Probabilistic description of nonlinear dynamics of domain walls

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Transition of a domain wall in a weak ferromagnet to a supersonic velocity is analyzed. The Lyapunov function is used to reduce a study of the dynamics of a nonequilibrium dissipative system to a study of an equilibrium conservative system. The velocities of steady-state motion of a domain wall correspond to extrema of the Lyapunov function which depends parametrically on the magnetic field. In fields corresponding to two comparable minima of the Lyapunov function the velocity changes abruptly, in analogy with the Maxwell rule in the theory of phase transitions. Allowance for fluctuations of the force acting on a domain wall suppresses a hysteresis of the dependence of the domain wall velocity on the magnetic field. The region of a constant near-sonic velocity of a domain wall depends weakly on dissipation in the elastic subsystem. The field dependence of the time of transition to a supersonic velocity, derived by the authors, is in better agreement with the experimental results than the corresponding dependence obtained earlier using a dynamic theory.

1. INTRODUCTION

Dynamics of domain walls in ferromagnets is strongly nonlinear. Among nonlinear effects in domain-wall dynamics there is special interest in multivalued dependences of the domain wall velocity on an external magnetic field. Such dependences are associated with the effects of additional drag forces of resonant nature on domain walls and they may occur, for example, in the region of the velocity of sound. This effect has been investigated both experimentally and theoretically in weak ferromagnets which are at present the only materials in which the velocity of domain walls has been found to exceed the velocity of sound. The experimentally determined dependence of the domain wall velocity on the magnetic field $V(H)$ obtained for a weak ferromagnet is plotted in Fig. 3 of Ref. 1. A corresponding theoretical curve is shown in Fig. 21 of Ref. 2. These dependences demonstrate clearly regions of constant domain wall velocity near the velocity of sound.

A considerable (of width amounting to tens or hundreds of oersteds) hysteresis of the domain wall velocity has been predicted theoretically[2] but it is not observed experimentally.[3] It is not clear from the existing theory how and in what magnetic field there is a transition from a sonic to a supersonic velocity of a domain wall. It follows from this theory that the width of the region of a constant domain velocity $\Delta H_{t,l}$ is inversely proportional to the attenuation coefficient of sound, which should depend strongly on temperature (the indices t and l refer to the longitudinal and transverse sound). Experiments indicate that $\Delta H_{t,l}$ depends very weakly on the sample temperature. Moreover, the existing theory predicts a very short time for crossing the sound barrier and an unrealistic dependence of this time on the magnetic field.[3]

The aim of the work reported below is to resolve these doubts by reducing the bifurcation problem of domain wall dynamics to the problem of a kinetic phase transition.

2. MAIN EQUATION. LYAPUNOV FUNCTION OF THE SYSTEM

A domain wall is a soliton-like object of the spin-density nonlinear field in a magnetic crystal. The dynamics of a domain wall is usually studied on the basis of the Landau-Lifshitz equations for such a field. Determination of the field dependence of the domain-wall velocity in the modern theory reduces to a nonlinear eigenvalue problem in which the eigenvalue is the velocity and the external parameter is the magnetic field. Subject to suitable boundary conditions, the Landau-Lifshitz equation yields a unique branch of eigenvalues of $V(H)$. An example of such a dependence is the familiar Walker solution. However, under certain conditions the solutions show branching and the dependence $V(H)$ becomes multivalued (see Fig. 21 in Ref. 2).

The problem of transition from one branch of $V(H)$ to another, formulated above, can be solved if we go beyond the eigenvalue framework. We are speaking here of an investigation of transient dynamics of domain walls. A suitable equation can be derived from the Landau-Lifshitz equations by the methods of perturbation theory for solitons (see, for example, Ref. 4). In the case of weak ferromagnets of the $YFeO_3$ type the relevant equation is (for its derivation see Refs. 5–7)

$$\partial P/\partial t+P/\tau=2M_sH+f(P), \tag{1}$$

where $P = m\dot{q}$ is the density of the momentum of a domain wall; q is the coordinate of the center of a domain wall; $m = m_0[1 - (\dot{q}/c)^2]^{-1/2}$ is the mass density of a moving domain wall; c is the limiting velocity of the wall, equal to the velocity of magnons in the linear part of the spectrum ($c = 2\times 10^6$ cm/s for $YFeO_3$); P/τ is the density of the viscous friction force; τ is the relaxation time (which is related to the domain wall mobility μ by $\mu = 2M_s\tau/m$); M_s is the saturation magnetization; $2M_sH$ is the pressure exerted on a domain wall by an external field H; $f(P)$ is the density of the additional drag force acting on a domain wall and related, in the present case, to a resonant interaction of the wall with acoustic phonons or generally with other quasiparticles in the medium. The relativistic factor $[1 - (\dot{q}/c)^2]^{-1/2}$ appears in the expression for the mass because the Landau-Lifshitz equations for weak ferromagnets reduce under our conditions to the sine-Gordon equation, which is known to be Lorentz-invariant. We dropped the term $\nabla\sigma\nabla q$, where σ

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is the surface tension (surface energy density), i.e., we ignored bending of a domain wall in the course of its motion. In the region of field corresponding to the transition between the branches of $V(H)$ ($H \approx H_M$ in Fig. 1a) the amplitude of bending of the plane front of a domain wall is 100 μm when the minimum observed radius of curvature of the wall is 1000 μm (Refs. 1 and 8). The effective contribution of the term $\nabla\sigma\nabla q$ is ~ 1 Oe, which is much less than the characteristic width of the near-sonic region of constancy of the domain wall velocity, amounting usually to a few tens of oersteds in orthoferrites. The form of Eq. (1) is identical with the Newton equation for a material particle (more exactly, for a flat membrane), which is a consequence of the particle-like properties of solitons. It can be represented conveniently in the form

$$\partial P/\partial t = -\partial\Phi(P)/\partial P, \tag{2}$$

where

$$\Phi = \frac{P^2}{2\tau} - 2M_s HP - \int_0^P f(\xi)\,d\xi \tag{3}$$

is the Lyapunov function of the system. If we regard P as a generalized coordinate, we can identify Φ with the potential function of the system. Writing down the equation of motion of a domain wall in the form of Eq. (2) using the Lyapunov function of Eq. (3) we can essentially reduce an investigation of the dynamics of a nonequilibrium dissipative system to a study of an equilibrium conservative system which has a potential and is described by a generalized coordinate P. The velocities of steady-state motion of a domain wall can be identified with extrema of the function Φ depending parametrically on H. The maxima of Φ correspond to absolutely unstable motion, whereas minima correspond to stable motion.

The task of finding the velocity of a domain wall in a given field when the wall has initially some specific velocity is equivalent to the task of finding a minimum of the function Φ which will be assumed by the system in this field.

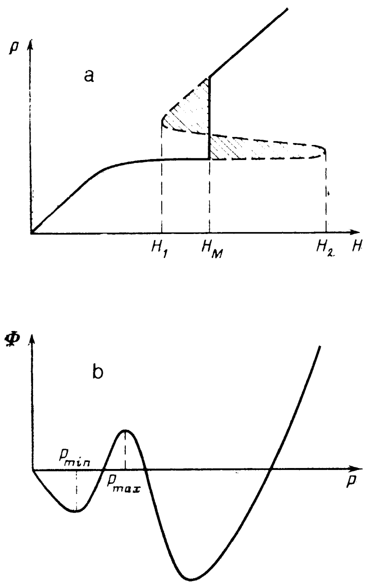

FIG. 1. Schematic representation of the formation of the dependence $V(H)$ in accordance with the Maxwell principle (a) and the Lyapunov function for the system considered in the range $H_M < H < H_2$ (b).

There is a different way of looking at Eq. (2). It can be regarded formally as the Landau-Khalatnikov equation of a system undergoing a first-order phase transition and described by the order parameter P.

3. FOKKER-PLANCK EQUATION AND THE MAXWELL PRINCIPLE

A change in the controlling parameters, such as the external field, alters the function Φ. The initial global minimum, which governs the state of the investigated system, can then become a metastable local minimum or can even disappear. In this case the system should go over from one local minimum to another. The moment of such a transition and the minimum in which the system is stable are found in the treatments of domain wall dynamics by adopting implicitly a principle known as the principle of maximum retardation. It can be formulated as follows[9]: a system, which is initially in a given local or global minimum, remains in this minimum as long as it exists. This assumption has led to predictions, based on a dynamic theory, that hysteresis should be exhibited by the velocity of such a system, but this is in conflict with the experimental evidence.

The adoption of this principle ignores the existence of noise, i.e., of fluctuations which undoubtedly occur in a system such as a domain wall moving across a real inhomogeneous sample. A satisfactory allowance of fluctuations can be made if Eq. (2) is supplemented on the right-hand side by a random force $F(t)$ and comparing it with the Fokker-Planck equation for the distribution function of the probability density $W(P)$ in the momentum space:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial P}\left(W\frac{\partial\Phi}{\partial P}\right) + \frac{\partial^2}{\partial P^2}(DW), \tag{4}$$

where the distribution function W depends on the momentum density, on time, and on the controlling parameters; D is the diffusion coefficient representing the level of noise in the system and governed by the correlation function of the random force:

$$\langle F(t)F(t')\rangle = 2D\delta(t-t').$$

Here, $F(t)$ is the resultant force, i.e, the sum of the effects on a domain wall of those defects which are being crossed by a domain wall of finite size at a given moment in time. The quantities P and D in Eq. (4) now describe the whole domain wall of finite size, i.e., we shall consider a wall as a system with one degree of freedom and ignore its multidimensional nature when considering the transition process. It is known from the experimental results that in the region of the field corresponding to a transition between two branches of $V(H)$ ($H \approx H_m$ in Fig. 1a) a domain wall exhibits bending of the plane front with an amplitude up to 50–100 μm when the radius of curvature is 1–2 mm and this happens in an observation region of 1 mm^2 (Refs. 1 and 8). In this region a domain wall can be regarded as having one degree of freedom.

In the simplest case if we assume that D is constant, we obtain the steady-state solution of Eq. (4):

$$W(P) = Ne^{-\Phi(P)/D}, \tag{5}$$

where N is the normalization constant.

In estimating the value of D we shall represent a random force $F(t)$ exerted by defects on a moving domain wall by

$$F(t)=S^{-1}\sum_i \pi_i\delta(t-x_i/V),$$

where x_i is the coordinate of the ith defect and π_i is the corresponding momentum of the force. As pointed out above, all the quantities P, $\Phi(P)$, and F should be normalized to a unit area S of the wall. In calculation of the correlation function $\langle F(t),F(t')\rangle$ we shall assume that $\langle\pi_i\pi_j\rangle = \langle\pi_i^2\rangle\delta_{ij}$, so that

$$\langle F(t)F(t')\rangle=\pi^2 nVS^{-1}\delta(t-t'), \tag{6}$$

where n is the concentration of defects. Let us assume that, for example, a defect is a local inhomogeneity of the magnetization δM_s; then, $\pi = 2\delta M_s HR^3/V$, where R is the characteristic size of the inhomogeneity. Substituting this value in Eq. (6), we obtain

$$D=\eta(2M_s/\mu)^2R^6nVS^{-1}, \quad \eta=(\delta M_s/M_s)^2.$$

An analogous estimate is valid also in the case of local inhomogeneities of other quantities such as the anisotropy constant K, the "exchange rigidity" A, etc.

Let us assume that $R\sim(2–5)\times10^{-4}$ cm, $n\sim10^8–10^{10}$ cm^{-3}, $S\sim10^{-4}$ cm^2, $\eta\sim10^{-2}–10^{-4}$, $\mu\sim10^3$ cm$^2\cdot$s$^{-1}\cdot$Oe^{-1}, and $V\sim10^6$ cm/s. Then, $D\sim10^{-7}–10^{-9}$ g$^2\cdot$cm$^{-2}\cdot$s^{-3}, which is in qualitative agreement with the experimental value.

According to Eq. (5), the velocity with the highest probability corresponds to a minimum of the function $\Phi(P)$, which is equivalent to the statement known as the Maxwell principle: the state of the system is governed by the global minimum of the potential function. Applying this principle, we can plot the dependence $P(H)$ or $V(H)$ and find the global minimum of the function $\Phi(P)$ for a field H varying from 0 to ∞. It is obvious that the field dependence of the velocity is single-valued, i.e., there is no hysteresis, in agreement with the experimental results reported in Ref. 3. In magnetic fields for which the values of Φ in the two lowest minima are comparable we can expect abrupt changes of the velocity tested in the dependences $V(H)$.

The velocities of stable steady-state motion of a domain wall can be found from a system of equations

$$\partial\Phi/\partial P=0, \quad \partial^2\Phi/\partial P^2>0, \tag{7}$$

which in the region of two minima of Φ has two solutions: $P_1(H)$ and $P_2(H)$.

The Maxwell principle is an equation for finding the bifurcation set, i.e., the set of points in the space of the controlling parameters in which a transition takes place from one local minimum to another:

$$\Phi(P_1(H))=\Phi(P_2(H)). \tag{8}$$

The solution of Eq. (8) is given by the field corresponding to the transition to a supersonic velocity in the $V(H)$ curve. The geometric equivalent of Eq. (8) is the equality of the areas of two regions shown shaded in Fig. 1a, which is analogous to the Maxwell rule in the theory of phase transitions.

4. CALCULATION OF THE WIDTH OF THE NEAR-SONIC REGION OF CONSTANT VELOCITY OF A DOMAIN WALL

It follows readily from the geometric interpretation of the Maxwell principle that the width of the near-sonic region of constancy of the domain wall velocity (i.e., the width of the plateau) manifested by the dependence $V(H)$ depends not only on the amplitude of a resonance peak, assumed earlier to be the dominant factor,[2,6,7,10] but also on the slope of the $P(H)$ curve, i.e., on the initial mobility of the wall. When the mobility increases the quantity ΔH_M (representing the width of the plateau) should increase, in agreement with the experimental results.

We shall consider the specific case of a singularity of the dependence $V(H)$ near the velocity of transverse sound in yttrium orthoferrite. We shall use an approximate expression for f proposed in Ref. 7 (all the following conclusions apply also to f given in Ref. 10):

$$f(P)=-\frac{bP/m_0}{[(P/m_0)^2-s_t^2]^2+\Delta^2}, \tag{9}$$

where

$$b=\frac{7\delta_t^2 s_t^3}{15c_t q_t}, \quad \Delta^2=\frac{7s_t^5}{20q_t^2c}, \quad q_t=\frac{\Delta_0 c_t}{2\eta_t c},$$

δ_t is the magnetoelastic coupling constant, s_t is the velocity of transverse sound, c_t is an elastic constant, and η_t is the dissipation in the elastic subsystem of the crystal. Equation (9) allows for the fact that $(s_t/c)^2\ll1$. If $f(P)$ is given by Eq. (9), the function $\Phi(P)$ is given by the following expression, apart from an additive constant:

$$\Phi=\frac{P^2}{2\tau}-2M_sHP+\frac{bm_0}{2\Delta}\operatorname{arctg}\frac{(P/m_0)^2-s_t^2}{\Delta}. \tag{10}$$

The area under the resonance peak of the function $f(P)$, equal to the difference between two calculated asymptotic values of the last term in Eq. (10) for the two opposite cases $P\gg m_0S$ and $P\ll m_0S$, is independent of η_t. Therefore, the width of the plateau ΔH_M deduced from the Maxwell rule depends weakly on the dissipation processes. On the other hand, the amplitude of the peak and, consequently, the width of the plateau ΔH_{mr}, deduced using the principle of maximum retardation, is inversely proportional to the dissipation.

This difference is illustrated in Fig. 2a. The values of ΔH_M were found by numerical solution of the system of equations (7)–(8) for the function Φ given by Eq. (10). It is clear from Fig. 2a that reduction in the dissipation in the elastic system, resulting in an increase in the width of the plateau by a factor of 10 compared with the dynamic theory, increases $\Delta N_{\max}$ only by one-third. This behavior agrees with the observation that cooling a sample from 300 to 4.2 K alters only slightly the real plateau,[1] although the value of ΔH_{mr} should then increase by several orders of magnitude.

As pointed out already, ΔH_M decreases on increase of the initial domain-wall mobility μ_0, in agreement with the experimental observation of disappearance of the plateau in the case of high-mobility samples, whereas ΔH_{mr} is independent of the domain wall mobility. The dependence $H(\mu_0)$ calculated numerically for a function $\Phi(P)$ described by Eq. (10) is shown in Fig. 2b.

5. TRANSITION (TUNNELING) TIME

The maximum-retardation principle and the Maxwell principle are extreme assumptions. The former holds in an ideal system free of fluctuations when the controlling pa-

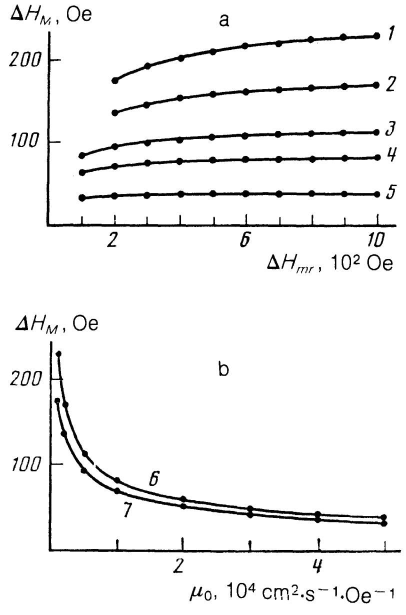

FIG. 2. Dependence of the width of the near-sonic region of constant domain wall velocity, calculated in accordance with the Maxwell principle, on the width of this region deduced using the dynamic theory (a) assuming various values of the initial mobility $\mu_0 = 10^3, 2\times 10^3, 5\times 10^3, 10^4$ and $5\times 10^4\ \mathrm{cm^2\cdot s^{-1}\cdot Oe^{-1}}$ (curves 1–5, respectively), and the dependence on the initial mobility (b) for different values of the width ΔH_{mr} deduced using the dynamic theory: 6) 1000 Oe; 7) 200 Oe.

rameters are altered slowly. The second is valid in systems in which the level of fluctuations characterized by the diffusion coefficient D is sufficiently high, i.e., when the diffusion coefficient is comparable with the height of the potential barrier separating minima of the function Φ. In general, the transition of a system from a metastable to the global minimum takes a certain time known as the tunneling time and this is a random quantity with the average value T_t. If in the range $H_M < H < H_2$ (see Fig. 1a) the value of T_t is less than the characteristic laboratory observation time T_l, which is close to the time taken by a domain wall to travel along a sample, amounting to about 10^{-7} s, and then the Maxwell principle applies. In the opposite case the maximum-retardation principle applies.

The dynamics of changes in the domain-wall velocity is described by the transient solutions of the Fokker-Planck equation (4). The presence of two terms on the right-hand side of this equation shows that the motion of a domain wall is characterized by two different time scales: a fast time T_r, associated with relaxation back to a local minimum after perturbation, and a slow time T_t associated with diffusion from a metastable minimum to the global minimum or, in other words, with tunneling of the domain wall velocity across a potential barrier. The time scales are given by the familiar expressions (see, for example, Ref. 9)

$$T_r = \left(\frac{\partial^2\Phi}{\partial P^2}\right)^{-1}_{P_{min}},$$

$$T_t = 2\pi\left|\left(\frac{\partial^2\Phi}{\partial P^2}\right)_{P_{min}}\left(\frac{\partial^2\Phi}{\partial P^2}\right)_{P_{max}}\right|^{-1/2} \times \exp\left[\frac{\Phi(P_{max}) - \Phi(P_{min})}{D}\right], \qquad (11)$$

where Φ is given by Eq. (10): $\partial^2\Phi/\partial P^2 = 1/\tau - \partial f/\partial P$; $P_{\max}$ corresponds to a maximum of the function Φ and $P_{\min}$ to a metastable minimum (Fig. 1b). The expressions given by the system (11) are not valid near the fields H_1 and H_2. However, in a probabilistic description the transition of a domain wall to the upper branch of the $V(H)$ curve after application of a magnetic field pulse H of intensity in the range $H_M < H < H_2$ represents a two-stage process. In the first stage the probabilistic distribution of $W(P)$ converges asymptotically in a time of the order of T_r to a Gaussian distribution with a curvature concentrated at a local minimum corresponding to the local branch of the $V(H)$ curve. In the second stage the value of $W(P)$ with the time scale T_t converges asymptotically to a steady-state solution (5), concentrated in the global minimum corresponding to the upper branch of the $V(H)$ curve. It is clear from Eqs. (10) and (11) that, in contrast to the dynamic theory,[3] the transition time has a finite value $T_t(H_M)$ in a field H corresponding to the edge of a plateau, which is confirmed experimentally.[3] Equating $T_t(H_M)$ to the maximum observed transition time, we find that $D \sim 10^{-7}\ \mathrm{g^2\cdot cm^{-2}\cdot s^{-3}}$ for $m_0 = 10^{-13}\ \mathrm{g/cm^2}$ and $\Delta H_{mr} \sim 10^3$ Oe. The value of T_t decreases on increase in H. For example, T_t decreases to half the initial value when the field is increased by 10 Oe, which is two orders of magnitude more slowly than in the dynamic theory, but in agreement (to within an order of magnitude) with the experimental results reported in Ref. 3. If for any reason the value of D decreases, then T_t corresponding to a given field rises rapidly and can exceed the laboratory observation time T_l. Then, the crossing of the sound barrier by a domain wall cannot be detected in a given field and the visible width of the plateau increases. On reduction in D the visible width of the plateau may increase up to ΔH_{mr} for a fixed value of T_l.

A reduction in the thickness of a sample from 100 to 10 μm increased the width of the magnetoelastic anomaly[11,12] for the transverse sound from 40 to 500 Oe. When a sample of thickness 20 μm was subjected to chemical etching of its surface, it was found[12] that the visible width of the plateau of the transverse sound decreased from 400 to 100 Oe. These results indicated that the roughness of the surface of a sample created by chemical etching eliminates the system "overheating" which was observed in the absence of the surface roughness.

The present paper proposes a new probabilistic description of the dynamics of a domain wall which allows for fluctuations of the system on the basis of the Fokker-Planck equation for the distribution function of the probability density in the momentum space. It is shown that we can introduce the Lyapunov function of a domain wall moving in a weak ferromagnet, so that the dynamics of this dissipative system can be reduced to that of a conservative potential system. The probabilistic description makes it possible to avoid the difficulties of the dynamic theory encountered in explaining why there is no hysteresis in the system and in deriving a single-valued field dependence of the domain wall velocity. The two-time nature of the process of overcoming of regions of resonant drag by a wall is established.

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