Contribution to the theory of the interaction between a magnetic charge and a medium

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It is shown that interaction between a moving monopole with atoms located along its trajectory renormalizes the effective charge of the monopole that determines the field far from the trajectory. This is in fact the physical reason why the true response functions in monopole electrodynamics differ from $\varepsilon(\omega)$ and $\mu(\omega)$. A model medium consisting of widely spaced quantum oscillators is considered, in which the renormalization changes the magnetic charge by a factor $\mu(\omega)$, so that, in particular, the equation for the Cherenkov-radiation intensity acquires an additional factor $\mu^2(\omega)$. A general estimate of the correction to the monopole charge is derived and shows that the renormalization in question is due mainly to diamagnetic effects.

1. INTRODUCTION

One of the deductions of present-day elementary-particle theories is that magnetic monopoles can exist,[1,2,3] although they have not been observed in experiment so far. A number of proposed methods of monopole detection are based on various effects due to electromagnetic interaction between a magnetic charge and a medium (see, e.g., Ref. 4). The analysis of this interaction usually begins with the Maxwell equation to which the magnetic-charge and current densities are added. The medium is then described by a dielectric constant $\varepsilon(\omega)$ and a magnetic permeability $\mu(\omega)$, determined from linear material equations. This approach was not subject to doubt for a long time, until a recent paper[5] revealed a fundamental difficulty encountered when an attempt is made to determine the magnetic field of a monopole moving in a continuous medium. It has been found that the Maxwell equations can be closed by many methods that are fully equivalent in monopole-free electrodynamics, but lead to different results for the field of a magnetic charge.[5,6] Since the choice of the particular forms of the material equations was in no way determined prior to publication of Ref. 5, it becomes necessary to review the main topics of magnetic-monopole electrodynamics.

A microscopic determination of the correct form of the material equations in linear monopole electrodynamics was the task undertaken in Ref. 5. Its implementation, however, calls for determining the response to fields that cannot be expressed in the usual manner in terms of a scalar and a vector potential.[6,7] If, on the other hand, the monopole field is described by a singular Dirac potential,[1] linear-response theory cannot be used, since the magnetic charge is quantized, and such a potential cannot be regarded as a small perturbation.

A somewhat different approach is proposed in the present paper. It is based on an analysis of the spatial distribution of the current induced in a medium by a moving monopole. We confine ourselves to the important particular case of a monopole in uniform rectilinear motion, and consider a medium with small spatial dispersion. In the actual calculations we use a model medium consisting of widely spaced and rigidly secured harmonic oscillators. These constraints allow us to explain many qualitative peculiarities of the problem without additionally assuming a weak interaction between the monopole and the medium.

We shall show that separate account must be taken of the contribution made to the induced current by atoms located in the immediate vicinity of the monopole trajectory. The point is that the atoms through which the monopole passes become excited and acquire an additional magnetic moment, so that the moving magnetic charge leaves in the medium a wake in the form of a "string" of magnetization. The current corresponding to this magnetization has the same structure as the current of a semi-infinite thin solenoid known to simulate the field of a monopole. Allowance for the additional current localized near the monopole trajectory leads therefore to a distinctive renormalization of the magnetic charge, and in the general case the renormalized charge $g^*(\omega)$ begins to depend on frequency. Also contributing to this frequency dependence are excited-atom relaxation processes that limit the length of the magnetization wake.

This renormalization effect changes the familiar results for the field of a magnetic charge in a medium. The expression for the Cherenkov-radiation spectral density differs from I. M. Frank's equation[9] in that the square of the charge is replaced by $|g^*(\omega)|^2$. The earlier expression remains in force at low frequencies for which the radiation wavelength exceeds greatly the magnetization-wake length, and also when the monopole moves in a channel cut through the medium and there is no extraneous current. It is curious that, in contrast to radiation by an ordinary charge, a channel influences the monopole radiation even if the channel radius is much smaller than the wavelength.

The approach proposed thus reduces the problem of correctly closing the Maxwell equations to the problem of calculating a function $g^*(\omega)$ that describes the interaction between a monopole and atoms directly in its path. So far, the renormalized magnetic charge has been calculated explicitly only in a few simple cases. Thus, for a system of quantum oscillators we find in the present paper that $g^*(\omega)=\mu(\omega)g$. It can be shown that the same relation holds also in classical-particle systems. In these models, however, $\mu(\omega)$ differs from unity only because of the diamagnetism, and the magnetic-charge renormalization is therefore actually small. A simple estimate made in the con-

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cluding section of this paper shows that in general the change of the monopole effective charge is indeed due to the diamagnetic effects.

2. RENORMALIZATION OF A MAGNETIC CHARGE IN A MEDIUM

We consider a magnetic monopole, with charge g, moving in a medium uniformly with velocity v in a straight line. The electromagnetic field produced by the monopole can be obtained with the aid of the Maxwell equations[2)]

$$\mathrm{rot}\,\mathbf{B}-\frac{1}{c}\dot{\mathbf{E}}=\frac{1}{c}\mathbf{j},\quad \mathrm{div}\,\mathbf{E}=\rho,$$
$$\mathrm{rot}\,\mathbf{E}+\frac{1}{c}\dot{\mathbf{B}}=-\frac{1}{c}\mathbf{j}_m,\quad \mathrm{div}\,\mathbf{B}=\rho_m, \tag{1}$$

where

$$\rho_m=g\delta(\mathbf{r}-\mathbf{v}t),\quad \mathbf{j}_m=g\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t),$$

and ρ and $\mathbf{j}$ are the charge and current densities induced in the medium by the fields $\mathbf{E}$ and $\mathbf{B}$. If we neglect spatial dispersion, the connection between the Fourier components of the induced charge and current and those of the field to first order in the field take the form

$$\rho_\omega(\mathbf{r})=(1-\varepsilon(\omega))\,\mathrm{div}\,\mathbf{E}_\omega(\mathbf{r}),$$
$$\mathbf{j}_\omega(\mathbf{r})=-i\omega(\varepsilon(\omega)-1)\mathbf{E}_\omega(\mathbf{r})+c\left(1-\frac{1}{\mu(\omega)}\right)\mathrm{rot}\,\mathbf{B}_\omega(\mathbf{r}), \tag{2}$$

where ε and μ are the dielectric constant and the magnetic permeability.[3)] This method of closing Eqs. (1) corresponds to the material equations $\mathbf{D}=\varepsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$ customarily used in the analysis of the Cherenkov radiation of a monopole[9,10] and of the total losses in the medium.[11,12] Of course, Eqs. (2) are not valid near a monopole, where the fields cannot be regarded as weak and slowly varying in space. This circumstance is as a rule not regarded as significant, and relations (2) are extrapolated to all of space, including the field source. When it comes to Cherenkov radiation, a justification for this extrapolation can be the intuitive notion that in a region much smaller than the radiation wavelength the changes of the properties of the medium have practically no effect on the radiation intensity (as first set fourth by L. I. Mandel'shtam). We shall see below, however, that a magnetic charge moving in a medium can leave along its trajectory an excited-atom wake not accounted for by Eqs. (2) and influencing the radiation intensity if the wake length is comparable with the wavelength λ. It is therefore necessary to consider separately the response of the medium located near the monopole trajectory.

We investigate first the general expression for the induced-current correction $\mathbf{j}(\mathbf{r},t)$ which we define as the difference between the exact averaged current density and the linear response (2). The arbitrary current density can be parametrized by three scalar functions. Under stationary conditions, with allowance for the symmetry of the problem (in which only one vector $\mathbf{v}$ is specified), it is convenient to use the following parametrization:

$$\delta\mathbf{j}(\mathbf{r},t)=c\,\mathrm{rot}[\mathbf{n}M(\mathbf{r}-\mathbf{v}t)]+\mathrm{rot}\,\mathrm{rot}[\mathbf{n}T(\mathbf{r}-\mathbf{v}t)]+\mathbf{v}Q(\mathbf{r}-\mathbf{v}t), \tag{3}$$

where $\mathbf{n}=\mathbf{v}/V$, $Q(\mathbf{r})$ is the correction to the charge density (2), while $M(\mathbf{r})$ and $T(\mathbf{r})$ are two unknown functions. In a cylindrical coordinate frame with Z axis along the monopole velocity, taking Fourier transforms with respect to time and factoring out next from all the functions the common z-dependence [the factor $\exp(i\omega z/v)$], we obtain from (3)

$$\delta j_\varphi=-c\frac{\partial M_\omega}{\partial\rho},$$
$$\delta j_z=vQ_\omega-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial T_\omega}{\partial\rho}\right), \tag{4}$$
$$\delta j_\rho=i\frac{\omega}{v}\frac{\partial T_\omega}{\partial\rho}.$$

If it is stipulated that $M_\omega(\rho)$ and $T_\omega(\rho)$ tend to zero as $\rho\to\infty$, then Eqs. (4), given the current, define uniquely all the scalar functions introduced to parametrize $\delta\mathbf{j}(\mathbf{r},t)$.

The physical meaning of Eq. (3) is quite clear: the three terms of the current describe respectively the additional contributions of the induced magnetic-moment, toroidal-moment,[13] and charge densities concentrated near the source and unaccounted for by the linear equation (2). We assume hereafter in the approximate estimates that the scalar functions contained in (3) differ noticeably from zero only at distances from the monopole trajectory on the order of the atom size a.

Let us dwell first on the last two terms of (3). In contrast to the first, which is seen from (4) to add an azimuthal component to the current, they should be present also when an ordinary charge moves through the medium. In a dielectric, however, the total charge corresponding to the $Q(\mathbf{r})$ distribution is zero, and the contribution of the dipole and higher electric moments to the magnetic field, as well as the contribution of the moving toroidal moment, is small compared with the field of a charge or a monopole if $\rho\gg a$. For example, the Cherenkov radiation from a dipole with $d\sim ea$ is small by virtue of $(a/\lambda)\ll 1$. The radiation due to the toroidal moments also contains this parameter.[14] The customary neglect of the additional current $\delta\mathbf{j}(\mathbf{r},t)$ in the analysis of fields at large distances from a charge trajectory is therefore apparently well justified.

To understand why similar reasoning does not hold in general for the first term of (3), we consider a medium consisting of isolated atoms in the ground state with zero angular momentum. The interaction between a slow monopole (of velocity lower than that of the atomic electrons) and such a medium was considered in Ref. 8. It was shown there that atoms through which a monopole passes become excited and acquire an angular momentum $\propto 2q\hbar$ ($q=eg/\hbar c$ is a half-integer) directed along the Z axis. The cause of this momentum can be readily perceived to be the twisting of the atomic electrons by the moving-monopole field (whose form, apart from the sign, is the same as that of the magnetic field of a moving charge).

Since the excited atoms thereby acquire a magnetic moment, the monopole should be followed by a magnetization wake whose thickness depends on the monopole velocities and is less than the atom size at low velocities. Disregarding the atomic relaxation, the current corresponding to this magnetization can be approximated roughly by

$$\delta\mathbf{j}(\mathbf{r},\ t)=c\,\mathrm{rot}(\mathbf{n}m(\rho))\theta(vt-z),\tag{5}$$

where $m(\rho)$ is some rapidly decreasing function. This is precisely the current described by the first term of Eq. (3). A fast monopole will interact much more strongly with the medium and ionize the atoms, but the action of its electric field, meaning also the general structure of the additional current (5), remains the same as before. Recognizing that the Fourier transform of the θ function is equal to $i/(\omega+i0)$ and comparing (5) with (3) and (4), we get

$$M_\omega(\rho)\approx im(\rho)/(\omega+i0)\tag{6}$$

at low frequencies. It will be shown below that it is just the appearance of the factor $1/\omega$ in $M_\omega(\rho)$ which makes the contribution of the current (3) to the field substantial at large distances from the monopole trajectory. The singularity in (6) as $\omega\to0$ is due, of course, to the infinite length of the wake, and is removed when relaxation is taken into account. Nonetheless, the magnetic-moment relaxation times, which depend on the actual model of the medium, can be quite large,[4] so that there are no *a priori* grounds for regarding the contribution of the first term in the additional current (3) to be small.

It can be assumed in the analysis of fields far from the moving monopole that $M_\omega(\rho)\sim\delta(\rho)$. Then taking only the first term of (3) into account, we write for the induced current

$$\mathbf{j}_\omega(\mathbf{r})=-i\omega(\varepsilon(\omega)-1)\mathbf{E}_\omega(\mathbf{r})+c(1-\mu^{-1}(\omega))\,\mathrm{rot}\,\mathbf{B}_\omega(\mathbf{r})+ic\gamma(\omega)\,\mathrm{rot}[g\mathbf{n}\delta(\rho)e^{i\omega z/v}],\tag{7}$$

where $\gamma(\omega)$ is defined by the integral[5]

$$\gamma(\omega)=-\frac{2\pi i}{g}\int_0^\infty d\rho\,\rho M_\omega(\rho).\tag{8}$$

The new response function $\gamma(\omega)$, which describes the monopole interaction with a medium close to its trajectory, can depend in general both on the charge of the monopole and on its velocity. A explicit determination of the function $\gamma(\omega)$ calls for a microscopic calculation.

Substituting the induced current (7) in the Maxwell equations (1), we obtain for the transverse electric field the equation

$$\Delta\mathbf{E}_\omega(\mathbf{r})+\varepsilon\mu\frac{\omega^2}{c^2}\mathbf{E}_\omega(\mathbf{r})=\frac{1}{c}g^*(\omega)\,\mathrm{rot}[\mathbf{n}\delta(\rho)e^{i\omega z/v}],\tag{9}$$

where

$$g^*(\omega)=g(1+\omega\mu(\omega)\gamma(\omega)).\tag{10}$$

Equation (9) has exactly the same form as that for the field of a monopole in a medium described by the linear material equations (2), but the role of the monopole charge is now assumed by a function $g^*(\omega)$ that depends on the properties of the medium near the monopole trajectory. Since $\mathbf{j}_m=0$ and $\mathbf{B}_\omega(\mathbf{r})=(c/i\omega)\nabla\times\mathbf{E}_\omega(\mathbf{r})$ outside the trajectory, the monopole magnetic field far from the monopole is also determined by the charge $g^*(\omega)$.

Allowance for the additional current (3) leads thus to a distinctive renormalization of the magnetic charge in the medium. We refer here, of course, not to the change of the true magnetic charge defined by the equation $\nabla\cdot\mathbf{B}=\rho_m$, but to the renormalized effective charge that determines the fields at large distances from the monopole trajectory ($\rho\gg a$). The cause of this behavior is that the correction to the induced current has the same structure as the current of a semi-infinite solenoid whose end coincides with the monopole location [this is most clearly seen from expression (5)]. Far from this solenoid, its field is well known to imitate that of a magnetic charge. Therefore even in a medium that contains no monopoles the magnetic charge can be partially screened, in the foregoing sense, by currents localized near its trajectory.

Knowing the fields far from the monopole, one can determine the loss to Cherenkov radiation. Note that a calculation of the loss from the force acting on the monopole encounters a fundamental difficulty. The point is that a quantized magnetic charge distorts strongly the ambient medium and calculation of the exact value of the magnetic field $\mathbf{B}$ on the monopole trajectory becomes a complicated problem (see Ref. 15 in this connection). We determine therefore the Cherenkov loss from the energy flux through a remote surface that encloses the monopole trajectory. Standard calculations yield for the radiation intensity in a frequency interval $d\omega$ the equation

$$dI_\omega=\frac{v}{c^2}|g^*(\omega)|^2\left(\varepsilon(\omega)-\frac{c^2}{v^2\mu(\omega)}\right)\omega\,d\omega,\tag{11}$$

which differs from the usual one[9,10] only in that the charge g is replaced by $g^*(\omega)$. It follows from (10) and (11) that for $\gamma(\omega)\sim1/\omega$ the contribution of the magnetization wake to radiation of wavelength λ does not contain the small parameter (a/λ). The difference between $g^*(\omega)$ and g can be neglected only if the wake is much shorter than λ.

The equations of Refs. 9 and 10 are valid also if the monopole moves in a channel cut through a medium and having a radius small compared with λ but larger than the characteristic length over which the function $M_\omega(\rho)$ decreases. In this case there is no magnetization wake and $g^*=g$. In this sense, radiation from a monopole is more similar to radiation from dipoles[16] than to radiation from an ordinary charge, which is not affected by the presence of a channel in the medium.

We conclude this section by establishing the correspondence between our approach and that of Ref. 5. Since the last term of (7) is proportional to the curl of the monopole current, it can be expressed, using the Maxwell equations, in term of the fields $\mathbf{B}$ and $\mathbf{E}$. The expression for the Fourier component of the transverse induced current then takes the general form

$$\mathbf{j}_\omega{}^t(\mathbf{k})=-i\omega(\tilde{\varepsilon}-1)\mathbf{E}_\omega(\mathbf{k})+ic\left(1-\frac{1}{\tilde{\mu}}\right)[\mathbf{k}\mathbf{B}_\omega(\mathbf{k})],$$

and the true response fucntions ε and μ introduced in Ref. 5 for the considered media are given by

$$\begin{aligned}1/\tilde{\mu}(\omega)&=1/\mu(\omega)+\omega\gamma(\omega),\\ \tilde{\varepsilon}(\omega,\ k)&=\varepsilon(\omega)+c^2k^2\gamma(\omega)/\omega.\end{aligned}\tag{12}$$

It follows hence, first, that the problem of calculating the correct response function reduces in fact to an analysis of the interaction between a monopole and a medium located near its trajectory (at distances on the order of the spatial-dispersion radius). Second, even if a medium containing no monopole can be described with spatial dispersion disregarded, this can generally not be done in monopole electrodynamics, because $\tilde{\varepsilon}$ becomes dependent on k. Since, finally, $\gamma(\omega)$ can readily become dependent on the magnetic charge, the medium in monopole electrodynamics can in general not be regarded as linear.

It was also shown in Ref. 5 that $\tilde{\mu} = 1$ in the particular case of an isotropic medium consisting of classical particles. We have then from (12) for such a medium

$$\gamma(\omega)=\omega^{-1}(1-\mu^{-1}(\omega)), \tag{13}$$

and therefore, according to (10), $g^*(\omega) = \mu(\omega)g$. We shall verify in the next section that the same relations hold also for a system of quantum oscillators.

THE QUANTUM-OSCILLATOR MODEL

We consider in this section the interaction between a fast monopole and a low-density medium made up of isolated atoms randomly distributed in space, with average density n. In the actual calculations the atoms will be simulated by nonrelativistic harmonic oscillators of frequency ω_0. The simplicity of the model permits a complete calculation of the renormalized charge g^* of a fast monopole.

To calculate the current induced near a monopole in a low-density medium it suffices to known the response to the fields produced by a magnetic charge in vacuum. We describe this field by means of singular vector potential. In a cylindrical coordinate frame with the Z axis parallel to the monopole velocity, it takes the form

$$A_\varphi=\frac{g}{\rho}(1-\cos\alpha),\quad A_\rho=A_z=0,$$
$$\operatorname{ctg}\alpha=\frac{z-vt}{\rho(1-v^2/c^2)^{1/2}}. \tag{14}$$

The potential (14) induces on the monopole trajectory an additional magnetic field

$$B_z=4\pi g\delta(\rho)\theta(vt-z)$$

meaning a string along which the magnetic flux $4\pi g$ returns to the monopole. If the monopole charge is so quantized that $q = eg/\hbar c$, the string is unobservable and (14) describes a magnetic charge. As $v\to 0$ the potential (14) is transformed into the Dirac static potential.[1]

If the monopole velocity v is much larger than the velocity v_0 of the electrons in the atom, the Schrödinger equation with the potential (14) can be solved by using the sudden approximation and assuming that the wave function $\psi_0(\mathbf{r})$ of the ground state of the atom does not manage to change during the monopole collision time. The interaction time $\rho v^{-1}(1-v^2/c^2)^{1/2}$ of an electron located at a distance ρ from the monopole trajectory should then be shorter than the characteristic time $1/\omega_0$ of electron motion in the atom, so that the admissible values of ρ are bounded from above. This bound, however, is immaterial for atoms through which a monopole having $v \gg v_0$ passes. Nor is the instantaneous approach valid in a narrow region near the monopole trajectory, where the fields are so strong that the wave function is substantially altered even in a short time of flight. This region is small by a factor $v_0/v \ll 1$ compared with the atomic size a, and will henceforth be disregarded. Since the instantaneous approximation does not take into account the evolution over a collision time, it yields the response of the medium only for frequencies $\omega \ll (v/a)(1-v^2/c^2)^{-1/2}$.

It follows from (14) that the passage of the molecule leaves in the atom a string that carries a quantized magnetic flux $\Phi = 2q(hc/e)$ and is described by a vector potential $A_\varphi = 2g/\rho$. The string can be eliminated from the Hamiltonian by a gauge transformation that changes the function $\psi_0(\mathbf{r})$ into

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})e^{-2iq\varphi}, \tag{15}$$

where φ is the angle in the cylindrical frame. The subsequent evolution of the state (15) is determined by the free Hamiltonian of the atom. The result (15) has a simple interpretation: the electron does not manage to change position during the time of flight of the monopole through the atom, but acquires under the influence of the electric field an angular momentum $-2q\hbar$ along the Z axis. The wave function (15) is not well defined at $\rho = 0$, since the instantaneous approximation is inapplicable for very small ρ. We point out also that sudden-approximation calculation, based on Eq. (15), of the ionization losses of a monopole leads to a known result[17] derived by another method (see Appendix 1).

We calculate now the induced current, replacing the atoms by harmonic oscillators, and consider first only one oscillator at the point $\mathbf{r}_0$. It is convenient to describe it not by the wave function $\psi(\mathbf{r},t)$ but by the Wigner function

$$f(\mathbf{r},\mathbf{p};t)=\int d^3\xi\,\psi(\mathbf{r}+\xi/2,t)\psi^*(\mathbf{r}-\xi/2,t)e^{-i\mathbf{p}\xi/\hbar}.$$

Using the explicit expression for the oscillator ground-state wave function and Eq. (15), we obtain for the Wigner function at $t = z_0/v + 0$, i.e., immediately after the passage of the monopole, a representation in the form

$$f_0(\mathbf{r},\mathbf{p})=\frac{2}{\pi a^2}\exp\left[-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{a^2}-\frac{a^2p_z^2}{\hbar^2}\right]\cdot\int d^2\xi\,W(\rho,\xi)\exp\left[-\frac{\xi^2}{4a^2}-i\frac{\xi\mathbf{p}_\perp}{\hbar}\right], \tag{16}$$

where

$$W(\rho,\xi)=\exp\left[-2iq\Omega(\rho,\xi)\right],$$

$\boldsymbol{\rho} = (x,y)$, $\mathbf{p}_\perp = (p_x,p_y)$, $a^2 = \hbar/m\omega_0$, and $\Omega(\rho,\xi)$ is the angle between the two-dimensional vectors $\boldsymbol{\rho} + \xi/2$ and $\boldsymbol{\rho} - \xi/2$. Straightforward calculations reduce the function $W(\rho,\xi)$ to the form

$$W(\rho,\xi)=\left(\frac{\rho^2-\xi^2/4-i[\rho\xi]_z}{|\rho^2-\xi^2/4-i[\rho\xi]_z|}\right)^{2q}. \tag{17}$$

The time dependene of the Wigner function for $t > z_0/v$ is determined in the instantaneous approximation only by the Hamiltonian of the harmonic oscillator and is described by the familiar equation

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial f}{\partial \mathbf{r}}-m\omega_0^2(\mathbf{r}-\mathbf{r}_0)\frac{\partial f}{\partial \mathbf{p}}=0, \tag{18}$$

which coincides with the classical equation for the distribution function. The function $f(\mathbf{r},\mathbf{p};t)$ is therefore governed by the equations of motion of a classical oscillator and the solution of Eq. (18) with the initial condition (16) can be written in the form

$$f(\mathbf{r},\mathbf{p};t)=f_0(\tilde{\mathbf{r}}(\mathbf{r},\mathbf{p},t),\tilde{\mathbf{p}}(\mathbf{r},\mathbf{p},t)),$$

where

$$\tilde{\mathbf{r}}=\mathbf{r}_0+(\mathbf{r}-\mathbf{r}_0)\cos\omega_0\tau-\frac{\mathbf{p}}{m\omega_0}\sin\omega_0\tau,$$

$$\tilde{\mathbf{p}}=\mathbf{p}\cos\omega_0\tau+m\omega_0(\mathbf{r}-\mathbf{r}_0)\sin\omega_0\tau.$$

In these equations $\tau=t-z_0/v$. Since the function (6) differs significantly from zero only if $|z-z_0|\sim a$, it can be assumed that for times $\tau\gg a/v$, the only ones considered in the sudden approximation, we have $\tau=t-z/v$.

Now we average the function $f(\mathbf{r},\mathbf{p};t)$ over the position of the center $\mathbf{r}_0$, assuming the oscillators to be randomly distributed in space with an average density n. It is convenient for this purpose to first expand $W(\boldsymbol{\rho},\boldsymbol{\xi})$ as a function of $\boldsymbol{\rho}$ in a two-dimensional Fourier integral. The integral over $d\mathbf{r}_0$ is then Gaussian and can be easily calculated. After a number of transformations we obtain for the averaged Wigner function

$$\overline{f(\mathbf{r},\mathbf{p};t)}=2a\pi^{1/2}n\exp\left(-\frac{a^2p_z^2}{\hbar^2}\right)\int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\boldsymbol{\rho}}$$

$$\cdot\int d^2\xi\,\tilde{W}(\mathbf{k},\boldsymbol{\xi}\cos\omega_0\tau)\cdot$$

$$\cdot\exp\Bigg[-\frac{\xi^2}{4a^2}-i\frac{\boldsymbol{\xi}\mathbf{p}_\perp}{\hbar}-i\frac{\mathbf{k}\mathbf{p}_\perp}{\hbar}a^2\frac{\sin\omega_0\tau}{\cos\omega_0\tau}$$

$$-\frac{k^2a^2}{4}\frac{(1-\cos\omega_0\tau)^2}{\cos\omega_0\tau}-$$

$$-\frac{1}{2}\mathbf{k}\boldsymbol{\xi}\frac{\sin\omega_0\tau(1-\cos\omega_0\tau)}{\cos\omega_0\tau}\Bigg],\qquad(19)$$

where $\tilde{W}(\mathbf{k},\boldsymbol{\xi})$ is the Fourier transform of $W(\boldsymbol{\rho},\boldsymbol{\xi})$. This expression must now be substituted in the equation for the induced-current density

$$\mathbf{j}(\mathbf{r},t)=\frac{e}{m}\int\frac{d^3p}{(2\pi\hbar)^3}\mathbf{p}\,\overline{f(\mathbf{r},\mathbf{p};t)}.$$

It is seen immediately that in this approximation the current along the Z axis is zero, and the integral over $\mathbf{p}_\perp$ yields the derivative of the δ function

$$i\hbar\frac{\partial}{\partial\boldsymbol{\xi}}\delta\left(\boldsymbol{\xi}+\mathbf{k}a^2\frac{\sin\omega_0\tau}{\cos\omega_0\tau}\right),$$

which eliminates the integration over $d\boldsymbol{\xi}$ in (19). It is convenient to change next to a two-dimensional Fourier transform of the current density as a function of $\boldsymbol{\rho}$ and to distinguish in it the transverse component $\mathbf{j}_t(\mathbf{k},\tau)$ which is proportional to $\mathbf{v}\times\mathbf{k}$. Taking into account next the explicit form (17) of $W(\boldsymbol{\rho},\boldsymbol{\xi})$ we obtain ultimately

$$\mathbf{j}_t(\mathbf{k},\tau)=2q\frac{ne\hbar}{m}\cos\omega_0\tau\,\Phi_q\left(\frac{k^2a^2}{2}\sin\omega_0\tau\right)$$

$$\cdot\exp\left[-\frac{k^2a^2}{2}(1-\cos\omega_0\tau)\right]\frac{2\pi i}{vk^2}[\mathbf{v}\mathbf{k}],\qquad(20)$$

where the function $\Phi_q(\zeta)$ is defined by the integral

$$\Phi_q(\zeta)=\frac{1}{4\pi}\int dx\,dy\sin x\,e^{-\delta|x|}$$

$$\cdot\left[\frac{x+\zeta}{(x+\zeta)^2+y^2}+\frac{x-\zeta}{(x-\zeta)^2+y^2}\right]\left(\frac{x^2+y^2-\zeta^2-2i\zeta y}{|x^2+y^2-\zeta^2-2i\zeta y|}\right)^{2q},$$

$$\delta\to+0.$$

It is possible to obtain in similar fashion the current in a system of oscillators also for a finite temperature T. The result for this case is obtained from (20) by replacing a^2 by

$$a^2(T)=\frac{\hbar}{m\omega_0}\operatorname{cth}\frac{\hbar\omega_0}{2T}$$

in the argument of the exponential (the function Φ_q contains $a^2=\hbar/m\omega_0$ as before).

To compare the expression obtained for the induced current with Eq. (7) of the preceding section, we expand (20) in powers of k. It is important here that the behavior of $\Phi_q(\zeta)$ for small ζ is given by

$$\Phi_q(\zeta)=1+q^2\zeta^2\ln|\zeta|+\ldots$$

Therefore Φ_q makes no contribution to the first two terms of the transverse-current expansion

$$\mathbf{j}_t(\mathbf{k},\omega)=-i\omega\frac{e^2n}{m}\frac{1}{\omega_0^2-\omega^2}g\frac{4\pi i[\mathbf{v}\mathbf{k}]}{vck^2}e^{i\omega z/v}$$

$$-g\left[\frac{\mathbf{v}}{v}\mathbf{k}\right]\frac{2\pi e^2na^2}{mc}\frac{1}{\omega}\left[\frac{\omega_0^2(\omega^2+2\omega_0^2)}{(\omega_0^2-\omega^2)(4\omega_0^2-\omega^2)}\right]e^{i\omega z/v}+\ldots$$

It is easy to verify that this expression becomes equivalent to the phenomenological equation (7) for the induced current upon substitution of the monopole field in vacuum (in this case the second term of (7) is small compared with the first in view of the smallness of v_0/c and can therefore not be obtained in the sudden approximation), while the response functions are equal to

$$\varepsilon(\omega)=1+\frac{\omega_p^2}{\omega_0^2-\omega^2},$$

$$\gamma(\omega)=-\frac{1}{\omega}\frac{\omega_p^2a^2}{2c^2}\frac{\omega_0^2(\omega^2+2\omega_0^2)}{(\omega_0^2-\omega^2)(4\omega_0^2-\omega^2)},\qquad(21)$$

where $\omega_p=(4\pi e^2n/m)^{1/2}$ is the plasma frequency. The magnetic permeability of a system of oscillators is calculated in Appendix 2, and

$$\frac{1}{\mu(\omega)}=1+\frac{\omega_p^2a^2}{2c^2}\frac{\omega_0^2(\omega^2+2\omega_0^2)}{(\omega_0^2-\omega^2)(4\omega_0^2-\omega^2)}.\qquad(22)$$

It follows from (21) and (22) that the relation $\omega\gamma=1-1/\mu$, which is valid also in classical-particle systems, holds in the model considered here [see (13)]. The renormalized monopole charge is likewise given in this case by

$$g^*(\omega)=\mu(\omega)g.\qquad(23)$$

The distinctive character of an oscillator model without atomic ionization is shown by the fact that $\gamma(\omega)$ does not depend on monopole charge and by the fact that $g^*(\omega)$ is determined by the simple equation (23). It is also known that a quantum oscillator is similar in many respects to a classical one for which relation (23) [with a different $\mu(\omega)$, naturally] is known to be valid. It must be emphasized at the same time that the result is not obvious, for in contrast to the analysis of classical systems, the derivation here made use of quantization of a magnetic charge which is nowhere assumed to be small. It turned out simply that the explicit form of the function $\Phi_q(\zeta)$ in (20), which describes essentially quantum and nonlinear effects in the response, is immaterial for the determination of $\gamma(\omega)$.

The reason for the singularity of $\gamma(\omega)$ as $\omega\to0$ is that there is no relaxation mechanism whatever in the model. In a real system, the induced current attenuates with time, and this can be accounted for in the simplest approximation by multiplying the current (20) by $\exp(-\nu\tau)$. The frequency ω in Eqs. (21) is then replaced by $\omega + i\nu$ and the renormalized charge for $\nu \ll \omega$ is, in accordance with (10),

$$g^*(\omega) = \frac{\omega\mu(\omega) + i\nu}{\omega + i\nu} g$$

with $\mu(\omega)$ from (22). Evidently, screening effects ($\mu < 1$) come into play only for $\omega \gg \nu$.

CONCLUDING REMARKS

We have demonstrated here that interaction of a monopole with atoms located on its trajectory renormalizes the effective magnetic charge, in view of the appearance of a magnetization wake in the medium. From the physical viewpoint, it is precisely this effect which leads to the difference between the true response functions introduced in Ref. 5 and the functions $\varepsilon(\omega)$ and $\mu(\omega)$. On a purely theoretical level, this difference leads to important consequences. For example, A. A. Kolmogorov's equation[10] that describes the Cherenkov radiation in media with $\mu \neq 1$ acquires an extra factor $\mu^2(\omega)$. This result, however, calls for a number of remarks.

First, magnetic-charge renormalization and all the concomitant effects occur only at frequencies higher than the magnetization-wake relaxation frequency. The relaxation mechanisms depend on the specific model of the medium and their analysis is outside the scope of the present paper. Second, in all the models for which[6)] $g^*(\omega) = \mu(\omega)g$ the value of μ differs from unity only on account of the diamagnetism, so that the real renormalization effects are rather small. This is true, unfortunately, not only in purely diamagnetic media.

To verify this, we consider a medium consisting of arbitrary atoms that can have a nonzero angular momentum in the ground state. From the angular-momentum conservation law it follows[8] that when a monopole passes precisely through the center of an atom the change of the atomic angular momentum is exactly equal to $-2q\hbar Z$, where Z is the number of electrons in the atom. This result depends neither on the initial angular momentum nor on the presence of electron spin. Multiplying $-2q\hbar Z$ by the gyromagnetic ratio and by the atom density n, we obtain the maximum magnetization in the wake of the monopole. Assuming further that $M_\omega(\rho)$ behaves like (6) and is decreased at a distance a from the monopole trajectory, we obtain from Eqs. (8) and (10) for the correction to the magnetic charge the estimate

$$\Delta g \sim -g\mu(\omega)\frac{Ze^2a^2}{mc^2}n. \qquad (24)$$

Disregarding for a moment the factor $\mu(\omega)$, it follows from (24) that the correction is determined not by the total magnetic susceptibility, which would lead to an equation such as (23), but only by its diamagnetic part, which is small in terms of the parameter $(v_0/c)^2 < 1$ ($v_0 \sim Z^{2/3}e^2/\hbar$) is the average electron velocity in the atom) even for $na^3 \sim 1$. This means that the magnetic-charge renormalization can be neglected for ordinary transparent media in which Cherenkov radiation is observed. The effect can be enhanced by choosing a condensed medium of sufficiently heavy atoms and with high $\mu(\omega)$.

This analysis pertains only to media with low spatial dispersion. In general outline, however, the analysis is meaningful also in the case of high spatial dispersion. To describe the interaction of a monopole with such media it is certainly necessary to introduce a renormalized charge $g^*(\omega,\mathbf{k})$ that depends on the frequency and on the wave vector $\mathbf{k}$. It is curious, that if Eq. (23) with μ replaced by $\mu(0,k) \approx k^2\lambda_L^2$ (λ_L is the field penetration depth) is applied now to a superconductor, the correct result $g^* \to 0$ is obtained for large scales, meaning total screening of the monopole's magnetic field (this field is entirely concentrated in a vortical filament trailing the monopole).

It is noteworthy, in conclusion, that effects similar to those considered in this paper could in principle take place also for an ordinary charge capable of leaving behind in some medium a polarization wake localized near its trajectory.

I am grateful to D. A. Kirzhnits and V. V. Losyakov for numerous stimulating discussions on the electrodynamics of a magnetic monopole.

APPENDIX 1

Let us calculate the monopole energy losses within the framework of the instantaneous approximation. In this approximation an atom located at a point $\mathbf{r}_0$ continues to remain in the ground state $\psi_0(\mathbf{r} - \mathbf{r}_0)$ directly after passage of the monopole, and the Hamiltonian takes the form

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + U(\mathbf{r}-\mathbf{r}_0), \qquad (1.1)$$

where $A_\varphi = 2g/\rho$, $A_\rho = A_z = 0$ (in a cylindrical coordinate frame with Z axis along the monopole velocity). Averaging the Hamiltonian (1.1) over the wave function $\psi_0(\mathbf{r} - \mathbf{r}_0)$, which we assume to be real and spherically symmetric, we obtain the change in the energy of the atom due to the interaction with the monopole:

$$\Delta E = \frac{2g^2e^2}{mc^2}\int d^2\rho\, dz \frac{1}{\rho^2}\psi_0^2(\mathbf{r}-\mathbf{r}_0).$$

Further averaging over the atom location in the $z_0 = \text{const}$ plane leads, with allowance for the normalization of the wave function, to the equation

$$\frac{dE}{dt} = vg^2\frac{4\pi e^2 n}{mc^2}\int\frac{d\rho}{\rho},$$

where n is the atomic density. The integral diverges at both limits because the sudden approximation is inapplicable for large as well as for very small ρ. The estimate given at the beginning of Sec. 3 yields for the a nonrelativistic monopole $\rho_{\max} \sim v/\omega_0$, where ω_0 is the characteristic atom-excitation frequency. We can estimate $\rho_{\min}$ by assuming that in the sudden approximation the distance that an electron initially located at a distance ρ can be displaced by a monopole field $E \sim gv/c\rho^2$ in an interaction time ρ/v should be small compared with ρ. We obtain then $\rho_{\min} \sim q\hbar/mv$. It appears that the probability of finding an electron in a region $\rho < \rho_{\min}$ immediately after the passage of a monopole is very low and tends to zero as $\rho \to 0$. With allowance for these estimates, the energy losses are given by

$$\frac{dE}{dt} = vg^2\frac{4\pi e^2 n}{mc^2}\ln\left(\frac{mv^2}{q\hbar\omega_0}\right), \qquad (1.2)$$

which is the result of Ref. 17.

APPENDIX 2

We consider the transverse response in a low-density system of harmonic oscillators. In the linear approximation, the current induced in one oscillator located at a point $\mathbf{r}_0$ can be expressed in the form

$$j_\omega^\alpha(\mathbf{r}) = \int d\mathbf{r}' \, \sigma_{\alpha\beta}^{(0)}(\mathbf{r}-\mathbf{r}_0, \mathbf{r}'-\mathbf{r}_0; \omega) E_\omega^\beta(\mathbf{r}'). \qquad (2.1)$$

The response is described in momentum space by the conductivity $\sigma_{\alpha\beta}^{(0)}(\omega;\mathbf{k},\mathbf{k}')$, which depends on two wave vectors. Assuming a random distribution of the oscillators in space, with an average density n, we average the current (2.1) over $\mathbf{r}$. It is easy to verify that the response is then

$$\sigma_{\alpha\beta}(\omega,\mathbf{k}) = n\sigma_{\alpha\beta}^{(0)}(\omega;\mathbf{k},-\mathbf{k}).$$

Using next the Kubo equation for $\sigma_{\alpha\beta}^{(0)}(\omega;\mathbf{k},-\mathbf{k})$, we get

$$\sigma_{\alpha\beta}(\omega,\mathbf{k}) = \frac{ine^2}{m\omega}\delta_{\alpha\beta} + \frac{ine^2\hbar^2}{m^2\omega} \cdot \sum_n \left[\frac{\langle 0|\nabla_\alpha e^{-i\mathbf{k}\mathbf{r}}|n\rangle\langle i|e^{i\mathbf{k}\mathbf{r}}\nabla_\beta|0\rangle}{\hbar\omega - E_n + E_0} - \frac{\langle 0|e^{i\mathbf{k}\mathbf{r}}\nabla_\beta|n\rangle\langle n|\nabla_\alpha e^{-i\mathbf{k}\mathbf{r}}|0\rangle}{\hbar\omega + E_n - E_0} \right], \qquad (2.2)$$

where the matrix elements relate the states of a harmonic oscillator located at the origin. To calculate the transverse part of the conductivity it suffices to consider $\sigma_{yy}(\omega,\mathbf{k})$ and to direct the vector $\mathbf{k}$ along the X axis. Since motions in the oscillator along different axes are independent, all the matrix elements factor out, after which

$$\sigma_t(\omega,k) = \frac{ine^2}{m\omega} + \frac{ine^2\hbar}{m^2\omega}\sum_{n_1n_2} \left|\left\langle 0\left|\frac{\partial}{\partial y}\right|n_1\right\rangle\right|^2 \cdot |\langle 0|e^{ikx}|n_2\rangle|^2 \left[\frac{1}{\omega-\omega_0(n_1+n_2)} - \frac{1}{\omega+\omega_0(n_1+n_2)}\right],$$

where the summation is now over the states of two one-dimensional oscillators. The equations for a one-dimensional oscillator are

$$\left\langle 0\left|\frac{\partial}{\partial x}\right|n\right\rangle = \frac{\delta_{n,1}}{2^{1/2}a},$$

$$|\langle 0|e^{ikx}|n\rangle|^2 = \frac{1}{n!}\left(\frac{k^2a^2}{2}\right)^n e^{-k^2a^2/2}, \quad a^2 = \frac{\hbar}{m\omega_0}$$

and allowance for them permits the transverse dielectric constant

$$\varepsilon_t(\omega,\mathbf{k}) = 1 + \frac{4\pi i}{\omega}\sigma_t(\omega,k)$$

to be written in our model in the form

$$\varepsilon_t(\omega,k) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}e^{-k^2a^2/2} \cdot \sum_{n=0}^{\infty} \frac{\omega_0^2(n+1)}{\omega^2-\omega_0^2(n+1)} \frac{(k^2a^2/2)^n}{n!}, \qquad (2.3)$$

where ω_p is the plasma frequency. This equation was obtained earlier by D. A. Kirzhnits and V. V. Losyakov, who found the linear response with the aid of the Heisenberg-operator equations of motion. According to the definition of $\varepsilon(\omega)$ and $\mu(\omega)$ used here, we have for small k

$$\varepsilon_t(\omega,k) = \varepsilon(\omega) + \frac{k^2c^2}{\omega^2}\left(1 - \frac{1}{\mu(\omega)}\right) + \dots$$

Expanding next (2.3) in powers of k we obtain $\varepsilon(\omega)$ from (21), and Eq. (22) for the magnetic permeability.

APPENDIX 3

We consider an atom located at a distance ρ from a monopole trajectory and calculate the change of its angular momentum as a result of interaction with the monopole. In contrast to the rest of the text, we use in the calculation a spherical coordinate frame with origin at the center of the atom.

An atom with zero angular momentum in the initial state acquires after the passage of a monopole an average angular momentum along the Z axis. In the instantaneous approximation this momentum can be calculated by merely averaging, over the ground-state wave function $\psi_0(r)$, the operator

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi} - r\sin\theta\frac{e}{c}A_\varphi(\mathbf{r}),$$

where $A_\varphi(\mathbf{r})$ is the potential of the string with quantized magnetic flux, which remains after the passage of the monopole. In the sperical coordinate frame we have

$$A_\varphi = 2g\frac{r\sin\theta - \rho\cos\varphi}{r^2\sin^2\theta + \rho^2 - 2R\rho\sin\theta\cos\varphi}.$$

The result of the averaging is quite simple:

$$L_z = -2q\hbar\int_0^\infty dr\, r^2\left(1 - \frac{\rho^2}{r^2}\right)^{1/2} R_0^2(r), \qquad (3.1)$$

where $R_0(r)$ is the radial part of the wave function. It can be seen that $L_z = -2q\hbar$ for $\rho = 0$, and that for ρ much larger than the atomic size the angular momentum decreases rapidly. Multiplying (3.1) by the gyromagnetic ratio and by the atom density n, we obtain the function $m(\rho)$ of Eq. (6). Taking next (8) into account and integrating (3.1) with respect to ρ we obtain $\gamma(\omega)$ for low frequencies in the form

$$\gamma(\omega) \approx -\frac{1}{\omega}\frac{2\pi e^2 n}{3mc^2}\langle r^2\rangle = \frac{-4\pi\chi}{\omega}, \qquad (3.2)$$

where $\langle r^2\rangle$ is the average over the ground state, and $\chi = -e^2n\langle r^2\rangle/6mc^2$ is the static diamagnetic susceptibility of the atomic gas. In a low-density medium, Eq. (3.2) leads to a renormalized magnetic charge $g^* = \mu g$ with $\mu = 1 + 4\pi\chi$.

[1]The result for a system of classical particles is given in Ref. 5.

[2]We use Heaviside units in this section.

[3]Here ε and μ are connected with the employed longitudinal and tranverse constants $\varepsilon_l(\omega,k)$ and $\varepsilon_t(\omega,k)$ by the relations

$$\varepsilon = \varepsilon_l(\omega,0) = \varepsilon_t(\omega,0), \quad 1 - \mu^{-1} = \lim_{k\to 0}[(\varepsilon_t(\omega,k) - \varepsilon(\omega))\omega^2/k^2c^2].$$

The quantity $\mu(\omega)$ defined in this manner is meaningful at all frequencies.

[4]If, for example, the relaxation time is assumed to be the lifetime $\tau \sim 10^{-8}$ s of an isolated excited atom, the wake length for a monopole with $v \sim 10^{-3}$ c is of the order of 0.3 cm.

[5]The convergence of this integral is not considered here. Convergence of the integral of the singular part of (6) is sufficient.

[6]It is shown in Appendix 3 that as $\omega \to 0$ this result is valid for any law of electron–nucleus interaction, and not only in the oscillator model.

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