Spin precession in a gravitational field

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We calculate spin precession rates in a gravitational field in a Lorentz frame that we construct via a rotationless Lorentz transformation from the laboratory frame. In deriving this tranformation, we obtain an exact expression for the rate of precession in both the Schwarzschild and Reissner-Nordstrøm metrics. Along the way, a transformation is derived which reduces the latter to "isotropic" form. Furthermore, our technique has been applied to an arbitrarily weak gravitational field. Our expression for the rate of precession in the limiting cases of low velocity and the post-Newtonian approximation is in agreement with previous results. As the speed of a particle tends to unity, the expression for the rate of precession has a finite limit, a part of which describes the rotation of the plane of polarization of a photon.

1. INTRODUCTION

Comparison of the predictions of general relativity with experimental data remains one of the most pressing problems of physics, due to a paucity of experimental results. Recently developed experimental techniques have made it possible to measure relativistic gravitational effects to high accuracy,[1] suggesting that calculations of specific new processes and measurements of effects that are already known might be refined to further test Einstein's theory.

In the present paper, we examine gravitational polarization effects involved in particle spin (or gyroscope axis) precession and the rotation of the plane of polarization of a photon, all moving freely in a gravitational field. These are all manifestations of exactly the same phenomenon, since the gyroscope axis, particle spin vector, and photon polarization vector all undergo parallel transport along their respective trajectories. De Sitter[2] was the first to point out, soon after the initial appearance of the general theory of relativity, that a gyroscope ought to precess in a gravitational field, while Fokker[3] provided the first correct calculation of uniform circular motion in the field of a centrally symmetric massive body (Schwarzschild field). Schiff[4] calculated the rate of precession of a nonrelativistic gyroscope in a weak field, but because it is so small, this effect has thus far not been observed. For example, a gyroscope mounted within a satellite orbiting the Earth would precess a total of seven arcseconds per year, an amount not yet accessible to existing instrumentation (see Ref. 1 for a review of the experimental situation and for further references). This phenomenon, however, could be detected if one were to investigate the behavior of polarized relativistic particles passing near a massive body like the Sun (relativity is required to prevent particle capture). The effect is proportional to the mass of the body responsible for the field, so the advantage that the latter experiment using polarized particles would have over a terrestrial gyroscope experiment would be of order 10^5 per revolution of the satellite. Obviously, the gyroscope torquing can be handled by increasing the length of the observation, which of course requires that the satellite maintain a fixed orientation throughout the entire measurement.

In calculating the spin precession rate, one assumes that the spin vector of a free particle undergoes parallel transport along a geodesic (if nongravitational forces act on the particle, the spin will undergo Fermi-Walker transport[5]). This statement was proved in Ref. 6 for the case of an electron in a weak field, and Ravndal proved it for a point particle with spin within the framework of supersymmetric classical mechanics.[7] One might hope that under appropriate circumstances the proposition would also be true in quantum field theory, and we shall provide qualitative arguments to support this view. In fact, the wave equation for a particle in a gravitational field is formally the same as the corresponding equation for a free particle in flat space described by curvilinear coordinates. Since in flat space the spin of a free particle maintains its orientation, and the straight line along which it moves is a geodesic, our previous statement about parallel transport of the spin along a geodesic is not surprising in this context. In curvilinear coordinates, as expected, we find

$$\frac{DS^i}{D\tau}=0, \tag{1}$$

where $D/D\tau$ represents the covariant derivative, $D\tau=d\tau$ is the element of length on the particle's world line, and S^i is its spin vector. It would seem that a similar equation ought to hold for a particle in a gravitational field as well, but we would be making an implicit assumption about the minimality of such a generalization. In a gravitational field, in general, terms can appear on the right-hand side of Eq. (1) that are proportional to the Riemann tensor R_{ijkl}, and to derivatives of the particle velocity $u_{;j}{}^{i}$, which vanish in free space. Examples include $R^{\ i}_{klm}u^l(Du^m/D\tau)S^k$; $R^{\ i}_{klm}u^{l;m}S^k$; $S_j(u^{i;j}-u^{j;i})$ etc.

Here $u_{i;j}$ denotes the covariant derivative with respect to x^j. In writing out possible nonminimal terms, we have observed the necessary conservation of the norm S^iS_i of the spin vector under parallel transport. If the dimensions of a particle wave packet are much less than the typical scale of variation of the gravitational field, one can treat the particle as a point particle to first order, so that only projections of the derivative $u_{;j}{}^{i}$ along the direction of particle motion survive, i.e., $Du^i/D\tau$, which vanish by virtue of the equation of motion. For these reasons, then, nominal terms drop out of the right-hand side of Eq. (1) for particle motion along a geodesic.

At this point, our discussion ceases to be rigorous, as

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there have been studies of the effects of spin on the equation of motion in which it was concluded that a particle with spin in a gravitational field does not move along a geodesic.[8] Under those circumstances, when the dimensions of the wave packet become comparable to the typical scale length of the gravitational field, it becomes impossible to localize the spin at a well-defined point, and S^i in Eq. (1) then describes a density for the spin vector. Furthermore, the situation is even more complicated when nonminimal terms are present, in that the norm S^iS_i of the spin density is then not conserved. In addition, in considering the exact equations of a particle in a gravitational field (a Dirac electron, for example), one gets the distinct impression that no closed set of equations exists to describe the evolution of the spin vector in an arbitrary gravitational field.

Our paper is organized as follows: in Sec. 2, using the Schwarzschild field, we devise a means of constructing a Lorentz frame, in which we then calculate the rate of spin precession. In Sec. 3, we obtain the rate of precession in a Schwarzschild field in a different way, in order to justify certain assumptions made in Sec. 2. In Sec. 4, the technique developed in Sec. 2 is applied to the Reissner-Nordstrøm field, and finally, in Sec. 5 we apply it to an arbitrary gravitational field.

2. MOVING REFERENCE FRAME

For the sake of definiteness, we shall deal with electrons. For a particle spin vector S^i and velocity u^j we have, as usual,[9]

$$S^iu_i=0. \tag{2}$$

Since the norm vector undergoing parallel transport is conserved, condition (2) implies that the norm of the spin vector is conserved in the electron's rest frame. Next, let us construct at any point in space a tetrad of orthonormal vectors $\lambda^i_{(a)}$, where $a = 0,1,2,3$ numbers the vectors of a tetrad; as the particle moves, we relate its spin to these vectors. It follows from (2) that we can only introduce a three-dimensional angular rate of precession in a reference frame (known as a comoving frame) in which one of the vectors, say $\lambda^i_{(0)}$, coincides with u^i. Clearly, though, the precession rate calculated in an arbitrary comoving frame generally has little to say about the spin precession itself, since the various comoving reference frames can rotate (three-dimensionally) about one another.

There is a single preferred reference frame in flat Minkowski space (call it Λ) which is derived from the laboratory frame L by a pure Lorentz transformation with no spatial rotation. Obviously, the rate of spin precession in that frame will also be the desired precession rate relative to the coordinate system under consideration. With a view to further generalization, let us describe the frame L and Λ in some detail.

Consider a Minkowski space with line element ds and Cartesian coordinates, t, x, y, and z such that $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$. The tetrad of normalized vectors which are directed, at every point in space, along constant-coordinate lines, forms a laboratory frame of reference with components

$$L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{matrix}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} -\lambda^i_{(\alpha)}.\\ -\lambda^i_{(0)}\end{matrix} \tag{3}$$

Here each row corresponds to a particular vector of the tetrad, and each column corresponds to a vector component (given by the superscript). Let us now operate on the frame of reference L with a rotationless Lorentz transformation matrix, and take the velocity $\mathbf{v}$ to be in an arbitrary direction. The resulting reference frame takes the form[10]

$$\Lambda=\begin{bmatrix}\delta_\beta{}^\alpha-(\gamma-1)\dfrac{v^\alpha v^\beta}{v^2} & \gamma v^\alpha\\ \gamma v^\beta & \gamma\end{bmatrix},\quad \gamma=(1-v^2)^{-1/2}. \tag{4}$$

This is precisely the frame in which one can obtain an expression for the Thomas precession (see below for details).

We now attempt to generalize these results to the case in which a gravitational field is present. Specifically, we consider a Schwarzschild field in "isotropic" coordinates[11]:

$$ds^2=\frac{g_-^2}{g_+^2}dt^2-g_+^4(d\xi^2+d\eta^2+d\zeta^2),$$
$$g_+=1+m/2\rho,\quad g_-=1-m/2\rho, \tag{5}$$

where $\boldsymbol{\rho} = (\xi,\eta,\zeta)$. In these coordinates, one can readily produce an analog of the laboratory reference frame in Minkowski space by choosing a tetrad of normalized vectors directed along the constant-coordinate curves, thereby forming a frame of reference that we may call the laboratory frame:

$$L=\begin{bmatrix}g_+^{-2}&0&0&0\\0&g_+^{-2}&0&0\\0&0&g_+^{-2}&0\\0&0&0&\dfrac{g_+}{g_-}\end{bmatrix}. \tag{6}$$

The analogy with Minkowski space is a perfect one, and we have no leeway in choosing this frame of reference.

Next, let us attempt to construct the analog of a Lorentz frame in the Schwarzschild field (5). If we go to a locally geodesic system, i.e., one in which the metric tensor is diagonal at the point in question, then it is perfectly natural to assume that the transformation matrix Λ^a_b,

$$\lambda^i_{(a)}=\Lambda_a{}^b l^i_{(b)};\quad \lambda^i_{(a)}\in\Lambda;\quad l^i_{(b)}\in L$$

is the same at the given point as the usual rotationless Lorentz transformation matrix in flat space [Eq. (4)]. Indeed, the tangent space at the given point is indistinguishable from a flat space, and the global properties of the gravitational field that result from a nonvanishing curvature tensor are not manifested under this transformation.

It is easy to see that the transformation

$$t'=\frac{1-m/2\rho_1}{1+m/2\rho_1}t;\quad \boldsymbol{\rho}'=\left(1+\frac{m}{2\rho_1}\right)^2\boldsymbol{\rho}, \tag{7}$$

where ρ_1 = const, diagonalizes the metric (5) at the points

$$t_1'=\frac{1-m/2\rho_1}{1+m/2\rho_1}t;\quad \boldsymbol{\rho}_1'=\left(1+\frac{m}{2\rho_1}\right)^2\boldsymbol{\rho}_1. \tag{8}$$

At these points, the laboratory frame L takes the form (3), while the Lorentz frame, and accordingly the Lorentz transformation matrix, take the form (4).

Now let us go back to isotropic coordinates at the indicated points. The laboratory reference frame is

$$L_1=\begin{bmatrix} \delta_\beta{}^\alpha g_{1+}^{-2} & 0 \\ 0 & \dfrac{g_{1+}}{g_{1-}} \end{bmatrix}, \quad g_{1\pm}=1\pm\frac{m}{2\rho_1},$$

and the rotationless Lorentz transformation matrix is

$$\Lambda_{1a}{}^b=\begin{bmatrix} \delta_\alpha{}^\beta+(g_{1-}\Gamma_1-1)\dfrac{v^\alpha v^\beta}{v^2} & g_{1+}{}^3\Gamma_1 v^\alpha \\ g_{1+}{}^3\Gamma_1 v^\beta & g_{1-}\Gamma_1 \end{bmatrix}.$$

$$\Gamma_1=(g_{1-}{}^{-2}-g_{1+}{}^6 v^2)^{-1/2}.$$

Here we have used the fact that $\mathbf{v}'=(g_{1+}^3/g_{1-})\mathbf{v}$. We thus obtain an expression for the Lorentz frame at the isolated points (8):

$$\Lambda_1=\begin{bmatrix} \left[\delta_\alpha{}^\beta+(g_{1-}\Gamma_1-1)\dfrac{v^\alpha v^\beta}{v^2}\right]g_{1+}^{-2} & \dfrac{g_{1+}{}^4\Gamma_1}{g_{1-}}v^\beta \\ g_{1+}\Gamma_1 v^\alpha & g_{1+}\Gamma_1 \end{bmatrix}. \quad (9)$$

Furthermore, note that ρ_1 in Eq. (9) is arbitrary, so that equation holds at any point in the Schwarzschild space. We emphasize once again that if we proceed by analogy with Cartesian coordinates in Minkowski space, there is no leeway in the choice of L [Eq. (6)] and Λ [Eq. (9)]—they are uniquely determined.

Let us use Eq. (9) to find the rate of spin precession. We expand the spin vector S^i in terms of the vectors of the comoving frame: $S^i=\lambda^i_{(a)}S^{(a)}$, where $S^{(0)}\equiv 0$ by virtue of Eq. (2). The fundamental equation (1) then implies that

$$dS^{(\beta)}=-\lambda_i^{(\beta)}D\lambda^i_{(\alpha)}S^{(\alpha)}, \quad (10)$$

where $\alpha,\beta=1,2,3$. In deriving (10), we have taken advantage of the orthonormality of the reference frame[12]:

$$\lambda^i_{(a)}\lambda_{i(b)}=\eta_{(ab)},$$

where $\eta_{(ab)}$ is the metric tensor of Minkowski space, $\eta_{(ab)}=\mathrm{diag}\,(1,-1,-1,-1)$. This last equation may be conveniently rewritten in the form

$$\lambda^i_{(a)}\lambda_i^{(b)}=\delta_a{}^b,$$

where $\lambda_i^{(b)}=\lambda_{i(a)}\eta^{(ab)}$, or in other words $\lambda_i^{(\alpha)}=-\lambda_{i(\alpha)}$, $\lambda_i^{(0)}=\lambda_{i(0)}$. Substituting (9) into Eq. (10), we obtain

$$\Omega_\alpha{}^\beta=-\lambda_i^{(\beta)}\frac{D\lambda^i_{(\alpha)}}{Dt}$$

$$=(\Gamma g_--1)\frac{v^\alpha\dot v^\beta-v^\beta\dot v^\alpha}{v^2}+\frac{m}{\rho^3}\frac{(1+g_-)}{g_+}\Gamma(v^\alpha\rho^\beta-v^\beta\rho^\alpha). \quad (11)$$

3. PROOF OF THE FUNDAMENTAL PROPOSITION[1)]

Before discussing Eq. (11) in more detail, let us derive (9) in a somewhat different way. We make no assumptions about the form of the Lorentz transformations, and suppose only that they exist in general. We introduce an auxiliary comoving frame A:

$$A=\begin{bmatrix} \dfrac{v_\eta}{g_+{}^2 v} & -\dfrac{v_\xi}{g_+{}^2 v} & 0 & 0 \\ \dfrac{v_\xi\Gamma g_-}{g_+{}^2 v} & \dfrac{v_\eta\Gamma g_-}{g_+{}^2 v} & 0 & \dfrac{\Gamma g_+{}^4}{g_-}v \\ 0 & 0 & g_+^{-2} & 0 \\ v_\xi\Gamma g_+ & v_\eta\Gamma g_+ & 0 & g_+\Gamma \end{bmatrix}, \quad (12)$$

where $v_\xi=d\xi/dt$; $v_\eta=\partial\eta/\partial t$; $v^2=v_\xi^2+v_\eta^2$; we have assumed that the particle moves in the (ξ,η) plane. This is not a significant assumption, as particle motion in a central force field is always planar, and a simple rotation suffices to bring the actual orbital plane into coincidence with any other plane. Making use of (10), we find that S^i precesses in the frame A at a rate

$$\omega_A=\frac{\Gamma g_-}{v^2}(\dot v_\xi v_\eta-\dot v_\eta v_\xi)+\frac{m}{\rho^3}\frac{\Gamma(1+g_-)}{g_+}(\xi v_\eta-\eta v_\xi). \quad (13)$$

In general, the frame A rotates in some way about the frame Λ. To ascertain the rate of rotation, let us consider a new reference frame R,

$$R=\begin{bmatrix} v_\eta/vg_+{}^2 & -v_\xi/vg_+{}^2 & 0 & 0 \\ v_\xi/vg_+{}^2 & v_\eta/vg_+{}^2 & 0 & 0 \\ 0 & 0 & g_+^{-2} & 0 \\ 0 & 0 & 0 & g_+/g_- \end{bmatrix}.$$

The reference frame A can obviously be obtained from the reference frame R by a rotationless Lorentz transformation. In fact, two of the three spatial vectors in the two frames are identical. Moreover, frame A moves at a velocity $\mathbf{v}$ relative to frame R, and this will provide the information we need. Frame A actually cannot rotate spatially with respect to R, since a frame of reference is completely defined by two of its three basis vectors. Recalling the coincidence between vectors and A and R and counting up the number of parameters that may be used in transforming from R to A, we are left with only one. Both reference frames are characterized by six free parameters: 16 (the total)—4 (for normalization)—6 (for orthogonality). Having specified one general vector, we have fixed three of those parameters [4—1 (for normalization)]; the second general vector fixes another two [4—1 (for normalization)—1 (for orthogonality to the first vector)]. That leaves but one parameter free. Since A moves at velocity $\mathbf{v}$ with respect to R ($\lambda^i_{(0)}=u^i$), R can be transformed into A via a rotationless Lorentz transformation.

We have thus shown that the frame of reference Λ in which we are considering the precession may be obtained from the frame L via a rotationless Lorentz transformation, and frame A may be obtained from frame R, but R is rotated by a certain angle φ with respect to L:

$$\sin\varphi=-\frac{v_\xi}{v}; \quad \cos\varphi=\frac{v_\eta}{v}. \quad (14)$$

Hence, we find that

$$\omega_{A\Lambda}=\omega_{RL}=\dot\varphi=\frac{1}{v^2}(\dot v_\eta v_\xi-\dot v_\xi v_\eta). \quad (15)$$

Finally, the desired rate of precession Ω is

$$\Omega=\omega_A+\omega_{A\Lambda}=\frac{(\Gamma g_--1)}{v^2}(\dot v_\xi v_\eta-\dot v_\eta v_\xi)$$

$$+\frac{m}{\rho^3}\frac{\Gamma(1+g_-)}{g_+}(\xi v_\eta-\eta v_\xi), \quad (16)$$

i.e., it is identical to (11). The coincidence between these two spin-vector precession rates means that we have found an explicit expression for the Lorentz frame Λ in the Schwarzschild field, and that our assumption of the local nature of Lorentz transformation is valid.

Taking the limit as $m\to 0$ in Eq. (16), i.e., going to flat space with rectilinear geodesics, we obtain the Thomas precession formula.[14] In order to obtain zero (the spin of a free particle naturally does not precess in flat space), it is necessary to make use of the equation of motion $Du^i/D\tau = 0$. This is of course reasonable, and relates to the fact that Eq. (11) holds for any arbitrary nongeodesic motion of a particle for which the spin vector undergoes Fermi-Walker transport along the trajectory.[5] Under the latter circumstances, S^i satisfies the equation

$$(g_{ij}-u_iu_j)\frac{DS^j}{D\tau}=0,$$

or equivalently

$$\frac{DS^i}{D\tau}=-u^iS^j\frac{Du_j}{D\tau}.$$

Introducing invariant components of S^i relative to some comoving frame of reference, we obtain

$$dS^{(b)}=-\lambda_i{}^{(b)}S^{(a)}D\lambda_{(a)}{}^i-\lambda_i{}^{(b)}u^iDu^j\lambda_{j(a)}S^{(a)}.$$

By virtue of the orthogonality of the vectors in this frame, the second term in the equations for the first three components vanishes. For the fourth equation ($b=0$), we have [recall that $S^{(0)}\equiv 0$]

$$dS^{(0)}=0=-u^jD\lambda_{i(a)}S^{(a)}-Du^j\lambda_{j(a)}S^{(a)}=-D(u^i\lambda_{i(a)})S^{(a)}=0,$$

i.e., the equation is satisfied identically, as we set out to prove.

Equation (16) combined with the equation of motion thus provides a complete solution, given arbitrary particle motion, for spin precession in a Schwarzschild field, with the proviso that the source of nongeodesic motion may not significantly distort the gravitational background field. A gyroscope in an airplane or rocket, for example, will satisfy this condition.

We now turn once more to the case of a geodesic trajectory. With $Du^i/D\tau = 0$, we have

$$\dot v_\xi v_\eta-\dot v_\eta v_\xi=-\frac{m}{\rho^3}\frac{1}{g_+}\left(\frac{g_-}{g_+^6}+v^2\right)(\xi v_\eta-\eta v_\xi).$$

The final expression for Ω then becomes

$$\Omega=\frac{m}{\rho^3}\frac{1}{g_+}\left[\Gamma+1-\frac{g_-}{v^2g_+^6}(\Gamma g_--1)\right](\xi v_\eta-\eta v_\xi). \quad (17)$$

For $m/\rho\ll 1$, we obtain

$$\Omega=\frac{m}{\rho^3}\left\{\frac{1}{v^2}[1-(1-v^2)^{1/2}]+1\right\}[\rho,\mathbf{v}]. \quad (18)$$

For $v\ll 1$, this reduces to the equation given by Schiff,[4]

$$\Omega_S=\frac{3m}{2\rho^3}[\rho,\mathbf{v}].$$

The precession rate obviously depends on the choice of reference frame L, which is determined by the coordinate system. In a hyperbolic trajectory, the total spin-rotation angle is the same for all metrics that become Cartesian at infinity. Integrating Eq. (18) along the straight line $\xi = vt$, $\eta = b$, $\zeta = 0$, we obtain

$$\vartheta=-\frac{2m}{b}\frac{2+(1-v^2)^{1/2}}{1+(1-v^2)^{1/2}}. \quad (19)$$

Note that in reference frame R, one of the axes points along the direction of v, so if we integrate ω_{RL} along the trajectory, we obtain a particle-velocity rotation angle after the field has passed[15]:

$$\varphi_R=-\frac{2m}{b}\frac{1+v^2}{v^2}.$$

This means that as $v\to 1$, the spin tends to turn jointly with the trajectory [see (19) for $v\to 1$].

4. SPIN PRECESSION IN A REISSNER-NORDSTRØM FIELD

The technique developed in the preceding sections can easily be generalized to the case of a test particle with spin moving in a Reissner-Nordstrøm field, which has a squared world-line element[16]

$$ds^2=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)dt^2-\frac{dr^2}{(1-2m/r+e^2/r^2)}-r^2(d\theta^2+\sin^2\theta\, d\Phi^2), \quad (20)$$

and an electromagnetic vector potential A_i equal to

$$A_i=(\varphi,0,0,0);\quad \varphi=e/r, \quad (21)$$

where m and e are the mass and charge of the central body, respectively. We assume here that $e\leqslant m$ in order not to have to deal with a naked singularity.

The reason it is possible to generalize the approach taken in Sec. 2 to the metric (20) is that there exists a coordinate transformation that takes the field (20) into a form analogous to (5): it is easy to show that after a transformation

$$r=\rho\left(1+\frac{m}{\rho}+\frac{m^2-e^2}{4\rho^2}\right) \quad (22)$$

the metric (20) acquires the form

$$ds^2=\frac{g_-^2}{\sigma^2}dt^2-g_+^2(d\xi^2+d\eta^2+d\zeta^2), \quad (23)$$

$$g_-=1-\frac{m^2-e^2}{4\rho^2};\quad g_+=1+\frac{m}{\rho}+\frac{m^2-e^2}{4\rho^2};\quad \rho=(\xi,\eta,\zeta)$$

and the vector potential becomes $A_i=(e/\rho g_+,0,0,0)$. By analogy with the Schwarzschild field, we call (t,ρ) isotropic coordinates. The metric (23) was apparently first developed in Ref. 17, although the gravitational field (23) of a charged point particle was obtained directly from the Einstein equations, and the coordinate transformation (22) relating the metrics (20) and (23) thus went undiscovered.

We see from the form of the metric (23) that all of the structures of Sec. 2 carry over into Reissner-Nordstrøm space in isotropic coordinates. Without dwelling on the details, we present here the most important results. The laboratory frame for this space takes the form

$$L=\begin{bmatrix}\delta_\beta{}^\alpha g_+^{-1} & 0\\ 0 & g_+/g_-\end{bmatrix}, \quad (24)$$

and the Lorentz frame obtained from (24) via a rotationless Lorentz transformation becomes

$$\Lambda=\begin{bmatrix}\left[\delta_\alpha{}^\beta+(g_-\Gamma-1)\dfrac{v^\alpha v^\beta}{v^2}\right]g_+^{-1} & \dfrac{g_+^3\Gamma}{g_-}v^\alpha\\ g_+\Gamma v^\beta & g_+\Gamma\end{bmatrix}$$

$$\Gamma=(g_-^2-g_+^4v^2)^{-1/2};\quad v^2=v_\xi^2+v_\eta^2+v_\zeta^2. \quad (25)$$

If we then make use of Eq. (10), which holds for an arbitrary gravitational field, Eq. (25) yields the rate of spin precession:

$$\Omega_\sigma{}^\varkappa=(g_-\Gamma-1)\,\frac{(v^\sigma\dot v^\varkappa-v^\varkappa\dot v^\sigma)}{v^2}$$
$$+\frac{\Gamma}{g_-}\left[2\frac{m}{\rho}+\frac{3}{2}\frac{(m^2-e^2)}{\rho^2}-\frac{(m^2-e^2)^2}{8\rho^4}\right]\frac{(\rho^\varkappa v^\sigma-\rho^\sigma v^\varkappa)}{\rho^2},\quad (26)$$

which reduces to (11) when $e=0$.

This latter expression can be sharpened somewhat if the equation of motion of the test particle is known. For definiteness, consider a charged particle with charge q, and moving freely (no coupling) in the field (23). The velocity vector u^i of this particle will satisfy the equation

$$\frac{Du^i}{D\tau}=qF^{ij}u_j,$$

whereupon we have

$$\dot v^\alpha v^\beta-\dot v^\beta v^\alpha=\frac{(\rho^\beta v^\alpha-\rho^\alpha v^\beta)}{\rho^2}\left[-\frac{eq}{\rho}\frac{g_-}{\Gamma g_+{}^3}\right.$$
$$\left.+\frac{g_-}{g_+{}^5}\left(\frac{m}{\rho}+\frac{m^2-e^2}{\rho^2}+\frac{m(m^2-e^2)}{4\rho^3}\right)+v^2\frac{(g_+-g_-)}{g_+}\right].\quad (27)$$

Substituting (27) into (26), we obtain

$$\Omega_\sigma{}^\varkappa=\frac{(\rho^\sigma v^\varkappa-\rho^\varkappa v^\sigma)}{\rho^2}\frac{1}{g_+}\left\{(g_-\Gamma-1)\left[(g_+-g_-)\right.\right.$$
$$\left.+\frac{g_-}{v^2g_+{}^4}\left(\frac{m}{\rho}+\frac{m^2-e^2}{\rho^2}+\frac{m(m^2-e^2)}{4\rho^3}\right)\right]$$
$$-\Gamma\left(2\frac{m}{\rho}+\frac{3}{2}\frac{(m^2-e^2)}{\rho^2}-\frac{(m^2-e^2)^2}{8\rho^4}\right)$$
$$\left.-\frac{eq}{\rho}(g_-\Gamma-1)\frac{g_-}{\Gamma g_+{}^2v^2}\right\}.\quad (28)$$

Since the charge on the central body enters into the metric expression (23) only the ratio in e^2/ρ^2 and we are assuming $e\leqslant m$, we find that to first order in m/ρ, the spin of a neutral particle precesses in a weak gravitational field at the same rate as given by (18) for the Schwarzschild field. In this approximation, the contrast with the Schwarzschild field appears only for a charged particle. For $e/\rho\ll 1$, the additional term $\mathbf{\Omega}_q$ in the precession rate becomes

$$\mathbf{\Omega}_{R-N}=\mathbf{\Omega}_S+\mathbf{\Omega}_q,\quad \mathbf{\Omega}_q=\frac{eq}{\rho^3}\frac{1}{v^2}\left[(1-v^2)^{1/2}-1\right][\boldsymbol{\rho},\mathbf{v}].\quad (29)$$

This equation describes the precession of the spin of a charged particle in a Coulomb field.

5. SPIN PRECESSION IN AN ARBITRARY WEAK GRAVITATIONAL FIELD

We shall now examine the motion of a particle with spin in an arbitary weak gravitational field described by the metric

$$g_{ik}=\eta_{ik}+h_{ik},\quad (30)$$

where $\eta_{ik}=\mathrm{diag}(1,-1,-1,-1)$ is the metric of Minkowski space, and $|h_{ik}|\ll 1$.

a) Choice of a laboratory frame. The principal property characterizing the laboratory frame is that its timelike vector $l^i_{(0)}$ coincides with the velocity vector u^i_L of some particle at rest in the given coordinate system: $u^i_L=(g_{00}^{-1/2},0,0,0)$. Thus,

$$l^i_{(0)}=\delta_0{}^i g_{00}^{-1/2}.\quad (31)$$

We now make use of the most restrictive possible orthonormality conditions for the remaining vectors $l^i_{(\alpha)}$. The requirement of orthogonality between the $l^i_{(\alpha)}$ and $l^i_{(0)}$ implies that

$$l_{(\alpha)0}=0.\quad (32)$$

If we take (32) into account, the remaining orthonormality relations take the form

$$l_{(\alpha)\mu}l_{(\beta)\nu}g^{\mu\nu}=\eta_{\alpha\beta},\quad (33)$$

where $\alpha,\beta,\mu,\nu=1,2,3$. Equation (33) cannot be solved for the $l_{(\alpha)\mu}$ in an arbitrary gravitational field. Instead, we attempt to carry out the appropriate calculations for the weak field (30), for which we obtain

$$g^{ik}=\eta^{ik}-\eta^{km}\eta^{in}h_{mn},\quad (34)$$

with $\eta^{ik}=\eta_{ik}$. We seek components $l_{(\alpha)\mu}$ of the reference frame in the form

$$l_{(\alpha)\mu}=\eta_{\mu\alpha}+B_{\mu\alpha},\quad (35)$$

where $B_{\mu\alpha}=O(h)$, such that in the absence of a gravitational field, the frame $l^{(i)}_a$ is transformed into the natural laboratory frame (3) for Minkowski space,

$$l^i_{M(a)}=\delta^i_{(a)}.$$

Substitution of (34) and (35) into (33) yields

$$B_{\alpha\beta}+B_{\beta\alpha}=h_{\alpha\beta}.\quad (36)$$

Separating $B_{\alpha\beta}$ into symmetric and antisymmetic parts, $B_{\alpha\beta}=B_{(\alpha\beta)}+B_{[\alpha\beta]}$, where $B_{(\alpha\beta)}=\frac{1}{2}(B_{\alpha\beta}+B_{\beta\alpha})$ and $B_{[\alpha\beta]}=\frac{1}{2}(B_{\alpha\beta}-B_{\beta\alpha})$, we find from (7) that

$$B_{\alpha\beta}=\frac{1}{2}h_{\alpha\beta}+B_{[\alpha\beta]},$$

in which $B_{[\alpha\beta]}$ is arbitrary. The simplest way to reduce the vectors $l_{(\alpha)\mu}$ to a specific usable form is to set $B_{[\alpha\beta]}$ equal to zero.[2] Thus, to $O(h^2)$, the laboratory frame that we have chosen takes the form

$$L=\begin{bmatrix}\delta_\alpha{}^\beta-\frac{1}{2}\eta^{\beta\nu}\eta_{\nu\alpha} & -h_{0\alpha}\\ 0 & 1-\frac{1}{2}h_{00}\end{bmatrix}\begin{matrix}-l^i_{(\alpha)}\\ -l^i_{(0)}\end{matrix}.\quad (37)$$

Note that because of the way in which we have broken out the quantities η_{ij} and h_{ij}, it is impossible to put together an antisymmetric second-rank tensor linear in h_{ik}. To do so would require other expressions—specifically, second derivatives of the metric tensor g_{ik}, such as $(h_{ik,jm}-h_{jk,im})\eta^{km}$.

In practical terms, a laboratory frame is characterized by the fact that it is stationary with respect to a selected frame of reference (the sun, for example), and does not rotate with respect to the fixed (i.e., infinitely distant) stars. The frame (37) naturally satisfies the first condition, but its interrelation with starlight remains to be elucidated. In this regard, we cannot guarantee that the precession rate that we may calculate at some time in the future will agree locally (in time) with feasible experimental measurements. For open trajectories in the gravitational field of an isolated system, however, the angle by which the spin rotates will perforce agree with the experimental value, since in the infinite depths of space, the frame (37) coincides with the natural laboratory frame for Minkowski space, which obviously does not rotate relative to the fixed stars.

b) Construction of the Lorentz frame. When we have chosen our laboratory frame, the construction of the corresponding Lorentz frame is unique, according to the results of Sec. 2. To make our exposition clear, we shall go into a fair amount of detail for each stage of this construction. We choose an arbitrary point in space-time $(\tilde t,\tilde x^\alpha)$ lying on the world line of a particle. The metric at this point is of the form $g_{ik}(\tilde t,\tilde x^\alpha)=\eta_{ik}+\tilde h_{ik}$. One can then easily verify that the linear transformation

$$t'=t\left(1+\frac{1}{2}\tilde h_{00}\right)+x^\alpha\tilde h_{0\alpha},\quad x'^\beta=x^\alpha\left(\delta_\alpha{}^\beta+\frac{1}{2}\eta^{\beta\mu}\tilde h_{\mu\alpha}\right)\qquad(38)$$

reduces the laboratory frame (37) at the point $(\tilde t',\tilde x'^\alpha)$ to the form (3), while the metric at that point is of course diagonalized. As we showed in Sec. 2, a rotationless Lorentz transformation at the given point and in the (t',x'^β) coordinate system looks just the same as it would in Minkowski space, that is, it is of the form given by Eq. (4).

Let us now return to the original coordinate (t,x^α). The Lorentz matrix Λ'^a_b in (4), being itself a scalar under coordinate transformations (since $\lambda^i_{(a)}=\Lambda^b_a l^i_{(b)}$), will only change under the present transformation by virtue of the fact that the velocities v'^α and v^β in the (t',x'^α) and (t,x^β) systems are related by

$$v'^\alpha=v^\alpha+\frac{1}{2}\eta^{\alpha\nu}\tilde h_{\lambda\rho}v^\rho-\frac{1}{2}v^\alpha\tilde h_{00}-v^\alpha v^\sigma\tilde h_{0\sigma}.\qquad(39)$$

If we then substitute (39) into Eq. (4), we find that the rotationless Lorentz transformation matrix at the point $(\tilde t,\tilde x^\alpha)$ in the (t,x^α) coordinate system takes the form

$$\Lambda_\beta{}^\alpha=\delta_\beta{}^\alpha-(\gamma-1)\frac{v^\alpha v^\rho}{v^2}+\frac{(\gamma-1)}{v^2}\tilde h_{\mu\nu}\left(\frac{1}{2}v^\alpha v^\mu\eta^{\beta\nu}+\frac{1}{2}v^\beta v^\mu\eta^{\alpha\nu}+\frac{1}{v^2}v^\alpha v^\beta v^\mu v^\nu\right)-\frac{\gamma^2}{2v^2}v^\alpha v^\beta[v^\varkappa v^\sigma\tilde h_{\varkappa\sigma}+(\tilde h_{00}+2v^\sigma\tilde h_{0\sigma})v^2],\qquad(40)$$

$$\Lambda_0{}^\alpha=\Lambda_\alpha{}^0=\gamma v^\alpha+\frac{\gamma}{2}v^\mu\eta^{\alpha\nu}\tilde h_{\mu\nu}-\frac{\gamma^3}{2}v^\alpha(\tilde h_{00}+2v^\sigma\tilde h_{0\sigma}+v^\varkappa v^\sigma\tilde h_{\varkappa\sigma}),$$

$$\Lambda_0{}^0=\gamma-\frac{1}{2}\gamma^3[(\tilde h_{00}+2\tilde h_{0\nu}v^\nu)v^2+v^\varkappa v^\sigma\tilde h_{\varkappa\sigma}].$$

Next let us operate with this matrix Λ^a_b (40) on the frame (37) at the point $(\tilde t,\tilde x^\alpha)$, obtaining as a result the Lorentz frame Λ at the point $(\tilde t,\tilde x^\alpha)$. Furthermore, note that we originally chose this point arbitrarily, so our construction is equally valid for any point in space-time. Thus, the Lorentz frame Λ obtained via a rotationless Lorentz transformation from the laboratory frame L (37) in the space (30) is, to order Oh^2,

$$\lambda^\beta_{(\alpha)}=\delta_\alpha{}^\beta+(\gamma-1)\frac{v^\alpha v^\beta}{v^2}+\frac{1}{2}(\gamma-1)\frac{v^\beta v^\mu}{v^2}\eta^{\alpha\nu}h_{\mu\nu}-\left(1-\gamma+\frac{v^2\gamma^3}{2}\right)\frac{v^\alpha v^\beta v^\mu v^\nu}{v^4}h_{\mu\nu}-\frac{1}{2}\eta^{\beta\nu}h_{\nu\alpha}-\frac{1}{2}v^\alpha v^\beta\gamma^3(h_{00}+2v^\sigma h_{0\sigma}),$$

$$\lambda^0_{(\alpha)}=v^\alpha\gamma+\frac{1}{2}v^\mu\eta^{\alpha\nu}\gamma h_{\mu\nu}-\frac{1}{2}\gamma v^\alpha h_{00}-\frac{\gamma^3}{2}v^\alpha(h_{00}+2v^\sigma h_{0\sigma}+v^\varkappa v^\sigma h_{\varkappa\sigma})-(\gamma-1)\frac{1}{v^2}v^\alpha v^\nu h_{0\nu}-h_{0\alpha},$$

$$\lambda^\alpha_{(0)}=v^\alpha\gamma-\frac{1}{2}v^\alpha\gamma^3(h_{00}+2v^\sigma h_{0\sigma}+v^\varkappa v^\sigma h_{\varkappa\sigma}),\qquad(41)$$

$$\lambda^0_{(0)}=\gamma-\frac{1}{2}\gamma^3(h_{00}+2v^\sigma h_{0\sigma}+v^\varkappa v^\sigma h_{\varkappa\sigma}).$$

c) Spin precession rate. Substituting the components of the Lorentz frame (41) into Eq. (10), we have to order h^2

$$\Omega_\sigma{}^\varkappa=\frac{v^\sigma\dot v^\varkappa-v^\varkappa\dot v^\sigma}{v^2}(\gamma-1)+\frac{\gamma}{2}[v^\varkappa(2h_{0\sigma,0}-h_{00,\sigma})-v^\sigma(2h_{0\varkappa,0}-h_{00,\varkappa})]$$
$$+\left(\gamma+\frac{\gamma}{v^2}-\frac{1}{v^2}\right)[v^\varkappa(h_{\nu\sigma,0}+h_{0\sigma,\nu}-h_{0\nu,\sigma})-v^\sigma(h_{\nu\varkappa,0}+h_{0\varkappa,\nu}-h_{0\nu,\varkappa})]\frac{v^\nu}{2}$$
$$+(\gamma-1)\frac{v^\mu v^\nu}{2v^2}[v^\varkappa(2h_{\sigma\mu,\nu}-h_{\mu\nu,\sigma})-v^\sigma(2h_{\varkappa\mu,\nu}-h_{\mu\nu,\varkappa})]\qquad(42)$$
$$+\frac{1}{2}(h_{0\sigma,\varkappa}-h_{0\varkappa,\sigma})+\frac{v^\nu}{2}(h_{\sigma\nu,\varkappa}-h_{\varkappa\nu,\sigma}).$$

Equation (10) for the rate of spin precession also holds for nongeodesic motion of a particle, so in combination with the particle's equation of motion, it gives a complete description of spin precession in an arbitrary gravitational field. Returning to the case of a test particle following a geodesic trajectory, we see that the equation of motion implies that

$$\dot v^\varkappa v^\sigma-\dot v^\sigma v^\varkappa=\frac{v^\sigma}{2}(2h_{\varkappa0,0}-h_{00,\varkappa})-\frac{v^\varkappa}{2}(2h_{\sigma0,0}-h_{00,\sigma})$$
$$+v^\sigma v^\nu(h_{\varkappa0,\nu}+h_{\varkappa\nu,0}-h_{0\nu,\varkappa})-v^\varkappa v^\nu(h_{\sigma0,\nu}+h_{\sigma\nu,0}-h_{0\nu,\sigma})$$
$$+\frac{1}{2}v^\sigma v^\mu v^\nu(2h_{\varkappa\mu,\nu}-h_{\mu\nu,\varkappa})-\frac{1}{2}v^\varkappa v^\mu v^\nu(2h_{\sigma\mu,\nu}-h_{\mu\nu,\sigma}).\qquad(43)$$

Substituting (43) into (42), we have

$$\Omega_\sigma{}^\varkappa=\frac{1-(1-v^2)^{1/2}}{2v^2}[v^\varkappa(2h_{0\sigma,0}-h_{00,\sigma})-v^\sigma(2h_{0\varkappa,0}-h_{00,\varkappa})]$$
$$+(v^\nu/2v^2)[1-(1-v^2)^{1/2}][v^\varkappa(h_{\nu\sigma,0}+h_{0\sigma,\nu}-h_{0\nu,\sigma})-v^\sigma(h_{\nu\varkappa,0}+h_{0\varkappa,\nu}-h_{0\nu,\varkappa})]$$
$$+{}^1/_2(h_{0\sigma,\varkappa}-h_{0\varkappa,\sigma})+{}^1/_2v^\nu(h_{\sigma\nu,\varkappa}-h_{\varkappa\nu,\sigma}).\qquad(44)$$

In some sense, Eqs. (42) and (44) are the ultimate end-products of the development in the present section. Let us examine various limiting cases of these two equations. Letting the particle velocity v approach zero in (42) and retaining terms linear in v, we obtain

$$\Omega_\sigma{}^\varkappa=\frac{1}{2}(v^\sigma\dot v^\varkappa-v^\varkappa\dot v^\sigma)+\frac{1}{2}[v^\varkappa(2h_{0\sigma,0}-h_{00,\sigma})-v^\sigma(2h_{0\varkappa,0}-h_{00,\varkappa})]$$
$$+\frac{1}{2}(h_{0\sigma,\varkappa}-h_{0\varkappa,\sigma})+\frac{v^\nu}{2}(h_{\sigma\nu,\varkappa}-h_{\varkappa\nu,\sigma}). \qquad (45)$$

In the post-Newtonian approximation for the gravitational field,[18] we have

$$g_{00}=1-2\varphi=1-\frac{2m}{r};\quad g_{0\alpha}=g_\alpha=-\frac{2}{r^3}\varepsilon_{\alpha\beta\gamma}x^\beta J^\gamma; \qquad (46)$$
$$g_{\alpha\beta}=-\delta_{\alpha\beta}(1+2\varphi),$$

where m is the mass and J^α the angular momentum of the body responsible for the field. Equation (45) then becomes

$$\Omega_\sigma{}^\varkappa={}^1/_2(v^\sigma\dot v^\varkappa-v^\varkappa\dot v^\sigma)+2(v^\varkappa\varphi_{,\sigma}+v^\sigma\varphi_{,\varkappa})+{}^1/_2(g_{\sigma,\varkappa}-g_{\varkappa,\sigma}).$$

This is the same as the corresponding equation for a gyroscope given in Ref. 19 (taking into account the definition of the acceleration a^α, of course). Finally, to conclude our examination of the post-Newtonian approximation, let us apply it to the general equations (42) and (44), giving the rate of spin precession of a relativistic particle in that approximation,

$$\Omega_\sigma{}^\varkappa=\frac{\gamma-1}{v^2}(v^\sigma\dot v^\varkappa-v^\varkappa\dot v^\sigma)+2\gamma(v^\varkappa\varphi_{,\sigma}-v^\sigma\varphi_{,\varkappa})$$
$$+\left(\gamma+\frac{\gamma}{v^2}-\frac{1}{v^2}\right)[v^\varkappa(g_{\sigma,\nu}-g_{\nu,\sigma})$$
$$-v^\sigma(g_{\varkappa,\nu}-g_{\nu,\varkappa})]\frac{v^\nu}{2}+\frac{1}{2}(g_{\sigma,\varkappa}-g_{\varkappa,\sigma}), \qquad (47)$$

and, taking the geodesic equation into consideration,

$$\Omega_\sigma{}^\varkappa=\left\{\frac{1}{v^2}[1-(1-v^2)^{1/2}]+1\right\}(v^\varkappa\varphi_{,\sigma}-v^\sigma\varphi_{,\varkappa})$$
$$+\frac{v^\nu}{2v^2}[1-(1-v^2)^{1/2}][v^\varkappa(g_{\sigma,\nu}-g_{\nu,\sigma})$$
$$-v^\sigma(g_{\varkappa,\nu}-g_{\nu,\varkappa})]+\frac{1}{2}(g_{\sigma,\varkappa}-g_{\varkappa,\sigma}). \qquad (48)$$

If in this equation we set the angular momentum J^α to zero, the remainder describes the precession of the spin of a relativistic particle in a Schwarzschild field. This calculation was first carried out in Ref. 13.

Equation (44) tends to a finite limit as the particle velocity v goes to unity:

$$\Omega_\sigma{}^\varkappa=\frac{1}{2}[v^\varkappa(2h_{0\sigma,0}-h_{00,\sigma})-v^\sigma(2h_{0\varkappa,0}-h_{00,\varkappa})]$$
$$+\frac{v^\nu}{2}[v^\varkappa(h_{\nu\sigma,0}+h_{0\sigma,\nu}-h_{0\nu,\sigma})$$
$$-v^\sigma(h_{\nu\varkappa,0}+h_{0\varkappa,\nu}-h_{0\nu,\varkappa})]$$
$$+\frac{1}{2}(h_{0\sigma,\varkappa}-h_{0\varkappa,\sigma})+\frac{1}{2}(h_{\sigma\nu,\varkappa}-h_{\varkappa\nu,\sigma})v^\nu. \qquad (49)$$

We have thus managed to find rate of precession of the spin of a massless particle, such as a neutrino, as well as the rate of precession of the polarization vector of a photon, since the latter also undergoes parallel transport along its trajectory (a geodesic).[20] The two final terms in (49) are the most interesting, since they contain the component of the three-dimensional rate of precession $\mathbf{\Omega}$ parallel to the photon velocity $\mathbf{v}$. It is easy to imagine that that is the part of $\mathbf{\Omega}$ that describes the rate at which the plane of polarization of the photon precesses. Consider, for example, a particle that at a given moment t is moving along the x axis. The component of $\mathbf{\Omega}$ parallel to $\mathbf{v}$ is then

$$\Omega_z{}^y={}^1/_2(h_{0z,y}-h_{0y,z})+{}^1/_2(h_{xz,y}-h_{xy,z}). \qquad (50)$$

This expression was first derived in Ref. 21 in a different manner, motivated by heuristic considerations. To order h^2, the total rotation angle θ of the plane of polarization for an open trajectory in the post-Newtonian approximation is zero, as reported in Ref. 20, and this result can readily be verified by using Eq. (50).

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[1)]A brief discussion of this point may be found in Ref. 13.
[2)]A discussion of similar ambiguities in a different reference frame may be found in Ref. 6.

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