp(-W_0[\rho_i(t)]) = \int D\psi^+ D\psi^- e^{-\Gamma_\psi}$$

$$= \mathrm{const} \exp\left(-\frac{1}{2}\sum_i \int_0^\beta \rho_i\, dt\right), \qquad (2.26)$$

and the integral over $D\rho$ is Gaussian. [Our regularization (2.11′) corresponds to the step-function value $\theta(0) = \frac{1}{2}$.] Performing this integration, we arrive at (2.23).

7. The functional integral of Ref. 6 for $S = \frac{1}{2}$ and $S = 1$ has a local action that depends on one complex number field, one real number field, and ghost fields of a Grassman nature. The boundary conditions with respect to the variable t that are obeyed by these fields are standard, as is the expansion of the functional integral itself in a perturbation-theory series.[6] But the integration over the Grassmann variables reduces to the calculation of the determinant of a four-dimensional operator, and is not carried out in the final form. In this sense, the difference between the representation in Ref. 6 and the representation (2.18) is radical.

It makes sense to elucidate the origin of our choice of fields (ρ, ψ). The sttes of a classical spin—vectors of a fixed length—form a sphere. As is well known, the sphere is covered by two complex planes. Thus, in order to define a state in this case it is necessary to specify one complex number ψ and one variable taking two values—the number of the plane. With this Ising spin degree of freedom we can associate, in the sence of (2.2), the real field ρ. This "trivialization" of the topology of the configuration space, accompanied by the appearance of nontrivial dynamics, also occurs in the quantum case.

3. DYNAMICS AT LOW TEMPERATURES

1. The low-temperture limit implies that $\beta J(0) \gg 1$ and $\bar{n} = \langle S^+ S^- \rangle \ll 1$. In addition, we shall use one further small parameter—the inverse range of the interaction, first introduced for the Heisenberg ferromagnet in Refs. 4. This means that $J_\mathbf{k}$ [the Fourier transform of the exchange matrix $J(\mathbf{r}_i - \mathbf{r}_j)$] is of the order of J_0 in a neighborhood of the point $\mathbf{k} = 0$ of linear dimensions $\sim 1/R$, and of order $J_0(a/R)^3$ in the rest of the Brillouin zone (here a is the lattice constant). Hence, in particular, it follows that

$$\sum_\mathbf{k} J_\mathbf{k}^2 \sim J_0^2 (a/R)^3.$$

In the case of nearest-neighbor interaction we have $(a/R)^3 = 1/z$, where z is the number of these neighbors, and for lattices with cubic symmetry this quantity is small (see also Ref. 13).

For the description of the effects contained in the picture of a gas of interacting magnons the representation (2.18) is less convenient than the explicit boson representations of Holstein and Primakoff[14] and Dyson and Maleev,[15,16] which neglect the finite-dimensionality of the space of spin states. Therefore, in the present paper we shall concentrate on the nonmagnon part of the dynamics of a magnet—namely, on the dynamics of the fluctuations of the longitudinal component of the spin.

2. In Ref. 4 it was found that the correlator $K_{ij}(t) = \langle S_i^z(0) S_j^z(t) \rangle$ contains two terms of different nature. The first ("dynamic") term arises because the magnons carry away magnetization, and thus give a contribution to $K_{ij}(t)$. This contribution has a power-law degree of smallness in the temperature, and is also small in the inverse range of the interaction. The second ("static") term corresponds to "frozen" longitudinal fluctuations of the Ising type, and is exponentially small for $\beta J_0 \gg 1$ but has no degree of smallness in R^{-1}. At low temperatures and for not very long times the second contribution can be neglected in comparison with the first. However, if the real time $t \to \infty$, the dynamic term decreases as t^{-1} (see Refs. 4 and 5), whereas the static term does not depend on t. The fact that terms ensuring the relaxation of the static contribution in $K_{ij}(t)$ are absent in the perturbation-theory series of Refs. 3 and 4 implies that this series is incomplete. This also applies to the equilibrium variant of the spin-operator diagram technique developed in Ref. 5a. Below we shall show that in the functional representation (2.18), which correctly takes into account the organization of the spin degrees of freedom, the "frozen" fluctuations "revive" and are described by a dynamic neutral scalar field with nonzero mass.

3. In (2.18) we cannot perform the integration over $\psi^\pm$ exactly, and, as usual, we divide the action into a principal part and a perturbation:

$$Z(h) = \int D\rho\, D\psi^+ D\psi^- e^{-\tilde{\Gamma}}, \quad \tilde{\Gamma} = \Gamma_0 + \Gamma_{int}$$

$$\Gamma_0 = \int_0^\beta dt \Big(\frac{1}{2} \rho_i J_{ij}^{-1} \rho_j + 2\psi_i^+ J_{ij}^{-1} \psi_j^- - 2\bar{\rho} J_{ij}^{-1} (\psi_i^- \psi_j^+ - \psi_j^+ \psi_j^-) \Big) + \sum_i \int_0^\beta \rho_i\, dt - \int_0^\beta h_i J_{ij}^{-1} (\rho_j + 2\psi_j^+ \psi_j^-)\, dt,$$

$$\Gamma_{int} = \int_0^\beta dt \big(-2\tilde{\eta}_i J_{ij}^{-1} (\psi_i^- \psi_j^+ - \psi_j^+ \psi_j^-) + 2\psi_i^+ \psi_i^- J_{ij}^{-1} \psi_i^+ \psi_j^- - 2\psi_i^+ J_{ij}^{-1} (\psi_j^-)^2 \psi_j^+ \big) - \Gamma_{np}.$$

$$\Gamma_{np} = \sum_i \ln \Big(1 + \exp \int_0^\beta \rho_i\, dt + \psi_i^-(\beta) \int_0^\beta \psi_i^+(t) \Big(\exp \int_0^t \rho_i\, dt' \Big) dt \Big). \tag{3.1}$$

Here we have introduced the notation Γ_{np} for the nonpolynomial part of $\Gamma_{\rm int}$ and the notation $\tilde{\eta}_i = \rho_i - \bar{\rho}$, where $\bar{\rho}$ is the average value of the field ρ_i, and have set $\mathbf{h}_i = (0,0,h_i)$, since we are interested in the dynamics of only the z component of the spin. The saddle-point value $\bar{\rho}_0$ is determined by minimizing the bare effective potential, equal to [see (2.26)]

$$V_0(\bar{\rho}) = \frac{1}{2} J_0^{-1} \bar{\rho}^2 + \bar{\rho} + W_0(\bar{\rho}) = \frac{1}{2} J_0^{-1} \bar{\rho}^2 + \frac{1}{2} \bar{\rho}. \tag{3.2}$$

Then

$$\bar{\rho}_0 = -{}^1/_2 J_0 \tag{3.3}$$

and the average spin $\langle S^z \rangle_0 = -\frac{1}{2}$ corresponds to the usual ferromagnetic vacuum. The bare propagator of the field ψ in the Fourier representation with respect to the spatial coordinates has the form

$$G_{\mathbf{k}}^0(t_1, t_2) = \langle \psi_{\mathbf{k}}^-(t_1) \psi_{\mathbf{k}}^+(t_2) \rangle = {}^1/_2 J_{\mathbf{k}} \theta(t_1 - t_2) \exp(\bar{\rho}(1 - J_{\mathbf{k}}/J_0)(t_1 - t_2)). \tag{3.4}$$

The contributions from the terms of $\Gamma_{\rm int}$ to the various averages and correlators are either small in the temperature or small in the inverse range of the interaction.

4. We shall obtain the correlation functions in real time directly from the representation (3.1) by replacing the segment $(0,\beta)$ on which the integration fields are specified by a rectangular contour in the complex t-plane and placing sources $h_i(t)$ on this contour in the necessary manner (for more detail, see Ref. 17 and also Ref. 7).

The correlator $K_{ij}(t)$ in our representation is the average

$$K_{ij}(t) = J_{il}^{-1} J_{jm}^{-1} \langle (\tilde{\eta}_l(0) + 2\psi_l^+(0) \psi_l^-(0)) \times (\tilde{\eta}_m(t) + 2\psi_m^+(t) \psi_m^-(t)) \rangle. \tag{3.5}$$

For $t \to \infty$ the magnon (dynamic) contribution to (3.5) vanishes (one can convince oneself of this by direct calculations), and there remains

$$K_{ij}(t) \xrightarrow[t \to \infty]{} J_{il}^{-1} J_{jm}^{-1} D_{lm}(t), \quad D_{lm}(t) = \langle \tilde{\eta}_l(0) \tilde{\eta}_m(t) \rangle. \tag{3.6}$$

By integrating over $\psi^\pm$, we obtain $Z(h)$ in the form

$$Z(h) = \int D\tilde{\eta} \exp \Big(-\int_0^\beta dt\, \tilde{\eta}_i J_{ij}^{-1} \tilde{\eta}_j + \sum_i W[\tilde{\eta}_i] + \int_0^\beta dt\, \tilde{\eta}_i J_{ij}^{-1} h_j \Big), \tag{3.7}$$

where the functional $W[\tilde{\eta}]$ is represented in the form of a series in $\tilde{\eta}$, starting from the quadratic terms: $W[\tilde{\eta}] = W_2[\tilde{\eta}] + W_3[\tilde{\eta}] + \dots$. The terms linear in $\tilde{\eta}$ are eliminated by a redefinition of $\bar{\rho}$, and for us are unimportant. The decisive contributions for the behavior of $K_{ij}(t)$ as $t \to \infty$ are the infrared-singular contributions to $W_2[\tilde{\eta}]$:

$$W_2[\tilde{\eta}] = \frac{1}{2} e^{\beta\bar{\rho}} \Big(\int_0^\beta \tilde{\eta}\, dt \Big)^2 + \int_0^\beta dt \Big(\sum_{\mathbf{k}} G_{\mathbf{k}}^0(\beta, t) e^{t\bar{\rho}} \Big) \Big(\int_0^t \tilde{\eta}\, dt' \Big)^2. \tag{3.8}$$

The terms omitted in (3.8) either are smaller in the temperature and R^{-1} than those taken into account, or do not play a role in the formation of the asymptotic form of the correlator for $t \to \infty$ (an example is the contribution that has arisen because of the local interaction of the fiels $\tilde{\eta}$ and ψ). Substituting (3.4) into (3.8) and assuming that

$$J_{\mathbf{k}}=\begin{cases} J_0, & ka<a/R \\ J_\infty \ll J_0, & ka>a/R, \end{cases} \qquad (3.9)$$

we obtain in the first nonvanishing order in R^{-1}

$$W_2[\tilde\eta]=\frac{1}{2}e^{\beta\bar\rho}\Big(\int_0^\beta\tilde\eta\,dt\Big)^2+J_0\Big(\frac{a}{R}\Big)^3\cdot\frac{4\pi}{3}\int_0^\beta dt\,e^{\bar\rho t}\Big(\int_0^t\tilde\eta\,dt'\Big)^2$$
$$+J_\infty e^{\beta\bar\rho}\int_0^\beta dt\Big(\int_0^t\tilde\eta\,dt'\Big)^2.$$

Again we retain only the infrared-singular terms:

$$W_2[\tilde\eta]=\frac{1}{2}e^{\beta\bar\rho}\Big(\int_0^\beta\tilde\eta\,dt\Big)^2+J_\infty e^{\beta\bar\rho}\int_0^\beta dt\Big(\int_0^t\tilde\eta\,dt'\Big)^2. \qquad (3.10)$$

If we now replace the segment $(0,\beta)$ by a contour C coinciding with this segment along the real axis and having ends going off to $+i\infty$, the trajectories for which

$$\int_C dt\,\tilde\eta\neq 0$$

will give an infinite contribution to the action $W_2[\tilde\eta]$ on account of the second term in (3.10), and these trajectories can be neglected. This means that it is legitimate to make the replacement

$$\tilde\eta=\dot\eta,$$

where the field η vanishes at the ends of the contour C. [More precisely, we change from $\tilde\eta(t)$ to the variables

$$\eta(t),\quad \xi=\int_C\tilde\eta\,dt.$$

The term quadratic in ξ appears in the action with an infinite coefficient, and fluctuations of this mode do not give any contribution to the dynamics of the other variables or to the observable correlators.] The generating functional (3.7) for the correlations functions takes the form

$$Z(h)=\int D\eta\exp\Big(-\sum_{\mathbf{k}}\frac{1}{J_{\mathbf{k}}}\int_C dt(|\dot\eta_{\mathbf{k}}|^2-J_{\mathbf{k}}J_\infty e^{\beta\bar\rho}|\eta_{\mathbf{k}}|^2)$$
$$+\sum_{\mathbf{k}}\frac{1}{J_{\mathbf{k}}}\int_C h_{\mathbf{k}}\dot\eta_{-\mathbf{k}}\,dt\Big). \qquad (3.11)$$

It can be seen from (3.11) that the neglect of terms in $W[\tilde\eta]$ that are small in R^{-1} is admissible only when one is studying fluctuations $\eta_{\mathbf{k}}$ with $ka>a/R$ (i.e., the dominant region of **k**-space), and $J_{\mathbf{k}}=J_\infty$. For such fluctuations,

$$Z(h)=\int D\eta\exp\Big(-\frac{1}{J_\infty}\sum_{\mathbf{k}}\int_C dt(|\dot\eta_{\mathbf{k}}|^2-m_\eta^2|\eta_{\mathbf{k}}|^2)$$
$$+\frac{1}{J_\infty}\sum_{\mathbf{k}}\int_C h_{\mathbf{k}}\dot\eta_{-\mathbf{k}}\,dt\Big), \qquad (3.11')$$

where

$$m_\eta^2=J_\infty^2e^{\beta\bar\rho}. \qquad (3.12)$$

The η-field propagator, satisfying zero boundary conditions at the remote ends of the contour for real time t, followss directly from (3.11'):

$$\langle\eta_{\mathbf{k}}(0)\eta_{-\mathbf{k}}(t)\rangle=\frac{J_\infty}{2m_\eta}e^{-m_\eta t}. \qquad (3.13)$$

Thus, the desired asymptotic form of the longitudinal correlator is

$$K_{\mathbf{k}}(t)\xrightarrow[t\to+\infty]{}\frac{1}{2}e^{\beta\bar\rho/2}e^{-m_\eta t} \qquad (3.14)$$

for $ka>a/R$; we note that in the calculation of, say, a single-site correlation function such values of **k** give the main contribution, and the asymptotic form for

$$K_{ii}(t)=\sum_{\mathbf{k}}K_{\mathbf{k}}(t)$$

coincides in the leading approximation in R^{-1} with the right-hand side of (3.14).

The exponential temperature factor in the expression (3.12) for m_η^2 has a simple explanation. The systematic perturbation theory describes a small transverse disturbance on the background of the "frozen" longitudinal fluctuations, and in no finite order of magnon perturbation theory will there be relaxations of these fluctuations. The destruction of the longitudinal correlations in time occurs owing to rapid flips of the spins at the lattice sites; the probability of such configurations is suppressed precisely by the factor $e^{\beta\bar\rho}$, since $-\bar\rho$ is the energy necessary to their appearance. A similar mechanism of restoration of symmetry for a particle in a two-humped potential has been described in detail in Ref. 18.

5. Reference 5b contains the statement that the longitudinal fluctuations acquire nontrivial dynamics only in the case of exact fulfillment of kinematic identities, one of which, in essence, is the relation (2.14). The authors of Ref. 5 suggest that one abandons the systematic expansion of each correlation function in a perturbation-theory series, and, instead of this, substitutes the perturbative result for the transverse correlator into the kinematic identities and solves the resulting equation for the longitudinal correlator exactly. This calculational scheme, in its construction, involves the fulfillment of a relation of the type (2.14), but contains a certain inconsistency. In addition, because of the prescriptive character of the procedure of Ref. 5, it is extremely difficult to point to even a formal reason for any particular phenomenon.

6. Our formalism is explicitly inhomogeneous in time; e.g., the averages $\langle\rho_i(t)\rangle$ and $\langle\psi_i^+(t)\psi_i^-(t)\rangle$ are nontrivial functions of t. However, the observable

$$\langle S_i^z\rangle=J_{ij}^{-1}(\langle\rho_j(t)\rangle+2\langle\psi_j^+(t)\psi_j^-(t)\rangle) \qquad (3.15)$$

is independent of t in each order of perturbation theory, and one can convince oneself by direct calculations that the expansion of (3.1) in a series in R^{-1} reproduces the result of Ref. 4. Here the Green function of the field ψ is determined by the bilinear part of the action Γ together with the term [see (3.1)]

$$\sum_i\psi_i^-(\beta)\int_0^\beta\psi_i^+(t)e^{\bar\rho t}\,dt. \qquad (3.16)$$

In place of (3.4) we obtain

$$G_{\mathbf{k}}(t_1,t_2)=G_{\mathbf{k}}^0(t_1,t_2)+\frac{G_{\mathbf{k}}^0(\beta,t_2)F_{\mathbf{k}}(t_1)}{1-F_{\mathbf{k}}(\beta)},$$

$$F_{\mathbf{k}}(t)=\int_0^\beta G_{\mathbf{k}}^0(t,t')e^{t'\bar{\rho}}dt'. \qquad (3.17)$$

Using for the calculation of $\langle\psi_j^+(t)\psi_j^-(t)\rangle$ the propagator (3.17) and taking into account in $\langle\rho_j(t)\rangle$ the contribution of the following term in $\Gamma_{\rm int}$:

$$\sum_i \psi_i^-(\beta)\int_0^\beta \psi_i^+(t)e^{t\bar{\rho}}\left(\int_0^t \eta_i(t')dt'\right)dt, \qquad (3.18)$$

we arrive at the expression in Ref. 4 for $\langle S_i^z\rangle$.

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[1]S. Leibler and H. Orland, Ann. Phys. (N.Y.) **132**, 277 (1981).
[2]A. Jevicki and N. Papanicolaou, Ann. Phys. (N.Y.) **120**, 107 (1979).
[3]I. V. Kolokolov, Phys. Lett. **114A**, 99 (1986).
[4]V. G. Vaks, A. I. Larkin, and S. A. Pikin, Zh. Eksp. Teor. Fiz. **53**, 281, 1089 (1967) [Sov. Phys. JETP **26**, 188, 647 (1968)].
[5]V. I. Belinicher an V. S. L'vov, a) Preprint No. 191, Institute of Automation and Electrometry, Siberian Division of the Academy of Sciences of the USSR, Novosibirsk (1982); b) Zh. Eksp. Teor. Fiz. **86**, 967 (1984) [Sov. Phys. JETP **59**, 564 (1984)].
[6]V. N. Popov and S. A. Fedotov, Zh. Eksp. Teor. Fiz. **94**, No. 3, 183 (1988) [Sov. Phys. JETP **67**, 535 (1988)].
[7]I. V. Kolokolov, Zh. Eksp. Teor. Fiz. **91**, 1313 (1986) [Sov. Phys. JETP **64**, 1373 (1986)].
[8]A. M. Polyakov, Zh. Eksp. Teor. Fiz. **57**, 271 (1969) [Sov. Phys. JETP **30**, 151 (1970)].
[9]J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959).
[10]S. Elitzur, E. Rabinovici, Y. Frishman, and A. Schwimmer, Nucl. Phys. B **273**, 93 (1986).
[11]V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
[12]S. N. Vergeles, Preprint No. 1987-22, Chernogolovka (1987).
[13]I. V. Kolokolov *et al.*, Preprint No. 253, Institute of Automation and Electrometry, Siberian Division of the Academy of Sciences of the USSR, Novosibirsk (1984).
[14]T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[15]F. J. Dyson, Phys. Rev. **102**, 1217 (1956).
[16]S. V. Maleev, Zh. Eksp. Teor. Fiz. **33**, 1010 (1957) [Sov. Phys. JETP **6**, 776 (1958)].
[17]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[18]A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).

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