Theory of the interaction of Q-ball solitons

T. I. Belova and A. E. Kudryavtsev

Institute of Theoretical and Experimental Physics, Moscow
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The existence of classically stable solitons with a time dependence of the $\exp(i\omega t)$ type is predicted by classical theories describing self-interacting charged fields. The authors show that there is a range of relative velocities of colliding solitons, $v_{cr} \leqslant v < v_{max}$, in which their interaction is almost elastic. In particular, in this range the interaction of a soliton (S) with an antisoliton (A) does not result in annihilation. Near v_{cr} two solitons merge into one strongly excited soliton of the breather type: $S + S \to S^*$. The nature of the breather solution S^* is discussed. Further reduction in the collision velocity may result in an "explosion" of the solution manifested by decay to a lower vacuum state.

1. INTRODUCTION

Properties of solitons in scalar charged fields have been discussed frequently in the literature (see Refs. 1–5). We shall adopt Coleman's terminology and call them Q-balls.[1] In theories of this kind the charge Q is conserved and this leads to the possibility of existence of stable solitons. The ranges of stability of Q-balls were investigated in detail in Refs. 4 and 5. We shall consider the problems of the interaction between Q-balls which are stable from the classical point of view.

We shall define a Q-ball as the solution of a nonlinear equation for a complex field Ψ

$$\Box\Psi+\Psi\frac{dU(|\Psi|^2)}{d|\Psi|^2}=0, \qquad (1)$$

which is of the form

$$\Psi_S(\mathbf{r},t)=\varphi(\mathbf{r})e^{i\omega t}, \qquad (2)$$

and we shall assume that $|\varphi(\mathbf{r})|_{r\to\infty}\to 0$. The quantity $U(|\Psi|^2)$ in Eq. (1) is the potential. Equation (1) conserves the motion–charge integral:

$$Q=i\int d^n x\left(\Psi^*\frac{\partial}{\partial t}\Psi-\Psi\frac{\partial}{\partial i}\Psi^*\right). \qquad (3)$$

The soliton solution given by Eq. (2) minimizes the energy functional for a given charge Q.

The equation for $\varphi_S(\mathbf{r})$ can be solved numerically in the three-dimensional case (see Ref. 2). Examples of an exact analytic solution are known for the one-dimensional case.[5] Not all the soliton solutions of equations of type (1) are classically stable. The range of stability is governed by the nature of the potential $U(|\Psi|^2)$ and by the values of the parameter ω.

Our aim will be to study the evolution of solutions of Eq. (1) that are more complex than that predicting one soliton (for example, two-soliton, etc. solutions). By way of example, we shall consider the properties of the one-dimensional nonlinear Klein–Gordon equation with a four-boson attraction. We shall show that in this case there exists a wide class of many-soliton solutions. Although our analysis will be based on a specific model, a discussion will show that the results are not greatly affected in the qualitative sense by the model itself and can be applied to a wide class of equations of type (1) which lead to soliton solutions.

2. NONRELATIVISTIC LIMIT

We shall consider the Klein–Gordon equation with self-interaction:

$$\partial^2\Psi/\partial t^2-\partial^2\Psi/\partial x^2+m^2\Psi-\mu^2\Psi|\Psi|^2=0, \qquad (4)$$

corresponding to attraction. The potential in this theory

$$U(|\Psi|)=m^2|\Psi|^2-\frac{\mu^2}{2}|\Psi|^4$$

is shown in Fig. 1. This theory has a soliton solution[5]

$$\Psi_S=\frac{\sqrt{2}}{\mu}(m^2-\omega^2)^{1/2}e^{-i\omega t}\,\mathrm{ch}^{-1}[x(m^2-\omega^2)^{1/2}], \qquad (5)$$

which is stable if

$$\omega_{cr}\leqslant\omega\leqslant m,$$

where ω_{cr} is found numerically to be $\omega_{cr}\approx 0.7$ (for $m=\mu=1$). The mass an charge for such a solution are, respectively,

$$M_S=\frac{8}{3}\frac{(m^2-\omega^2)^{1/2}}{\mu^2}(m^2+2\omega^2), \qquad Q_S=8\frac{\omega(m^2-\omega^2)^{1/2}}{\mu^2}. \qquad (6)$$

The subsitution $\omega\to-\omega$ reverses the sign of the system $Q\to -Q$ and we then obtain a solution which would be natural to call an antisoliton (A).

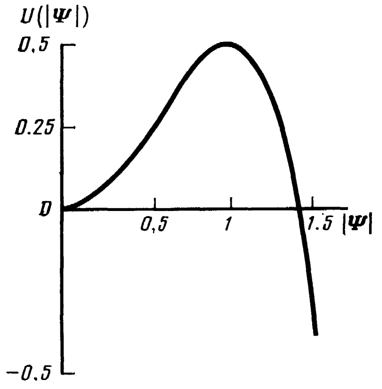

FIG. 1. Potential energy $U(|\Psi|)$ for Eq. (4). The units selected are such that $m=\mu=1$.

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Among the solutions of Eq. (4) there are some which are nearly nonrelativistic. We shall first go to the nonrelativistic limit in Eq. (4). Applying the standard procedure of Ref. 6, we shall introduce functions $\varphi(x,t)$ and $\chi(x,t)$ related to Ψ by

$$\varphi=\frac{1}{2}\left(\Psi+\frac{i}{m}\Psi_t\right),$$
$$\chi=\frac{1}{2}\left(\Psi-\frac{i}{m}\Psi_t\right).$$

Next, making the substitution $(\varphi,\chi)\to e^{-imt}(\tilde\varphi,\tilde\chi)$ we can readily show that the functions $\tilde\varphi$ and $\tilde\chi$ satisfy the following system of equations:

$$\begin{aligned} i\frac{\partial\tilde\varphi}{\partial t}&=-\frac{\nabla^2}{2m}(\tilde\varphi+\tilde\chi)-\varkappa(\tilde\varphi+\tilde\chi)|\tilde\varphi+\tilde\chi|^2,\\ i\frac{\partial\tilde\chi}{\partial t}&=-2m\tilde\chi+\frac{\nabla^2}{2m}(\tilde\varphi+\tilde\chi)+\varkappa(\tilde\varphi+\tilde\chi)|\tilde\varphi+\tilde\chi|^2,\end{aligned}\tag{7}$$

where $2m\varkappa=\mu^2$. Close to the nonrelativistic limit, we have $|\chi|\ll|\varphi|$ and in the zeroth (nonrelativistic) approximation we obtain

$$i\frac{\partial\tilde\varphi^{(0)}}{\partial t}=-\frac{\nabla^2}{2m}\tilde\varphi^{(0)}-\varkappa\tilde\varphi^{(0)}|\tilde\varphi^{(0)}|^2.\tag{8}$$

Equation (8) is the nonlinear Schrödinger equation investigated in Ref. 7. It belongs to a class of fully integrable equations and the properties of its solutions had been investigated in detail. The one-soliton solution of Eq. (8) is the nonrelativistic limit of the soliton Ψ_S of Eq. (5). Equation (8) also has many-soliton solutions. The two-soliton solution of Eq. (8) describes elastic scattering of solitons with a nontrivial dependence of the scattering phase of the initial velocity. Therefore, we may hope that for parameters close to the nonrelativistic limit the collisions of solitons in Eq. (1) are also nearly elastic.

We checked this by a numerical experiment in which we considered the process of collision of two solitons separated by a distance exceeding their dimensions. The results of calculations for the soliton velocity $v=0.4$ and the equal frequency ω of both solitons are presented in Fig. 2. The main conclusion of this numerical calculation is that the collision process is nearly elastic in the range $v\lesssim0.8$ and an increase in v clearly results in growth of perturbations in the region of the continuous spectrum. In view of the stability of one-soliton solutions such waves subsequently quit a soliton. Therefore, the interaction is in the form of an almost elastic collision and some of the energy and charge is transferred to small wave oscillations which leave the interaction region. An investigation of collisions of two solitons of different frequencies $\omega_1\neq\omega_2$ shows that solitons do indeed pass through one another without experiencing backscattering. The range of stability of two-soliton solutions is narrower than that of one-soliton solutions. An estimate of ω_{cr} is readily obtained from the requirement that the maximum of the modulus of the solution should be to the left of $|\Psi_{max}|$ in Fig. 1 ($|\Psi|=1$ when $m=\mu=1$). Hence, in the case of an n-soliton solution with the same frequencies ω, we find that

$$\tilde\omega_{cr}{}^2>1-1/2n^2.\tag{9}$$

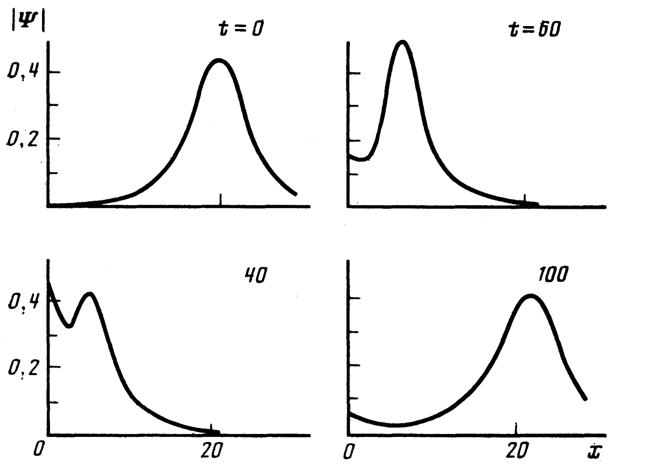

FIG. 2. Collisions of two solitons with identical frequencies $\omega_1=\omega_2=0.95$ traveling at a velocity $v=0.4$.

For example, if $n=1$, we have $\omega_{cr}^{(1)}=1/\sqrt{2}\approx0.707$, which is very close to the exact estimate given in Ref. 5. If $n=2$, we find that $\omega_{cr}\gtrsim0.935$. It should be noted that the estimate given by Eq. (9) for $n\geqslant2$ does not allow for the absence of the principle of superposition for solitons. We shall return to the problem of stability (see Sec. 5).

3. INTERACTION OF A SOLITON WITH AN ANTISOLITON

We have described above an extensive class of solutions of Eq. (4) which are close to many-soliton solutions of Eq. (8). Following a similar procedure, but separating solutions proportional to $\exp(+imt)$, we obtain solutions for nonrelativistic antisolitons described by an equation which is a complex conjugate of Eq. (8). Consequently, among all the solutions of Eq. (4) there are two separate subclasses which are nearly nonrelativistic for particles and antiparticles. The question arises of what we can say about the interaction in solitons belonging to these two different subclasses. In this case the initial condition in the form of a soliton and an antisoliton separated in space does not allow us to separate a general rapidly varying factor proportional to $\exp(\pm imt)$.

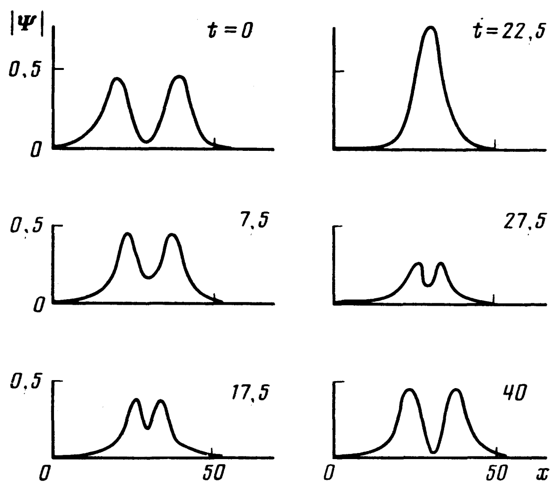

FIG. 3. Soliton–antisoliton collisions for $v_1 1-v_2=0.4$ and $\omega_1=\omega_2=0.95$.

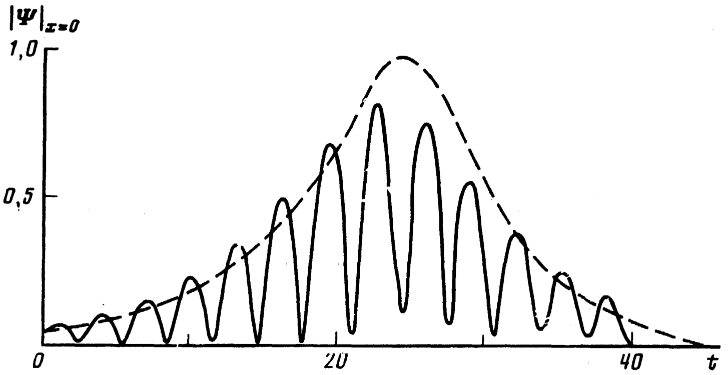

FIG. 4. Behavior of the field amplitude $|\Psi(0, t)|$ for soliton-soliton collisions (dashed curve) and for soliton-antisoliton collisions (continuous curve) in the case when $v = 0.4$ and $\omega = 0.95$.

An attempt to obtain an analytic solution of this problem is discussed in Sec. 4. However, we first tackled this problem by numerical solution of Eq. (4).

The results of our calculations are presented in Fig. 3 and can be summarized as follows: when the approach velocities of S and A are small ($v \sim 0.3$–0.4), a soliton and an antisoliton pass practically elastically through one another. An increase in the relative velocity increases the inelasticity of the interaction and the losses, by analogy with the case of S–S collisions. However, it follows from the numerical results that in the principal approximation there is no S–A (or S–S) interaction in the scattering process. The difference between the behavior of the solutions for the cases of S–S and S–A interactions is demonstrated in Fig. 4, which shows how the modulus of the field $|\Psi(0, t)|$ behaves as a function of time.

4. LINEARIZATION OF THE PROBLEM. CHARACTERISTIC TIME

Equation (4) is strongly nonlinear. However, a stable many-soliton solution is concentrated mainly in the range of small values of $|\Psi|$ to the left of the maximum of the potential in Fig. 1. In this range the potential $U(|\Psi|)$ is dominated by the term $m^2|\Psi|^2$. We can thus hope that for short times the evolution of a soliton can be described approximately as the solution of the linear equation

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} + m^2\Psi = 0. \qquad (10)$$

We shall therefore consider the evolution in time of a wave packet obeying Eq. (10) and identical at $t = 0$ with $\Psi_s(x, 0)$ of Eq. (5) (such an identity is stipulated also for the first derivatives with respect to t at $t = 0$). Expanding the solution of Eq. (10) as a Fourier integral and assuming the initial condition just stated, we obtain the following Klein–Gordon (KG) solution

$$\Psi_{KG}(x, t) = \frac{\sqrt{2}}{4\mu}\int_{-\infty}^{+\infty} \frac{dk\, e^{ikx}}{\operatorname{ch}(\pi k/2\Omega)} \left\{\left[1 - \frac{\omega}{(m^2+k^2)^{1/2}}\right] \times \exp[i(m^2+k^2)^{1/2}t] + \left[1 + \frac{\omega}{(m^2+k^2)^{1/2}}\right]\exp[-i(m^2+k^2)^{1/2}t]\right\}, \qquad (11)$$

where $\Omega = (m^2 - \omega^2)^{1/2}$. It should be noted that particles and antiparticles are represented differently in the case of solitons. In the limit $\omega \to m$ the contribution of antiparticles disappears. In the nonrelativistic limit such a wavepacket is narrow in the momentum space: $k_{\text{char}} \lesssim \Omega \ll m$.

Naturally, the function $\Psi_{KG}(x, t)$ of Eq. (11), which is identical with $\Psi_s(x, t)$ of Eq. (5), begins to diverge from $\Psi_s(x, t)$ at $t \neq 0$. However, this divergence is not very important as long as the phases of the harmonics dominating $\Psi_{KG}(x, t)$ are the same. In other words, the solution given by Eq. (11) reproduces mainly a soliton up to a characteristic moment in time t_{char} described by the relationship

$$[(m^2+k_{\text{char}}^2)^{1/2} - m]\, t_{\text{char}} \sim \pi/2,$$

which yields

$$t_{\text{char}} \sim m/\Omega^2. \qquad (12)$$

Any other solution of the problem (1) close to the nonrelativistic limit also differs little from the solution of the KG equation (10) when $t < t_{\text{char}}$. For example, in the case of the two-soliton (S–S) solution when the relative velocity is v and the characteristic dimensions of a soliton are Ω^{-1}, the collision time of solitons is $t_{\text{coll}} = 1/\Omega v$. If the condition $t_{\text{coll}} \ll t_{\text{char}}$ is satisfied, the interaction of solitons in the process of collision can be ignored in the zeroth approximation.

The soliton-antisoliton (S–A) scattering case is also described by the above scheme. In fact, if in this case we expand the initial state in terms of the solutions of Eq. (10), which are given by

$$\Psi_{SA}^{KG}(x, t) = \int \{A_k \exp[i(m^2+k^2)^{1/2}t] + B_k \exp[-i(m^2+k^2)^{1/2}t]\} e^{ikx}\, dk,$$

we can easily show that the contributions to the main harmonics of the function A_h come from an antisoliton and the contribution to B_h come from a soliton. In view of the nonrelativistic conditions, packets remain narrow in both cases. Therefore, the estimate of the characteristic time given by Eq. (12) remains valid. For all the cases under consideration (S–S and S–A) the relative velocity in collisions has a lower limit:

$$v \gtrsim v_{\text{cr}} \sim (m/\Omega)^{-1}.$$

5. LAGGING AND LEADING EFFECTS IN SOLITON COLLISIONS

We can thus see that there is a range of soliton velocities in which they interact weakly. This range is defined by the inequalities

$$v_{c1} < v \leqslant v_{max}. \tag{13}$$

The value of v_{max} is defined only approximately. We can show that an increase in the soliton velocity increases the contribution of antiparticles to the expansion of Eq. (11) and the wave packet broadens. Consequently, the interaction process becomes increasingly inelastic.

However, nonlinear effects appear also in collisions of solitons traveling at velocities in the range defined by Eq. (13). One of these effects is the phase lead or lag of solitons in the course of their interaction. In the *S–S* interaction case the numerical results predict a phase lead in soliton collisions. This process is in qualitative agreement with a phase lead in the case of collisions of solitons described by the nonlinear Schrödinger equation.[8] In the case of the *S–A* interaction, we find that for some values of the collision parameters (velocities v and frequencies ω) again we can expect a phase lead, but it is less than in the case of the *S–S* interactions. The results of calculations of the lead in the case of the *S–S* and *S–A* collisions of solitons of frequency $\omega = 0.975$ are given below for some of the soliton velocities v:

v	$S+S$	$S+A$
0.20	1.13	1.11
0.15	1.52	1.18
0.10	1.42	1.35
0.05	2.125	–

The results are presented in the form of the ratios of the path x traveled by a soliton to the theoretical path x_{th} that would have been traversed by a freely moving soliton in the same time interval. It is worth noting that the lead is less for the *S–A* collisions than for the *S–S* collisions. This difference can be explained as follows. If we write down the solution of Eq. (4) in the form

$$\Psi = \Psi_1(x, t) + \Psi_2(x, t) + \chi(x, t) = \Psi_0 + \chi(x, t), \tag{14}$$

where Ψ_1 and Ψ_2 are the solutions for the first and second solitons, we can linearize the solution of Eq. (4) with respect to $\chi(x, t)$ within the limits defined by Eq. (13). Then, the equation for $\chi(x, t)$ becomes

$$\chi_{tt} - \chi_{xx} + m^2\chi + V_1\chi + V_2\chi^* = Q(x, t), \tag{15}$$

where $V_1 = -2\mu^2\Psi_0\Psi_0^*$, and $V_2 = -\mu^2\Psi_0\Psi_0$. The source function is determined by the overlap of the solutions Ψ_1 and Ψ_2 and is given by

$$Q(x, t) = \mu^2(\Psi_1^2\Psi_2^* + \Psi_1^*\Psi_2^2 + 2|\Psi_1|^2\Psi_2 + 2\Psi_1|\Psi_2|^2). \tag{16}$$

The behavior of the function $Q(x, t)$ is different in the case of the *S–S* and *S–A* collisions. In the former case all the terms in Eq. (16) have the same fast phase: $Q(x, t) \propto \exp(i\omega t)$. In the *S–A* case the time dependence is different for different terms in Eq. (16). Therefore, the effect of the source on the function $\chi(x, t)$ in the *S–A* case is weaker than that in the *S–S* case because of oscillations.[1)] Since the source acts on all the soliton modes, this applies in particular to the zeroth shift mode. Therefore, the phase shift in the *S–A* case is less. For some values of the parameters it may even show reversal of the sign. This last effect was observed by us. For the *S–A* interaction the frequency ω changes at a fixed soliton velocity $v = 0.4$. Then, a lead is observed for $\omega = 0.870$ and a lag for $\omega = 0.875$. Such a reversal of the sign may be due to the fact that up to the moment of full approach of solitons they acquire different phases (depending on the velocity and frequency) proportional to ωT_{app}, where T_{app} is the approach time.

6. CRITICAL VELOCITY AND "EXPLOSION" OF A SOLUTION

When the soliton velocity is reduced, the collision time increases and the role of the nonlinear effects should become greater. Numerical experiments demonstrate that there is a certain critical velocity v_{cr} below which the behavior of the solution changes radically. In the case of Eq. (4) and some values of the parameter ω we can then expect an "explosion" of a two-soliton solution. This can be described as follows. The collision of solitons gives rise to a strong peak at $x = 0$ in a graph describing $|\Psi(x, t)|$. The width of this peak is small and it is less than the soliton width. The quantity $|\Psi(0, t)|$ grows and becomes greater than unity, i.e., the peak shifts to the right of the hump of the graph representing the potential energy of the field. The rate of rise of $|\Psi(0, t)|$ then rises strongly although the width of the peak does not change. In other words, a singularity of the δ-function type forms in the field.

The values of v_{cr} depend on the frequency ω. In the case of the *S–S* interaction an instability occurs at frequencies $\omega > \omega_{cr}^{(2)} = 0.935$, which follows from an estimate given by Eq. (9). For example, if $\omega = 0.95$, then $v_{cr} = 0.240$. At high values of $\omega = 0.96$, an estimate[2)] gives $v_{cr} < 0.05$. On the other hand, in the case of the *S–A* collisions the critical velocity is reached only for $\omega < \omega_{cr}^{(2)} = 0.935$. For example, $v_{cr} = 0.4$ for $\omega = 0.85$, whereas $v_{cr} < 0.05$ for $\omega = 0.95$.

These differences in the critical velocity v_{cr} are clearly associated also with the difference between the expressions for the source $Q(x, t)$ in the cases of the *S–S* and *S–A* collisions.

7. LARGE ANHARMONIC OSCILLATIONS NEAR v_{cr}

Calculations reveal a new nonlinear effect near the critical velocity v_{cr} in the *S–S* collision case. For example, if

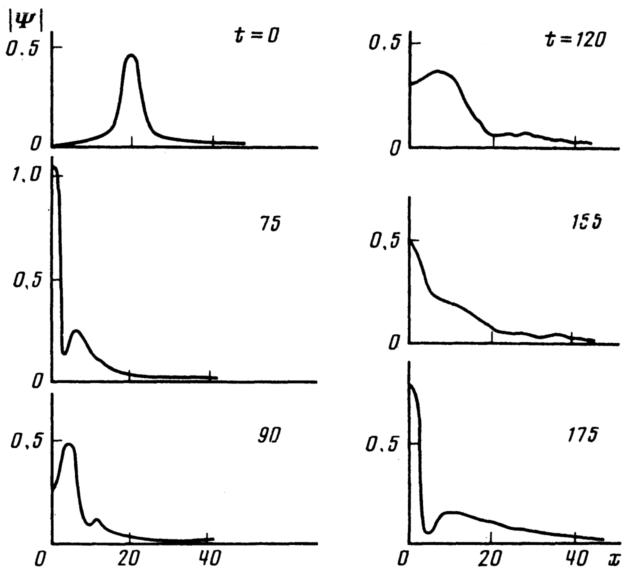

FIG. 5. Behavior of the function $|\Psi(x, t_i)|$ in the range of oscillations when $\omega_1 = \omega_2 = 0.95$ and $v_1 = -v_2 = 0.2 = 0.250$.

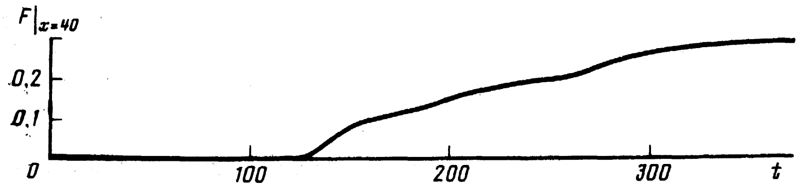

FIG. 6. Energy flux from the region of pulsations across the boundary ($x = 40$) calculated for $\omega_1 = \omega_2 = 0.95$ and $v_1 = -v_2 = 0.250$.

$\omega = 0.95$, the critical values $v \approx 0.240$. Collisions at a velocity $v = 0.255$ are characterized by the usual passage of solitons. When the velocity is $v = 0.250$, solitons do not pass through one another but coalesce. Then, near the point $x = 0$ a strong perturbation of the type described in Sec. 6 is observed. This solution is shown in Fig. 5. Figure 6 demonstrates the energy flux across the boundary at $x = 40$; the values of the flux are given in absolute units and they should be compared with the energy of the system amounting to 2.4122 (on the semiaxis $x \geqslant 0$); we can see that $\sim 12\%$ of the energy is lost after four oscillations. Our computer calculations were stopped at $t = 360$.

Our analysis thus revealed a long-lived pulsating solution. The modulus of the field amplitude at $x = 0$ is shown in Fig. 7. Clearly, the solution should be regarded as a strong excitation of a discrete mode above a soliton. The presence of a discrete mode in a spectrum of small oscillations, relative to the soliton solution, is discussed in Ref. 5 in connection with the soliton stability. The spectra of the eigenvalues of small oscillations around the soliton of Eq. (5) are discussed in Ref. 5. In this soliton stability region there is a frequency $\tilde{\Omega}$ which modulates the amplitude and profile of a soliton. It follows from Ref. 5 that in the case of small oscillations and values of ω close to unity, we have

$$\tilde{\Omega} \approx 1 - \omega. \tag{17}$$

Therefore, when a given mode is excited, a solition is converted into a breather with a period $T_0 = 2\pi/\tilde{\Omega}$. For $\omega = 0.95$ we find[3)] that $T_0 \approx 125$. The average value of the period $\overline{T}$ is 95 for a nonlinear oscillation shown in Fig. 5. A high value of $\overline{T}$ is close to T_0. We must bear in mind that the solution obtained here is far from a harmonic oscillation, because the oscillation amplitude is not small. Consequently, anharmonicity is important in the theory, particularly the interaction of the investigated mode with other modes. The existence of such a nontrivial interaction between different modes is manifested clearly in the case of S–S collisions characterized by $\omega = 0.95$ when the velocity is $v = 0.245$, lying half-way between $v_{cr} = 0.240$ and the value $v = 0.250$ corresponding to a breather-type solution. In this case the solution "explodes," but this is not an instantaneous process. It is preceded by one large oscillation. Therefore, a redistribution of the energy between various modes plays here a role. Some of the energy of the collision excites other modes. After a period $\overline{T}$ some fraction of the energy returns to the singular mode and the energy of this mode is sufficient for an explosion. Figure 8 shows, in terms of the variables v and ω, a curve separating the region where solitons fly apart from a region of a breather and an explosion.

Our hypothesis on the nature of the breather is supported by the fact that solutions of this type have not been obtained for the S–A collisions. In fact, in this case the classical object from which deviations should be studied is uncharged vacuum. It has no discrete frequencies and all the excitations belong to the continuum. In our case a breather solution is obtained for nonrelativistic velocities v and frequencies ω close to the nonrelativistic limit $|\omega - m| \ll m$. Under these conditions it would seem that the solution with the S–S collisions should reproduce the behavior of two-soliton solutions of the nonlinear Schrödinger equation (8), which corresponds to elastic soliton scattering. However, in the case of the nonlinear Schrödinger equation there are also solutions[7] corresponding to bound soliton states. The discrete level found[5] in the spectrum of excitations above a soliton of Eq. (5) [which is the solution of Eq. (4)] is identical in the nonrelativistic limit with a two-soliton bound state obtained in Ref. 5 for the case when the relationship between the parameters[4)] is given by $\eta_1^2 \gg \eta_2^2$ Then the two-soliton bound state obtained from the nonlinear Schrödinger equation represents an excitation of small amplitude $\sim \eta_2$ above a soliton

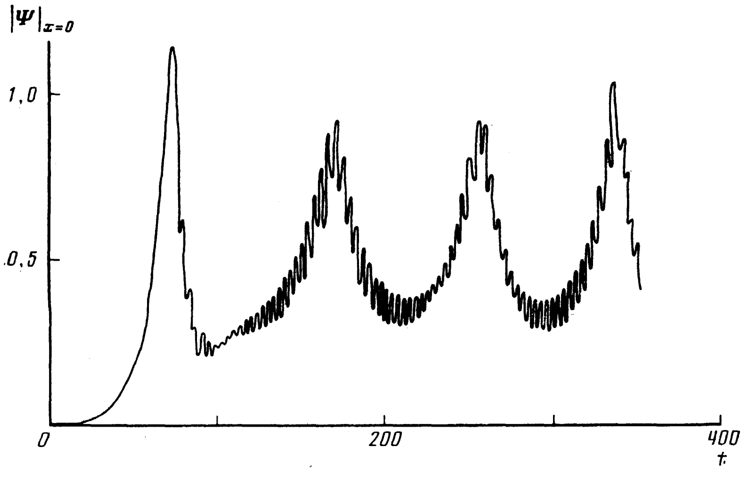

FIG. 7. Function $|\Psi(0, t)|$ in the oscillation case: $\omega_1 = \omega_2 = 0.95$; $v_1 = -v_2 = 0.250$.

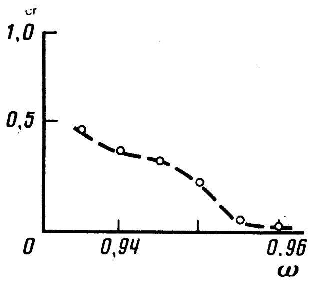

FIG. 8. Boundary between the region where solitons fly apart and the region where they coalesce, plotted using variables ω and v.

of amplitude $\sim\eta_1$ and its frequency $\Omega = 4\eta_1^2$ is identical with the frequency $\tilde{\Omega} = 1 - \omega$ calculated in Ref. 5.

This bound state of solitons obtained from the nonlinear Schrödinger equation is absolutely unstable against any small perturbations[8] and in the presence of such perturbations it should split into two solitons. In our case such a perturbation is in the form of a small difference between the initial and nonrelativistic conditions, as well as a difference of Eq. (4) from the nonlinear Schrödinger equation, so that the bound state mentioned above appears as a resonance resulting from coalescence of two solitons, i.e., due to the process

$$S+S\rightarrow S^{*}.$$

CONCLUSIONS

The reported results represent an initial investigation of the interaction of solitons described by Eq. (4). Undoubtedly, a more detailed discussion is needed of the range near v_{cr} for different frequencies, a more detailed study should be made of the soliton delay times, etc. However, the pattern of the interaction of solitons has now been investigated in the first approximation.

We demonstrated that in the case of charged solitons of the Q-ball type there is a range of parameters (frequencies ω and velocities v) in which these classical objects interact weakly (almost elastic scattering). Although our attention was concentrated mainly on the one-dimensional equation (4), this conclusion applies to other equations of the (1) type which predict Q-ball solitons. In fact, the basis for the theoretical approach is the proximity of each of the interacting objects to the nonrelativistic limit. Therefore, our conclusion of the existence of a characteristic time t_{char} applies also in the multidimensional case. Numerical values of the critical velocity and of other parameters can naturally depend on the adopted model.

An important result is also the discovery of a large-amplitude breather-type solution. The existence of such a solution is related to the existence of a discrete excitation mode of a soliton. The large period of the oscillations in question can be regarded as a characteristic feature suitable for identification of the object when the theory based on Eq. (1) is compared with concrete physical objects.

By way of physical applications, we must mention primarily an interesting possibility that collisions of tiny droplets of liquid helium can be described in terms of the soliton interaction. Another example is an attempt to describe collisions of nuclei in the soliton collision language.[9] In this context the stopping of colliding solitons near v_{cr} discovered by us may indicate feasibility of energy transfer from translational motion to internal degrees of freedom, which is of importance in the search for a quark-gluon plasma in collisions of nuclei. Finally a suitable physical object can be cold boson entities in outer space.[10]

We shall now consider the relationship between our results and those reported by other authors. The role of the nonrelativistic limit in the stability conditions of some nontopological soliton solutions of the field equations was pointed out in Ref. 11. In Ref. 12 the problems of the interaction of Q-ball solitons were investigated numerically for certain types of the $U(|\Psi|)$ potentials. The authors of Ref. 12 discovered a strong perturbation of vacuum at the point $x = 0$ after the passage of solitons, when such a perturbation behaves as a localized breather. The existence of a breather is not related to threshold phenomena, but appears in a wider range of relative velocities. It is quite likely that the occurrence of breathers in Ref. 12 is also related to the presence of a discrete mode in the soliton excitation spectrum, although the problem of the possible origin of the breather solution is not considered from this point of view in Ref. 12. It should be noted that, in contrast to our calculations, in Ref. 12 a breather is predicted for the *S–S* and *S–A* collisions and it is pointed out that the period of an *S–S* breather is considerably greater than that of an *S–A* breather. One should mention also the results of Ref. 13, where a report was given of numerical calculations supplemented by the results of Ref. 12. In particular, the problem of the interaction of breathers was considered in Ref. 13. It should be mentioned moreover that some problems investigated by us are discussed in Refs. 14 and 15. Our preliminary results were published in Ref. 16.

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1) The initial phases of solitons in collisions are fixed by the condition (5). An additional constant phase, permitted by the equation of motion, is not introduced.

2) Studies of soliton collisions in the case of a low relative velocity are difficult because of the large computation time which is needed.

3) It is understood here that the frequency ω does not change in the course of a soliton coalescence process. In fact, the interaction conserves the charge Q but not the frequency ω. Allowance for this fact ensures a better agreement between T_0 and the numerical value of $\overline{T}$.

4) The parameters η_i are described in terms of the notation of Ref. 7.

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