Virtual statistical equilibrium and quasistationary homogeneous turbulence spectra

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We show that the directivity (non-equilibrium) of a cascade process (splitting or merging of vortices) leads to a very strong nonlinearity in the description of homogeneous turbulence. The "thermodynamics" of the degrees of freedom of such turbulence turns out to be nonlinear for arbitrarily low pulsation energies. We propose a method of virtually balanced cascade processes which makes it possible to apply in this situation the maximum entropy principle to a virtual system in which there is, apart from the real cascade, a virtual cascade in the opposite direction. We find, for the inertial range of scales, spectral densities of the pulsation energy of the form k^γ, $k^\gamma \ln(kL)$, $k^\gamma \ln^2(kL)$,..., where L is the correlation (integral) turbulence scale. We establish agreement of the first two dependences with the experimental data of various authors.

A physical system isolated from external actions tends with time to a state of equilibrium. This state is characterized by maximum entropy. Usually the transition of the system to equilibrium takes place in two stages: the formation of quasistationary nonequilibrium states and the evolution of the quasistationary states to complete statistical equilibrium. We shall in the present paper be interested in quasistationary nonequilibrium states in homogeneous turbulence. However, although one can use for stationary equilibrium states the maximum entropy condition, there is no such universal extremum principle for a quasistationary nonequilibrium situation.[1] When one is dealing with advanced turbulence one usually stipulates that this is a system with a very large number N of degrees of freedom. There is an estimate of how N depends on the Reynolds number (Ref. 2):

$$N \sim \mathrm{Re}^{9/4}.$$

In fact, one is dealing here with the number of vortices.[3] Formally one can take as degrees of freedom, for instance, the modes of the Fourier expansion for the velocity field—discretization of the model in this case is usually accomplished by considering a large but finite region of motion. The dynamical equations in this model are the Navier-Stokes set. The applicability of the Liouville theorem leads to a simplified nonviscous set of equations for the Fourier expansion modes, and for a special choice of variables one can also change to a Hamiltonian system.[4]

1. In the case of uniform isotropic turbulence without external energy supply, a monotonic damping with time (degeneration) of the average kinetic pulsation energy, proportional to $\langle u^2 \rangle$ (u is the characteristic velocity of the pulsations) will occur. Kolmogorov and Obukhov (see Ref. 5) advanced the idea of existence in advanced turbulence, of an extended range of wave numbers characterized by a quasistationary behavior. The basis of this idea was the assumption that for that range of scales the characteristic times for the processes were small compared to the characteristic time for the total degeneracy. Such a ratio of the characteristic times makes it possible for the degrees of freedom from that range to be easily adapted to a slow change in the integral turbulent regime. The quasistationary state of these degrees of freedom depends on a few integral parameters of motion and only through them on the time. The use of these ideas led, in particular, to finding the Kolmogorov-Obukhov law for the spectral density of the energy E:

$$E = c\varepsilon^{2/3} k^{-5/3} \tag{1}$$

(k is the absolute magnitude of the wave number, $\varepsilon = \frac{1}{2} d\langle u^2 \rangle / dt$, and c is a constant). In the above-mentioned range a cascade process of subdividing the vortices takes place; this consists of the following.[5] Large-scale (energy-containing) vortices split up and in that way transfer their energy to smaller vortices which, in turn, also split up, and so on, until the size of the vortices produced becomes so small that they vanish rapidly under the action of the viscosity (and in them the energy dissipation is realized). The reality of the splitting-up process, and also of the inverse process of the merging of vortices, has been confirmed by direct observations.[5]

2. If we introduce the entropy S of the system of degrees of freedom in the quasi-equilibrium range it will, in the case of uniform isotropic turbulence, be a functional of the energy E and will depend on the time only through $E(k,t)$.

The first question in which we shall be interested is the question of whether S is a regular functional of $E(k)$ in the vicinity of $E = 0$.[6] This question is nontrivial because of the essential nonlinearity of the hydrodynamic equations. However, a negative answer to this question means that for small E one cannot use an expansion of $S(E)$ in regular functional series, and thus the "thermodynamics" of the turbulent degrees of freedom in the quasi-equilibrium range will be nonlinear even for small E.

We consider the functional derivative $\delta S / \delta E(k)$ as a functional of $E(k)$ in the vicinity of $E = 0$ (i.e., for small E). We assume that this functional can be expanded in a regular functional power series in E,[6] and we restrict ourselves for sufficiently small R to the first term in that expansion:

$$\frac{\delta S}{\delta E(k)} = \left.\frac{\delta S}{\delta E(k)}\right|_{E=0} + \int_0^\infty G(k', k) E(k') \, dk'. \tag{2}$$

The function $G(k',k)$ describes the action of the degrees of freedom characterized by the wave number k' on the degrees of freedom of wave number k. If the main contribution to the

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entropy comes from the interaction realized through the splitting up (cascade), $G(k',k)$ can be roughly approximated by a singular generalized function with its peak at $k' = \alpha k$, where α is the multiplicity of the scale splitting in the cascade (if the scale is divided into N parts, $\alpha = N^{-1}$). It follows from the well known theorem about such functions[7] that $G(k',k)$ can be written uniquely in the form

$$G(k', k)=\sum_{p\leqslant N}(-1)^p g_p(k)\frac{d^p\delta(k'-\alpha k)}{d(k')^p}, \tag{3}$$

where the $g_p(k)$ are functions of k and $\delta(x)$ is the Dirac delta function. The number N is called the order of the function G (Ref. 7, p. 22) and is determined by the differential properties of the functions $E(k)$ in the vicinity of $k=0$ (Ref. 5, p. 151 of original). One notes easily that the cascade representation (3) of $G(k',k)$ is not symmetric under an exchange of k' and k (this is clearly connected with the preferred cascade direction—splitting up). However, it is clear from (2) that

$$G(k', k)=\delta^2 S/\delta E(k')\delta E(k), \tag{4}$$

and hence, the function $G(k',k)$ must be symmetric under an exchange of k' and k. The idea of a directed cascade (splitting up) is thus incompatible with a regular nature of the functional $S(E)$ and, hence, in the case of such a cascade the functional S is nonregular. This, in turn, leads to the nonlinearity of the thermodynamics in the quasistationary range even for small E.

3. We consider a virtual system in which we combine a reverse cascade with the real cascade, i.e., together with the splitting up of vortices a merging which statistically is the inverse process of the splitting up occurs (one should note that the inverse cascade is in fact observed in turbulent flows[3]). In that case, we can write instead of (2) the entropy S_v in the form

$$\frac{\delta S_v}{\delta E(k)}=\frac{\delta S_v}{\delta E(k)}\bigg|_{E=0}+\int_0 G_v(k', k)E(k')\,dk', \tag{5}$$

where

$$G_v(k', k)=G(k', k)+G^*(k', k), \tag{6}$$

$G(k',k)$ is given by Eq. (3), and

$$G^*(k', k)=\sum_{p\leqslant N}(-1)^p g(k')\frac{d^p\delta(k-\alpha k')}{dk^p}, \tag{7}$$

α is the multiplicity of the real splitting up (see above), and $1/\alpha$ the multiplicity of the virtual merging of the vortices, which is statistically the inverse of the real splitting up. $G_v(k',k)$ which is given by Eq. (6) is now symmetric under an exchange of k' and k and the equation

$$G_v(k', k)=\frac{\delta^2 S_v}{\delta E(k')\delta E(k)}\bigg|_{E=0} \tag{8}$$

does not contradict the representation (6). For the virtual system we can therefore use an expansion of $S_v(E)$ in a regular series in integral powers of E and the thermodynamics of the virtual system is linear for small E.

By virtue of the additivity of the energy the spectral energy density for the regular system differs from the spectral energy density for the virtual system merely by a factor 1/2. We shall therefore use below the same symbol E for the spectral energy density of the virtual system as for the real system, since this is unimportant for what follows.

4. The virtual system is not only thermodynamically linear (for small E) but also in equilibrium. Indeed, the presence in it of the virtual cascade which is statistically the inverse of the real cascade enables us to consider it to be closed and to formulate for it a maximum entropy extremum principle (virtual statistical equilibrium):

$$\delta S_v/\delta E(k)=0. \tag{9}$$

First of all, it is clear that the state with $E(k)\equiv 0$ is the state of virtual statistical equilibrium, i.e.,

$$\delta S_v/\delta E(k)\,|_{E=0}=0. \tag{10}$$

It follows from (10) and (5) that

$$\frac{\delta S_v}{\delta E(k)}=\int_0^\infty G_v(k', k)E(k')\,dk'. \tag{11}$$

Substituting the representation (6) into (11) and using (3) and (7) we get the condition for virtual statistical equilibrium in the form of a differential equation

$$\sum_{p\leqslant N}\left\{g_p(k)\alpha^{-p}\frac{d^pE(\alpha k)}{dk^p}+(-1)^p\frac{d^p}{dk^p}\left[g_p\left(\frac{k}{\alpha}\right)E\left(\frac{k}{\alpha}\right)\right]\right\}=0. \tag{12}$$

Considering in the quasiequilibrium range a scale-invariant subrange,[8] we impose on $g_p(k)$ the requirement of uniformity of order p in the variable k [in order that Eq. (12) in that range be scale invariant]. We must note that Eq. (12) can be scale invariant while its solutions will not possess scale invariance. This phenomenon is connected with spontaneous symmetry breaking and is due to the degeneracy of the equation. It enables us to use this equation in a wavenumber range which is much broader than the scale-invariant one (see below).

It follows then from Euler's theorem about homogeneous functions that $g_p(k) = a_p k^p$, where the a_p are constants. We get then from (12)

$$\sum_{p\leqslant N}\left[(a_p\alpha^{-p})k^p\frac{d^pE(\alpha k)}{dk^p}+(-1)^p a_p\alpha^{-p}\frac{d^pk^pE(k/\alpha)}{dk^p}\right]=0. \tag{13}$$

Equation (13) is a linear homogeneous differential equation with a divergent argument of the Euler equation type (Ref. 9, p. 96). This equation has a solution of the form (Ref. 9, p. 96)

$$E(k)\propto k^\gamma \ln^\beta(kL), \qquad \beta=0, 1, \ldots, (q-1), \tag{14}$$

where q is the multiplicity of the root γ of the characteristic equation and L a constant with the dimensions of length. The case $\beta = 0$, i.e., power law solutions $E\propto k^\gamma$, is known for the scale invariant range [see Eq. (1) and Ref. 8]. But there are solutions (14) with $\beta = 1,...$ which are unusual for the theory of uniform turbulence. It is interesting to compare solution (14) with $\beta \neq 0$ with the modified Kolmogorov-Obukhov theory.[5,8] In the present approach the spontaneous symmetry breaking (scale invariance) leads to violation of the power law nature of the spectrum, but in the modified Kolmogorov-Obukhov theory the spectrum continues to obey a power law.[5,8]

We turn to experiments. It has been possible in experiments to observe power-law solutions of the form (1) (Kolmogorov-Obukhov law) over a rather broad range of wave numbers in shear flow for very large values of the Reynolds number. However, in experiments behind hydrodynamic grids (which model uniform isotropic turbulence) so far one has not succeeded in doing so. A verification of the fact that power laws are satisfied experimentally is accomplished by choosing a doubly logarithmic scale. In these graphs the data fit a straight line if they obey a power law. To verify the dependence (14) with $\beta = 1$ we must choose semilogarithmic scales: $k^{-\gamma}E$ and $\ln k$. In these scales the experimental data lie on a straight line, if they satisfy the relation

$$E(k) \propto k^{\gamma} \ln(kL). \tag{15}$$

For γ we choose the value $\gamma = -5/3$ (corresponding to the Kolmogorov-Obukhov hypothesis[5]). We show in the figure the data described in Refs. 10 and 11. The straight lines correspond to sections where the relation (15) is satisfied.

5. Vortices with scales larger than the correlation (integral[5]) scale have low probability in uniform turbulence so that their contribution to the integral characteristics of uniform isotropic turbulence is unimportant. In the model considered this is reflected by the fact that $E(k) \to 0$ as $k \to 1/L$, if we use solution (15). Therefore, in the given model it will be natural to identify L with the correlation (integral) scale. That this identification is adequate can be checked by comparison with experiments. Indeed, the correlation (integral) scale of uniform isotropic turbulence in an incompressible fluid is connected with the spectral energy density through the equation (Ref. 5, p. 54 of original)

$$L = \frac{3\pi}{2\langle u^2\rangle} \int_0^\infty k^{-1}E(k)\,dk \approx 3\pi \int_{L^{-1}}^\infty k^{-1}E(k)\,dk/2\langle u^2\rangle \tag{16}$$

(longitudinal scale).

We substitute into (16) the representation (15) reconciled with the Kolmogorov-Obukhov approach, i.e.,

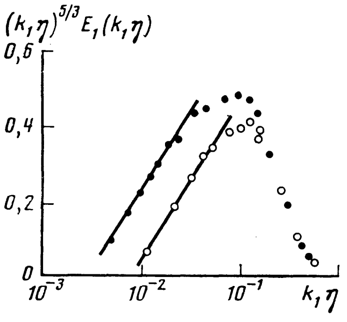

FIG. 1. Verification of the relation $E \propto k^{-5/3}\ln(kL)$ by experiments behind grids: ●—Ref. 10, Re $= 1.05\times 10^4$; ○—Ref. 11, Re $= 3.4\times 10^4$; (η is the Kolmogorov length scale, $E_1(k_1)$ the one-dimensional density; one proves easily that if $E(k)$ is described by a relation of the form (15), $E_1(k_1)$ is also described by a relation of that form).

$$E(k) = c\varepsilon^{2/3}k^{-5/3}\ln(kL), \tag{17}$$

where $c \approx 1.5$ (Refs. 5 and 12) and $\varepsilon = \frac{1}{2}d\langle u^2\rangle/dt$. We get

$$-\frac{L d\langle u^2\rangle/dt}{\langle u^2\rangle^{3/2}} = A_0, \tag{18}$$

where

$$A_0 = 2\left(\frac{3\pi c}{2}\int_1^\infty x^{-8/3}\ln x\,dx\right)^{-3/2}.$$

In experiments one usually measures the longitudinal component of the velocity pulsations $\langle u_1^2\rangle$ which is connected with $\langle u^2\rangle$ through the relation $3\langle u_1^2\rangle = \langle u^2\rangle$. For it we can rewrite Eq. (18) in the form

$$-\frac{L d\langle u_1^2\rangle/dt}{\langle u_1^2\rangle^{3/2}} = A, \tag{19}$$

where $A = \sqrt{3}A_0$. Substituting for c its value 1.5 we find that $A = 0.85$. In a recent experiment described in Ref. 13 $\langle u_1^2\rangle$ and L were measured in flow behind a grid which models uniform isotropic turbulence. In this experiment the quantity $L(d\langle u_1^2\rangle/dt)\langle u_1^2\rangle^{-3/2}$ was indeed practically constant and the value of this constant was in that experiment $A \approx 0.78$.

We must note that Eq. (19) has been known (as a semi-empirical relation) for a very long time[14]—although the value of the constant A had not been evaluated theoretically. Earlier experiments gave for the values of the constant A a spread from 0.8 to 1.4, with A tending to 0.8 as the Reynolds number increased.[14] However, the experiments indicate also a dependence of the exponent in the energy degeneracy law on the conditions at the grid ("initial conditions") and changes in the law for the increase in L are not always correlated with changes in the energy damping law in such a way that Eq. (19) is satisfied. This problem has not yet been cleared up.

6. The problem of whether some solution or other from the set (14) is realized can apparently be solved theoretically by a study of the stability. However, to do this one must know the form of $\delta^2 S_v/\delta E(k')\delta E(k)$ not only for $E = 0$, but also for $E(k)$ given by (14), since the stability condition is given by the relation[15]

$$\delta^2 S_v = \int \frac{\delta^2 S_v}{\delta E(k')\delta E(k)}\delta E(k')\delta E(k)\,dk'\,dk < 0$$

in the situation of virtual equilibrium which we studied.

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