# Nature of the symmetry group of the 6*j*-symbol

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The symmetries of the 6*j*-symbol form a group, related to the transposition of roots of the characteristic polynomial of a quadratic algebra.

## 1. INTRODUCTION

The 6*j*-symbols (Racah coefficients), which are the recoupling coefficients between two schemes for adding three angular momenta, are widely used in theoretical physics. In addition, the 6*j*-symbols play an important role in various problems of mathematical physics, related to the theory of group representations. In particular, Racah coefficients arise in the "tree" problems and representations of the $SU(3)$ group.[1] The connection of the 6*j*-symbols with the theory of special functions is also of interest; in particular it turns out that they are expressible in terms of a new class of orthogonal polynomials.[1] Of particular interest is the study of the symmetries, i.e. transformations of the parameters that leave invariant the 6*j*-symbol

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$

(here $j_1, j_2, j_3$ are the values of the angular momenta being added, $j_4$ is the total angular momentum, and $j_{12}$ and $j_{23}$ are intermediate angular momenta.)

In addition to the "classical" symmetries, discovered already by Racah,[2] the so-called Regge symmetries were found in 1958.[3] Their discovery was a kind of scientific sensation, since no simple interpretation of these symmetries existed. In the 30 years that have passed since many attempts were made to give a simple explanation or a different treatment of these mysterious transformations (see, e.g., Ref. 4). However, all these explanations are rather complicated and awkward from the point of view of calculations.

In this paper we show that the 6*j*-symbols are uniquely determined by a representation of a quadratic algebra with three generating operators: the invariance properties of the 6*j*-symbols are a trivial consequence of the invariance of the structure parameters of the algebra.

We note that the well-known method of Vilenkin[5] of group-theoretical construction of special functions does not permit the inclusion of 6*j*-symbols (and the orthogonal polynomials corresponding to them[1]), because the latter are not the matrix elements of a transformation operator of a Lie group. In particular, the Vilenkin method does not cover a sufficiently wide class of functions—classical orthogonal polynomials of discrete argument.

An approach that permits the inclusion of precisely this class of functions was proposed in Ref. 6. The main idea of the method consists of the observation that the three-term recursion relation for the special functions (specifically—orthogonal polynomials) is obtained as a consequence of the commutation relations of a Lie algebra. The orthogonal polynomials themselves arise as the transformation function between the bases of two hermitian operators that enter the algebra. Moreover, the eigenvalues of one of the operators serve as the order of the polynomial, while the eigenvalues of the other serve as the argument of the polynomial. From the very construction scheme it is clear that the form of these polynomials is independent of the specific nature of the representation and is determined just by the commutation relations of the algebra. Using this approach it is possible to include in the scheme of Lie algebras with three generators several known classes of polynomials of discrete as well as of continuous argument.[6]

The 6*j*-symbols cannot be included in this scheme, because a Lie algebra with three generators has too few independent parameters. However this can be accomplished, if the commutator of two operators is expressed nonlinearly in terms of the other operators, i.e., it is necessary to drop the requirement that the resultant algebra be a Lie algebra. If one takes the simplest generalization—an algebra with quadratic commutators—then it becomes possible to construct ladder representations of such an algebra for which the transformation function between the bases of the two operators is the 6*j*-symbol.

Algebras with quadratic commutation relations were first introduced by Sklyanin in papers[7] devoted to the lattice model in the theory of magnetism. However the Sklyanin type algebras possess too complicated a structure, in particular they have four generating operators in place of the three needed in our case.

In this paper it is shown that a quadratic algebra with a simple structure can successfully be applied to the study of 6*j*- and 3*j*-symbols, as well as of the Wilson-Racah orthogonal polynomials, which are a generalization of these physical objects.[1] Moreover, practically all the important properties of these polynomials follow just from the commutation relations, which makes possible a new simple algebraic treatment of these important objects of theoretical physics.

## 2. 6*j*-SYMBOLS AND THE QUADRATIC ALGEBRA

We recall that 6*j*-symbols appear in the theory of addition of three angular momenta $\mathbf{J}_k$ ($k = 1,2,3$) with eigenvalues $j_k(j_k + 1)$ as follows. We form the squares of the intermediate momenta according to the schemes $\mathbf{J}_{12}^2 = (\mathbf{J}_1 + \mathbf{J}_2)^2$ and $\mathbf{J}_{23}^2 = (\mathbf{J}_2 + \mathbf{J}_3)^2$ and we introduce the square of the total angular momentum $\mathbf{J}_4^2 = (\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3)^2$, which commutes with all the other operators. The eigenfunctions of the operators

$$\mathbf{J}_{12}^2\psi_p^{(12)} = p(p+1)\psi_p^{(12)}, \qquad p = j_{12},$$

$$\mathbf{J}_{23}^2\varphi_q^{(23)} = q(q+1)\varphi_q^{(23)}, \qquad q = j_{23}$$

form two independent bases in the space in which the quantum numbers $j_1, j_2, j_3$ and $j_4$ have been fixed. The transforma-

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tion matrix $\langle \varphi_q | \psi_p \rangle$ forms the set of $6j$-symbols.

The main idea of the proposed method reduces to the following. Since $\psi_p, \varphi_q$ are eigenfunctions of two noncommuting operators, information on the structure of the matrix element $\langle \varphi_q | \psi_p \rangle$ may be obtained by studying the commutation properties of these operators. In particular, if the commutation relations form a closed algebra then the properties of the matrix elements are uniquely determined by the properties of the algebra.

We therefore introduce the three operators

$$K_1 = \mathbf{J}_{12}^2, \quad K_2 = \mathbf{J}_{23}^2, \quad K_3 = [K_1, K_2] = -4i\mathbf{J}_1 \cdot (\mathbf{J}_2 \times \mathbf{J}_3). \quad (1)$$

An uncomplicated direct verification shows that these operators indeed form the algebra

$$[K_2, K_3] = AK_2^2 + B\{K_1, K_2\} + CK_2 + D_1 + E_1K_1,$$
$$[K_3, K_1] = BK_1^2 + A\{K_1, K_2\} + CK_1 + D_2 + E_2K_2, \quad (2)$$

which is closed under commutation. Here $\{K_1, K_2\}$ denotes the anticommutator, and the structure constants have the following form:

$$A = B = -2, \quad C = 2\sum_{k=1}^{4} \alpha_k,$$
$$D_1 = 2(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_2), \quad D_2 = 2(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4), \quad (3)$$
$$E_1 = E_2 = 0, \quad \alpha_k = j_k(j_k + 1).$$

The algebra (2) is quadratic, i.e., the right-hand sides of the commutation relations contain squares and bilinear combinations of the original operators. It is interesting to note, however, that this circumstance is not an obstacle in the construction of a representation of the algebra.

Indeed, let us choose a basis $W_p$, in which the operator $K_1$ is diagonal:

$$K_1 W_p = p(p+1) W_p. \quad (4)$$

It is then easy to see that in the basis $W_p$ we have a ladder representation, in which the operators $K_2$ and $K_3$ are tridiagonal:

$$K_2 W_p = a_{p+1} W_{p+1} + a_p W_{p-1} + b_p W_p,$$
$$K_3 W_p = 2(p+1) a_{p+1} W_{p+1} - 2p a_p W_{p-1}. \quad (5)$$

To obtain the explicit form of the matrix elements $a_p, b_p$ it is sufficient to make use of the commutation relations (2), with the result

$$b_p = [-2p^2(p+1)^2 + Cp(p+1) + D_2]/4p(p+1), \quad (6)$$
$$a_p^2 = R(p)R(-p)/p^2(4p^2-1), \quad (7)$$

where

$$R(p) = \prod_{k=1}^{4} (p - \xi_k) \quad (7a)$$

is the characteristic polynomial of fourth degree. Its roots $\xi_k$ determine the endpoints in the range of variation of $p$ and can be obtained starting from the angular momentum composition scheme $|j_1 - j_2| \leqslant j_{12} \leqslant j_1 + j_2, |j_{12} - j_3| \leqslant j_4 \leqslant j_{12} + j_3$:

$$\xi_1 = j_1 - j_2, \quad \xi_2 = 1 + j_1 + j_2, \quad \xi_3 = j_3 - j_4, \quad \xi_4 = 1 + j_3 + j_4. \quad (8)$$

A complete description of the representation of the algebra (2), as in the case of a Lie algebra, requires further the specification of the value of the Casimir operator $Q$, commuting with all the generators of our algebra. Direct verification shows that the Casimir operator for the algebra (2) has the form

$$Q = B\{K_1^2, K_2\} + A\{K_2^2, K_1\} + (B^2 + E_1)K_1^2$$
$$+ (A^2 + E_2)K_2^2 + K_3^2 + (AB + C)\{K_1, K_2\}$$
$$+ 2(D_2 + AC)K_2 + 2(D_1 + BC)K_1. \quad (9)$$

For a particular realization of the algebra (2) for which the operators have the form (1), the Casimir operator is not arbitrary and is expressed in terms of the quantum numbers $j_1, j_2, j_3$ and $j_4$ (see the following Section).

In this way specification of the constants $C, D_1, D_2$ and $Q$, which we shall call the parameters of the algebra, completely describes the representation of the algebra (2).

The $6j$-symbol $W_p(q)$ is now uniquely determined from the recursion relation

$$K_2 W_p(q) = a_{p+1} W_{p+1}(q) + a_p W_{p-1}(q) + b_p W_p(q)$$
$$= q(q+1) W_p(q), \quad (10)$$

where $|\xi_1| \leqslant p \leqslant \xi_2 - 1$ and $W_{|\xi_1|-1}(q) \equiv 0$. We note that $W_p(q)$ is determined from the recursion relation to within a normalization factor. In this way the full number of variables in the symbol $W_p(q)$ equals six—the four structure parameters of the algebra (2) and the two quantum numbers $p, q$, which precisely corresponds to the number of independent parameters of the $6j$-symbol.

### 3. ANALYSIS OF SYMMETRIES OF THE ALGEBRA AND THE 6j-SYMBOL

As was shown in the preceding Section, the $6j$-symbol is determined by specifying the parameters $C, D_1, D_2$ and $Q$ of the algebra (2). Therefore, if the algebra (2) is invariant under some transformation of its parameters, then the $6j$-symbol will also be invariant under this transformation. Let us find all such transformations.

From formulas (3) and (9) one readily finds that the parameters of the algebra are symmetric function of the roots $\xi_k$ of the characteristic polynomial:

$$C = S_1 - 2, \quad D_1 = -{}^1/_2 S_1^2 + {}^1/_2 S_2 + S_4^{1/2},$$
$$D_2 = 2S_4^{1/2}, \quad Q = -{}^1/_4 S_1^2 + S_1(1 + S_4^{1/2}) + S_3 - 1, \quad (11)$$

where

$$S_N = \sum_{i_n} \xi_{i_1}^2 \xi_{i_2}^2 \ldots \xi_{i_N}^2, \quad i_1 < i_2 < \ldots < i_N. \quad (12)$$

This means that any transposition of the roots among themselves, as well as changing the sign of an even number of the roots (taking into account the fact that the algebra parameters depend on the root $S_4^{1/2}$) leaves the parameters of the algebra, and therefore the $6j$-symbols, invariant. Moreover, since the number of parameters of the algebra coincides with the number of the roots of the characteristic polynomial, all the symmetries of the algebra are contained in these transformations.

In this fashion we obtain an invariant subgroup of the $6j$-symbol, which leaves unchanged its last column. This subgroup has dimension $4!\cdot 12 = 288$, where 4! is the dimension of the permutation group of the roots and 12 is the dimension of the group of sign changes of two or four roots.

Let us consider some of these transformations in more detail:

a) the transposition $\xi_1 \rightleftarrows \xi_2$ is the "mirror"[2] symmetry $j_2 \to -1 - j_2$;

b) the transposition $\xi_1 \rightleftarrows \xi_3$ is the Regge symmetry

$$j_k \to \sigma - j_{5-k}, \quad \sigma = \frac{1}{2}\sum_1^4 j_k;$$

c) the change of the pair of signs $\xi_{1,3} \to -\xi_{1,3}$ is equivalent to the classical Racah symmetry $j_1 \rightleftarrows j_2 j_3 \rightleftarrows j_4$.

The above described three types of transformations generate the entire invariant subgroup of the 6*j*-symbol.

To obtain the full symmetry group it is necessary to add to the subgroup considered so far the group of dual transformations, which mix the rows of the 6*j*-symbol. In algebraic language such a transformation means passing from the realization given by Eqs. (1), to a different realization corresponding to a different choice of the pair of angular momenta being added. The number of such transformations altogether is $C_4^2 = 6$. For example, the simplest dual transformation $K_1 \rightleftarrows K_2$, $K_3 \to -K_3$, $D_1 \rightleftarrows D_2$ is equivalent to one of the classical Racah symmetries: $j_1 \rightleftarrows j_3, j_{12} \rightleftarrows j_{23}$.

In this manner the full invariance group of the 6*j*-symbol has dimension $4!\cdot 12\cdot 6 = 144\cdot 12$.

We note that usually the discussion of the invariance problem of the 6*j*-symbol is confined to the enumeration of the 144 symmetries—the classical and the Regge symmetries. However by themselves these symmetries do not form a group. As we have seen, it is necessary to include in this group the so-called mirror transformations $j_k \to -1 - j_k$, there being precisely 12 such independent ones.

The fact that all possible symmetries of the 6*j*-symbol form a group was, apparently, not noticed previously. In our opinion the most interesting circumstance is the connection between this symmetry group and the group of permutations of the roots of the characteristic polynomial of fourth degree. This is one more testimony to the fact that the 6*j*-symbols have (rather mysterious) connections to the most varied fields of physics and mathematics.[8]

## 4. SYMMETRIES OF THE 3*j*-SYMBOL

The 3*j*-symbols are simpler objects than the 6*j*-symbols. They arise in the addition of two angular momenta $\mathbf{J}_1, \mathbf{J}_2$ as follows. Let us fix in the space of the angular momenta being added the quantum numbers $j_1, j_2$ and $m = (\mathbf{J}_1 + \mathbf{J}_2)_z$. Then the eigenfunctions of the operators $(\mathbf{J}_1 + \mathbf{J}_2)^2$ and $(\mathbf{J}_1 - \mathbf{J}_2)_z$ form in this space two independent bases. The transformation matrix between these bases is the 3*j*-symbol. In other words, the 3*j*-symbol is the Clebsch-Gordan coefficient $\langle jm | j_1 m_1 j_2 m_2 \rangle$ in the expansion of the total angular momentum in terms of the direct product of the angular momenta being added.

The symmetry of the 3*j*-symbol may be obtained from the symmetries of the 6*j*-symbol by making use of the existence of an asymptotic relation that connects these two objects.[2] We shall indicate another, more constructive, approach, which is once again related to a quadratic algebra.

We introduce three operators in the space with fixed quantum numbers

$$N_1 = (\mathbf{J}_1 + \mathbf{J}_2)^2, \quad N_2 = (\mathbf{J}_1 - \mathbf{J}_2)_z, \quad N_3 = [N_1, N_2] = 4i(\mathbf{J}_1 \times \mathbf{J}_2)_z. \tag{13}$$

One verifies that these operators form a quadratic algebra of the form (2), but with a simpler structure. In contrast to the case of the 6*j*-symbol, the matrix element $a_p$ of the operator $N_2$ is expressed in terms of the characteristic polynomial of third order

$$R(p) = \prod_1^3 (p - \xi_k),$$

where the roots $\xi_1, \xi_2$ have the previous meaning (8), and $\xi_3 = m$. The structure parameters of the algebra are again symmetric functions of the roots:

$$A = -2, \quad B = C = E_2 = 0, \quad E_1 = -4,$$
$$D_1 = 2S_1, \; D_2 = 2S_3^{1/2}, \; Q = 4S_2 \tag{14}$$

[the functions $S_N$ are given, as before, by expression (12)]. Consequently, all possible transformations of parameters that leave the algebra and the 3*j*-symbols invariant, consist of permutation of roots and sign changes of two roots.

Let us consider some of these transformations:

a) the transposition $\xi_1 \rightleftarrows \xi_2$, as in the case of the 6*j*-symbol, is equivalent to the mirror symmetry $j_2 \to -1 - j_2$;

b) the transposition $\xi_1 \rightleftarrows \xi_3$ is the Regge symmetry

$$j_1 \to {}^1/_2(j_1 + j_2 + m), \quad j_2 \to {}^1/_2(j_1 + j_2 - m), \quad m \to j_1 - j_2;$$

c) the change of two signs $\xi_{1,3} \to -\xi_{1,3}$ is equivalent to the exchange of the momenta $j_1 \rightleftarrows j_2$, $m \to -m$.

In this manner the stationary (i.e., not changing the values $j$ and $m_1 - m_2$) subgroup of symmetries of the 3*j*-symbol consists of $3!\cdot 3 = 18$ transformations, which contains classical, Regge and mirror symmetries.

As in the case of the 6*j*-symbol, to obtain the full group of symmetries one must include dual transformations. Indeed, the 3*j*-symbols are obtained by the addition of three angular momenta to a form of zero sum. It is clear that there exist six ways to choose a pair of angular momenta to form the operators $N_1$ and $N_2$. For example, the interchange of the original operators $\mathbf{J}_1$ and $\mathbf{J}_2$, $N_1 = (\mathbf{J}_1 + \mathbf{J}_2)^2$, $N_2 = (\mathbf{J}_2 - \mathbf{J}_1)_z$ is equivalent to the symmetry $j_1 \rightleftarrows j_2$, $m_1 \rightleftarrows m_2$. Thus the full symmetry group of the 3*j*-symbol contains $18\cdot 6 = 108$ transformations.

Thus the symmetries of the 6*j*- and 3*j*-symbols are related to representations of quadratic algebras of the same type. The difference lies in the specific values of the parameters of the algebra. This explains the "Regge mystery." He observed: "There is no simple relation between the symmetries of the 6*j*- and 3*j*-symbols".[3]

## 5. CONCLUSION

We have established that the symmetries of the 6*j*-symbol (3*j*-symbol) form finite groups, isomorphic to the invariance group of the characteristic polynomial of respectively fourth and third degree. This polynomial arises as the matrix element (7) in the representation of a quadratic algebra, with the explicit form of this matrix element being determined by specifying the parameters of the algebra (i.e., the structure constants and the Casimir operator). It is important to note that invariance of the algebra parameters under the transformations of the roots includes not only the classical and the Regge symmetries, but also the mirror symmetries, without which the transformations of the 6*j*- and 3*j*-symbols do not form a group.

We note that the effectiveness in using the quadratic algebra is not limited to the case of the rotation group, discussed in the present paper. It is not hard to show that the same quadratic algebra is formed by the operators of the 0(2,1) algebra (from the discrete series representations), which permits one to immediately conclude that the 6*j*-symbols of the 0(2,1) group coincide functionally with the 6*j*-symbols of the rotation group (previously this fact had no simple explanation). Moreover, an algebra of the same structure is formed by the squares of the *T*- and *U*-spin in the *SU*(3) algebra, which helps to understand the reason for the appearance of the same 6*j*-symbols in the theory of representations of that group.[1]

Finally we note that the quadratic algebra of the type (2) has greater universality—it may serve as basis for the construction of the Wilson-Racah orthogonal polynomials introduced in Ref. 9. Indeed, it can be shown that the algebra (2) with arbitrary structure parameters produces generalized Wilson-Racah polynomials (of continuous as well as discrete argument). This answers the question left open in Ref. 6 on the inclusion of these polynomials into the algebraic scheme with three generators.


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