Plasma condensation formation in a fluctuating strong magnetic field

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We suggest a method for solving the equations of ideal magnetohydrodynamics in the strong field approximation using the frozen-in coordinate technique. We study plasma flows near a varying magnetic dipole. We obtain exact analytical solutions for the case of a linearly varying magnetic moment. We discover a new mechanism for magnetic bunching caused by small fluctuations in the magnetic field.

1. INTRODUCTION

The strong-magnetic-field approximation is applicable when the magnetic force dominates the other ones: the gravitational force, the gradient of the gas pressure, and so on. It is of great interest, in particular for astrophysical applications of plasma physics.[1] In the framework of this approximation Syrovatskiĭ developed the concepts of a neutral current layer[2–4] and of magnetic bunching.[5,6]

Notwithstanding the considerable simplification of the equations of magnetohydrodynamics (MHD) in the strong field approximation, their solution remains all the same a serious problem. Existing methods[1] allow one only to study two-dimensional problems. Analytical solutions are then only obtained by the small-perturbation method[1,2] which is applicable at initial times for small changes in the magnetic field and under neglect of inertial effects. The authors are not aware of any analytical solution of a global nonlinear problem with physically realizable boundary and initial conditions. As to three-dimensional problems, their solution is difficult even numerically.

We propose in the present paper a new method for solving MHD problems in the strong field approximation using the frozen-in coordinate method.[7,8] The proposed method is given in the second section. In Section 3 it is used to study the nonlinear problem of plasma flow near a variable magnetic dipole. We consider two-dimensional (planar) and three-dimensional (axisymmetric) statements of the problem. For the case of a linearly varying magnetic moment we obtain analytical solutions which are valid for arbitrary times.

We study in Section 4 the plasma flow near a dipole with a dipole moment which fluctuates around a constant value. We show that on the background of fast vibrational motions there appears a systematic precession-type flow which leads to bunching of the plasma towards the dipole equator. We briefly discuss possible astrophysical applications of this effect in the Conclusion.

2. METHOD FOR SOLVING THE MHD EQUATIONS IN THE STRONG FIELD APPROXIMATION

The dimensionless MHD equations for an ideal medium have the following form[1]:

$$\varepsilon^2 \frac{d\mathbf{v}}{dt} = -\gamma_0{}^2 \frac{\nabla p}{\rho} - \frac{1}{\rho}[\mathbf{H}\,\mathrm{rot}\,\mathbf{H}], \tag{1}$$

$$\mathrm{div}\,\mathbf{H}=0, \tag{2}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}[\mathbf{vH}], \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\,\mathrm{div}\,\rho\mathbf{v}, \tag{4}$$

$$\frac{\partial}{dt}(p\rho^{-\beta}) = 0. \tag{5}$$

Here $\mathbf{v}$ and ρ are the plasma velocity and density; p and $\mathbf{H}$ are the pressure and magnetic field strength; $\varepsilon = V/V_A$, $\gamma_p^2 = p_0/\rho_0 V_A^2$ are dimensionless parameters characterizing the problem; $V_A = H_0(4\pi\rho_0)^{-1/2}$ is a characteristic value of the Alfvén velocity; V, ρ_0, p_0, H_0 are characteristic values of the velocity, the density, the pressure, and the magnetic field, respectively; β is the polytropic index.

The strong field approximation corresponds to the conditions[1]

$$\gamma_0{}^2 \ll \varepsilon^2 \ll 1. \tag{6}$$

In zeroth order in ε^2 and γ_0^2 the set of MHD equations takes the form

$$[\mathbf{H}\,\mathrm{rot}\,\mathbf{H}]=0, \tag{7}$$

$$\mathrm{div}\,\mathbf{H}=0, \tag{8}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}[\mathbf{vH}], \tag{9}$$

$$\frac{\partial \rho}{\partial t} = -\,\mathrm{div}\,\rho\mathbf{v}, \tag{10}$$

$$\mathbf{H}\frac{d\mathbf{v}}{dt} = 0. \tag{11}$$

The derivation and the conditions of applicability of these equations were discussed in detail in Ref. 1. Their solution has been split into two parts: a) the determination of the magnetic field from Eqs. (7) and (8) for given boundary conditions which vary in time; b) calculation of the plasma velocity and density from Eqs. (9)–(11) for a known magnetic field. In the present paper we restrict ourselves to the second part of the problem and assume that the magnetic field is known.

We note that it is far from being possible to find continuous solutions of the set (7)–(11) for arbitrary continuous deformations of the magnetic field.[4] For instance, in the two-dimensional case such solutions exist only provided there are no singular zeroes of the magnetic field in which the electric field $E \neq 0$. When such points exist it is necessary to introduce cuts (current layers).[4]

In order that a solution of Eqs. (9)–(11) for a given evolution of the magnetic field be completely determined

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inside some region, one must specify the initial conditions[1]

$$v_{\parallel}(0, \mathbf{r}) = f_1(\mathbf{r}), \tag{12}$$

$$\rho(0, \mathbf{r}) = f_2(\mathbf{r}), \tag{13}$$

where $v_{\parallel}$ is the velocity component along the field lines [the velocity component at right angles to the field lines is determined from (9)]. For the sake of simplicity we assume in what follows that the initial density distribution is uniform ($f_2 = \rho_0 = \text{const}$).

The particular feature of Eqs. (7)–(11) consists in that in the solution there can appear regions of mutual interpretation of plasma fluxes.[1] The appearance of such regions is connected with the fact that in Eq. (1) the gas pressure gradient was neglected. As a result fluid particles moving along the field line tubes do not "feel" one another and their trajectories can intersect. In that case the results obtained in the approximation applied cease to be correct.

The frozen-in coordinate method is useful for solving problems of ideal MHD.[7,8] One introduces a doubly Lagrangian system: with respect to a parameter along the line of the flow flux (the time t) and with respect to a parameter α along the magnetic field line. As in any Lagrangian approach (see, e.g., Ref. 9) the function required is the radius vector $\mathbf{r} = \mathbf{r}(t,\alpha,\xi,\psi)$ of the fluid particle, where t,α,ξ,ψ are the frozen-in coordinates. The set of MHD Eqs. (1)–(5) can be written as follows in the frozen-in coordinates[8]:

$$\varepsilon^2\mathbf{r}_{tt} - (\rho\mathbf{r}_\alpha)_\alpha = -\left([\mathbf{r}_\xi\mathbf{r}_\psi]\frac{\partial}{\partial\alpha} + [\mathbf{r}_\psi\mathbf{r}_\alpha]\frac{\partial}{\partial\xi} + [\mathbf{r}_\alpha\mathbf{r}_\xi]\frac{\partial}{\partial\psi}\right)\left(\frac{1}{2}\rho^2\mathbf{r}_\alpha^2 + \gamma_0^2 p\right), \tag{14}$$

$$\rho[\mathbf{r}_\alpha\mathbf{r}_\xi]\mathbf{r}_\psi = 1, \tag{15}$$

$$p/\rho^\beta = \text{const}, \tag{16}$$

where $\mathbf{r}_t$, $\mathbf{r}_\alpha$, $\mathbf{r}_\xi$, $\mathbf{r}_\psi$ indicate partial derivatives with respect to the corresponding parameters. Equation (14) is Eq. (1) written in the frozen-in coordinates. Condition (15) guarantees that (2) and (4) are satisfied.[7] The frozen-in condition (3) is satisfied by virtue of the special choice of the Lagrangian coordinates. It is shown in Ref. 7 that Eq. (3) is the necessary condition that we can introduce a twice Lagrangian system of coordinates. This is just the reason why these coordinates are called frozen-in coordinates

If we have solved the set (14)–(16), i.e., have found the functions $\mathbf{r}$, ρ, and p, we can evaluate the velocity and the magnetic field:

$$\mathbf{v} = \mathbf{r}_t, \tag{17}$$

$$\mathbf{H} = \rho\mathbf{r}_\alpha. \tag{18}$$

In order to write down the strong-field approximation MHD equations (7)–(11) in frozen-in coordinates we must put $\varepsilon^2 = 0$, $\gamma_0^2 = 0$ in (14) and add Eq. (11) written in frozen-in coordinates. As a result we get

$$(\rho\mathbf{r}_\alpha)_\alpha = \left([\mathbf{r}_\xi\mathbf{r}_\psi]\frac{\partial}{\partial\alpha} + [\mathbf{r}_\psi\mathbf{r}_\alpha]\frac{\partial}{\partial\xi} + [\mathbf{r}_\alpha\mathbf{r}_\xi]\frac{\partial}{\partial\psi}\right)\left(\frac{1}{2}\rho^2\mathbf{r}_\alpha^2\right), \tag{19}$$

$$\rho[\mathbf{r}_\alpha\mathbf{r}_\xi]\mathbf{r}_\psi = 1, \tag{20}$$

$$\mathbf{r}_\alpha\mathbf{r}_{tt} = 0. \tag{21}$$

We shall work out in what follows a method for solving this set of equations.

We have already noted that we are interested in plasma flow arising when there is a known continuous deformation of the magnetic field. We therefore assume that the solution $\mathbf{H}(t,\mathbf{r})$ of Eqs. (7) and (8) with the appropriate boundary conditions is known at any time with, perhaps, the introduction of necessary cuts. We write the magnetic field at any time in parametric form

$$\mathbf{H}(t, \mathbf{R}) = \rho_0\mathbf{R}_\alpha(t, \alpha, \xi, \psi) \tag{22}$$

with the transition Jacobian

$$\rho_0[\mathbf{R}_\alpha\mathbf{R}_\xi]\mathbf{R}_\psi = 1, \tag{23}$$

where ρ_0 is a constant. The function $\mathbf{R}(t,\alpha,\xi,\psi)$ thus introduced satisfies the equation

$$\rho_0\mathbf{R}_{\alpha\alpha} = \left([\mathbf{R}_\xi\mathbf{R}_\psi]\frac{\partial}{\partial\alpha} + [\mathbf{R}_\psi\mathbf{R}_\alpha]\frac{\partial}{\partial\xi} + [\mathbf{R}_\alpha\mathbf{R}_\xi]\frac{\partial}{\partial\psi}\right)\left(\frac{1}{2}\rho_0^2\mathbf{R}_\alpha^2\right). \tag{24}$$

We shall look for a solution of the set (19)–(21) in the form

$$\mathbf{r}(t, \alpha, \xi, \psi) = \mathbf{R}(t, \sigma(t, \alpha, \xi, \psi), \xi, \psi), \tag{25}$$

where $\sigma(t,\alpha,\xi,\psi)$ is an unknown function. One can easily verify by a direct check that if we relate $\sigma(t,\alpha,\xi,\psi)$ to the density $\rho(t,\alpha,\xi,\psi)$ through the formula

$$\rho = \rho_0/\sigma_\alpha, \tag{26}$$

Eqs. (19) and (20) will be satisfied identically [if we use Eqs. (23) and (24)]. As $\rho\mathbf{r}_\alpha = \rho_0\mathbf{R}_\sigma$ the transformation (25), (26) with an arbitrary function σ leaves the magnetic field configuration unchanged.

To complete the construction of the solution of the set (19)–(21)) we must determine the unknown function $\sigma(t,\alpha,\xi,\psi)$ from Eq. (21). After substituting (25) into (21) the problem reduces to solving one ordinary second order differential equation:

$$\mathbf{R}_\sigma^2\sigma_{tt} + 2(\mathbf{R}_\sigma\mathbf{R}_{\sigma t})\sigma_t + (\mathbf{R}_\sigma\mathbf{R}_{\sigma\sigma})\sigma_t^2 + (\mathbf{R}_\sigma\mathbf{R}_{tt}) = 0 \tag{27}$$

with the initial conditions

$$\sigma(0, \alpha, \xi, \psi) = \alpha, \tag{28}$$

$$\mathbf{r}_t\mathbf{r}_\alpha|_{t=0} = f(\alpha, \xi, \psi). \tag{29}$$

If we take (26) into account, condition (28) guarantees that (13) is satisfied (for $f_2 = \rho_0 = \text{const}$) and (29) determines the velocity component along the field lines at $t = 0$.

We thus obtain the following algorithm for solving the set (9)–(11) under the conditions that $\mathbf{H}(t,\mathbf{r})$ is known.

1. Representation of the field $\mathbf{H}(t,\mathbf{r})$ at each time in the parametric form (22), (23).

2. Determination of the function $\sigma(t,\alpha,\xi,\psi)$ from Eq. (27) with the initial conditions (28), (29).

3. Obtaining the solution (25) and evaluating the density and the velocity using, respectively, Eqs. (26) and (17).

We apply the method developed here to solving an actual nonlinear problem.

3. PLASMA FLOW IN A VARIABLE DIPOLE FIELD

The problem of the motion of a plasma in a strong magnetic dipole field has been formulated and solved in the approximation of small changes in the dipole moment and hence small displacements of the plasma in a two-dimensional planar statement of the problem in Ref. 1 and in a three-dimensional axisymmetric statement in Ref. 5. As there are no qualitative differences between them we shall in what follows consider the planar case. In the Appendix we give the results referring to the three-dimensional statement. In the two-dimensional case the set (19)–(21) takes the form

$$(\rho \mathbf{r}_\alpha)_\alpha = \left([\mathbf{r}_\xi \mathbf{e}_z]\frac{\partial}{\partial \alpha} + [\mathbf{e}_z \mathbf{r}_\alpha]\frac{\partial}{\partial \xi}\right)\left(\frac{1}{2}\rho^2 \mathbf{r}_\alpha^2\right), \tag{30}$$

$$\rho[\mathbf{r}_\alpha \mathbf{r}_\xi]\mathbf{e}_z = 1, \tag{31}$$

$$\mathbf{r}_\alpha \mathbf{r}_{tt} = 0, \tag{32}$$

where

$$\mathbf{r}(t, \alpha, \xi) = \{x(t, \alpha, \xi), y(t, \alpha, \xi), 0\}, \quad \mathbf{e}_z = \{0, 0, 1\}.$$

Writing the given two-dimension field $\mathbf{H}(t,\mathbf{r})$ in the parametric form (22) with the transition Jacobian

$$\rho_0[\mathbf{R}_\alpha \mathbf{R}_\xi]\mathbf{e}_z = 1 \tag{33}$$

is made easier thanks to the following fact. Condition (33) will be satisfied automatically, if we choose

$$\xi = A(t, x, y), \tag{34}$$

where $A(t,x,y)$ is the z component of the vector potential (the only one which is nonvanishing). Indeed, the equations $H_x = \partial A/\partial y$, $H_y = -\partial A/\partial x$ and (22) enable us to evaluate the Jacobian

$$\frac{\partial(\alpha, \xi)}{\partial(X, Y)} = \frac{\partial \alpha}{\partial X}\frac{\partial A}{\partial Y} - \frac{\partial \alpha}{\partial Y}\frac{\partial A}{\partial X} = (\mathbf{H}\nabla_\mathbf{R})\alpha = \rho_0,$$

where X, Y are components of the vector $\mathbf{R}$. Therefore

$$\rho_0[\mathbf{R}_\alpha \mathbf{R}_\xi]\mathbf{e}_z = \rho_0 \frac{\partial(X, Y)}{\partial(\alpha, \xi)} = 1.$$

We consider the problem of plasma flow in the field of a dipole which varies with time and which is positioned at the origin with its axis along the x axis. The magnetic field is

$$\mathbf{H} = \left\{\frac{m(t)(x^2 - y^2)}{(x^2+y^2)^2}, \frac{2m(t)xy}{(x^2+y^2)^2}, 0\right\}. \tag{35}$$

We introduce in the usual way polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. Since the vector potential is $A = r^{-1}m(t)\sin\varphi$, we have

$$\mathbf{R}(t, \alpha, \xi) = \left\{\frac{m(t)}{2\xi}\sin(2\varphi(t, \alpha, \xi)), \frac{m(t)}{\xi}\sin^2(\varphi(t, \alpha, \xi)), 0\right\}. \tag{36}$$

Substituting (36) into (22) we get the equation

$$\varphi_\alpha \sin^2\varphi = \xi^3/\rho_0 m^2(t). \tag{37}$$

whence after elementary integration we find

$$\varphi - {}^1/_2 \sin 2\varphi = 2\xi^3\alpha/\rho_0 m^2(t). \tag{38}$$

The last equation determines $\varphi(t,\alpha,\xi)$ implicitly. Equations (36) and (38) solve the problem of writing down the two-dimensional dipole field in parametric form.

Following the algorithm expounded in the preceding section we look for a solution of the set (30), (32) in the form

$$\mathbf{r}(t, \alpha, \xi) = \mathbf{R}(t, \sigma(t, \alpha, \xi), \xi), \tag{39}$$

where $\sigma(t,\alpha,\xi)$ is an unknown function. In the given actual case it is convenient to change to another unknown function

$$\eta(t, \alpha, \xi) = \varphi(t, \sigma(t, \alpha, \xi), \xi), \tag{40}$$

where $\varphi(t,\alpha,\xi)$ is determined from (38). The function η has a simple physical meaning—it is the polar angle of a Lagrangian particle with coordinates (t,α,ξ). It follows from (38) and (40) that

$$\eta(t, \alpha, \xi) - {}^1/_2 \sin 2\eta(t, \alpha, \xi) = 2\xi^3\sigma(t, \alpha, \xi)/\rho_0 m^2(t). \tag{41}$$

Using this relation we get from (26) an expression for the density:

$$\rho = \rho_0/\sigma_\alpha = \xi^3/m^2(t)\eta_\alpha \sin^2\eta. \tag{42}$$

In the given case Eq. (27) takes the form

$$m\eta_{tt} + 2m_t\eta_t + {}^1/_2 m_{tt}\sin 2\eta = 0. \tag{43}$$

The initial conditions follow from (28) and (29)

$$\eta(0, \alpha, \xi) = \eta_0(\alpha, \xi), \tag{44}$$

$$m\eta_t + {}^1/_2 m_t \sin 2\eta_0 = f(\alpha, \xi), \tag{45}$$

where $\eta_0(\alpha,\xi)$ is implicitly given by the equation

$$\eta_0 - {}^1/_2 \sin 2\eta_0 = 2\xi^3\alpha/\rho_0 m^2(0), \tag{46}$$

while (45) determines the velocity component along the field lines at $t = 0$. For the sake of simplicity we shall put $f \equiv 0$; this means that initially the plasma velocity has no component along the field lines.

It follows from the results of the preceding section that if we have solved Eq. (43) the formulae

$$x(t, \alpha, \xi) = \frac{m(t)}{2\xi}\sin 2\eta, \quad y(t, \alpha, \xi) = \frac{m(t)}{\xi}\sin^2\eta \tag{47}$$

determine the solution of the set (30)–(32). The density and the velocity can be found from (42) and (17), respectively. It is convenient for the calculations to use (46) and rewrite (42) in the form

$$\rho = \rho_0\left(\frac{m(0)\sin\eta_0}{m(t)\sin\eta}\right)^2 \frac{(\eta_0)_\alpha}{\eta_\alpha}. \tag{48}$$

We consider the motion of the plasma near the dipole axis ($\eta \ll 1$). In that case Eq. (43) simplifies: $(m\eta)_{tt} = 0$. Its solution with the initial conditions (44), (45) has the form $\eta = \eta_0 m(0)/m(t)$, and (48) gives the expression for the density

$$\rho = \rho_0 m(t)/m(0). \tag{49}$$

In Ref. 1, Eq. (49) was obtained from other considerations.

It is valid for any function $m(t)$ (for $\eta \ll 1$).

One can also integrate Eq. (43) elementarily if the dipole moment changes according to a linear law

$$m(t)=m_0+bt \tag{50}$$

(see Refs. 1, 5 and 6 for astrophysical applications of this problem). The solution has the form

$$\eta=\eta_0-\frac{1}{2}\frac{bt}{m(t)}\sin 2\eta_0, \tag{51}$$

$$\rho=\rho_0\left(\frac{m_0\sin\eta_0}{m(t)\sin\eta}\right)^2\frac{m(t)}{m_0\cos 2\eta_0+2m(t)\sin^2\eta_0}. \tag{52}$$

The density changes in this case on the dipole equator ($\eta=\eta_0=\frac{1}{2}\pi$) as follows

$$\rho=\rho_0\frac{m_0^2}{(m_0+bt)(m_0+2bt)}. \tag{53}$$

We note that we can consider (51) as an implicit definition of the function $\eta_0=\eta_0(t,\eta)$ and the density which is given by (52) is thus a function of merely two parameters: the time t and the polar angle η.

If the dipole moment decreases ($b<0$) the quantity η_0 becomes for $t>m_0/2|b|$ a nonunique function of η, i.e., there occur intersections of the trajectories of the liquid particles. In that case the solution (51), (52) ceases to be correct.

It follows from (49) and (53) that when the magnetic dipole increases the density increases on the dipole axis ($\eta=0$) and decreases on the equator ($\eta=\frac{1}{2}\pi$). The opposite process occurs when the magnetic moment decreases. This effect was called magnetic bunching in Ref. 1.

4. MAGNETIC FIELD FLUCTUATIONS AS CAUSE OF PLASMA CONDENSATION FORMATION

Let the dipole moment change as follows with time:

$$m(t)=m_0+af(t), \tag{54}$$

where m_0 and a are constant quantities, and $f(t)$ is an oscillating function for which $\langle f\rangle=0$, $\langle f_t^2\rangle\neq 0$ (the brackets indicate averages). Without loss of generality we can put $\langle f_t^2\rangle=1/2$. Let the condition $\lambda\equiv a/m_0\ll 1$ also be satisfied. This means that we wish to study plasma flow in an almost constant weakly fluctuating strong magnetic field. In this case Eq. (43) and the initial conditions take the form

$$(1+\lambda f)\eta_{tt}+2\lambda f_t\eta_t+{}^1/_2\lambda f_{tt}\sin 2\eta=0, \tag{55}$$

$$\eta=\eta_0,\quad (1+\lambda f)\eta_t+{}^1/_2\lambda f_t\sin 2\eta_0=0\quad \text{for}\quad t=0. \tag{56}$$

Equation (55) is weakly nonlinear, i.e., it contains a small parameter λ such that when it is zero the equation degenerates into a linear differential equation with constant coefficients. This kind of equations has been well studied in the theory of nonlinear oscillations.[10] It is impossible to use for their solution the usual expansion in powers of the small parameter to obtain results applicable for large time intervals. However, using the averaging method[10] we can get an equation for the smoothly varying part of the complete solution.

Following Ref. 10 we bring (55) to standard form, i.e., to a set of first-order equations in which the right-hand side is proportional to the small parameter λ. To do this we introduce instead of the unknown function η two new functions γ and Ω using the formulac

$$\eta=\gamma-{}^1/_2\lambda f\sin 2\gamma, \tag{57}$$

$$\eta_t=\lambda\Omega-{}^1/_2\lambda f_t\sin 2\gamma. \tag{58}$$

Differentiating (57) and comparing it with (58) we find

$$(1-\lambda f\cos 2\gamma)\gamma_t=\lambda\Omega,$$

or

$$\gamma_t=\lambda\Omega+O(\lambda^2). \tag{59}$$

Differentiating (58) and substituting it into (55) we get

$$\Omega_t=\lambda\{f_t\Omega\cos 2\gamma-2f_t\Omega+\sin 2\gamma[ff_{tt}\cos^2\gamma+(f_t)^2]\}+O(\lambda^2). \tag{60}$$

We can apply to Eqs. (59), (60) the averaging principle according to which we must average the right-hand side over the explicitly appearing time. Since $\langle f_{tt}\rangle=-\langle f_t^2\rangle=-1/2$ we get to first approximate the equations

$$\gamma_t=\lambda\Omega,\quad \Omega_t={}^1/_2\lambda\sin^2\gamma\sin 2\gamma \tag{61}$$

with the initial conditions $\gamma=\eta_0$, $\Omega=0$.

The function γ changes slowly with time; the exact solution η will contain additional small oscillating terms which do not affect the systematic change of η.

We determine the time dependence of the density on the axis and at the equator of the dipole. It follows from (49) that near the axis $\rho=\rho_0$ apart from small vibrational terms, i.e., there will be no systematic change in the density on the axis. The situation is completely different near the dipole equator. For γ close to $\frac{1}{2}\pi$ we get from the linearized set (61) $\gamma=\frac{1}{2}\pi-(\frac{1}{2}\pi-\eta_0)\cos\lambda t$. The density therefore has the form

$$\rho|_{\gamma=\pi/2}=\rho_0/\cos\lambda t. \tag{62}$$

Hence it follows that there is a slow systematic bunching of the plasma towards the dipole equator.

We consider this process in more detail. We get the qualitative behavior of the solutions most simply if we note that the set (61) can be written in Hamiltonian form:

$$\gamma_t=\partial\mathscr{H}/\partial\Omega,\quad \Omega_t=-\partial\mathscr{H}/\partial\gamma, \tag{63}$$

where

$$\mathscr{H}={}^1/_2\lambda\Omega^2-{}^1/_4\lambda\sin^4\gamma.$$

The lines $\mathscr{H}=\text{const}$ (Fig. 1) are the phase curves of Eqs. (61). The solutions depend periodically on the time and particles with γ initially cloes to $\frac{1}{2}\pi$ oscillate with the smallest period ($T=2\pi/\lambda$). When $t>\pi/2\lambda$, intersections of the trajectories of liquid particles, which move along one force tube, will occur and the solution becomes incorrect.

To obtain the density distribution at different times as function of the polar angle γ we integrated Eqs. (61) numerically. The results are shown in Fig. 2. For high densities it is no longer possible to neglect the gas pressure gradient in Eq. (1) and therefore when $t\gtrsim\pi/2\lambda$ we need other methods for studying plasma flows near the dipole equator.

We emphasize the difference in principle between the magnetic bunching mechanism described in section 3 (see

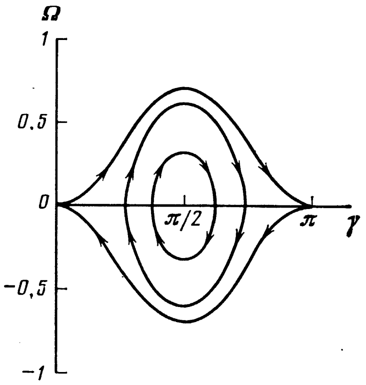

FIG. 1. Phase plane of the set (61).

also Refs. 1, 5, and 6) and the one considered in this section. For the latter, in contrast to the former, one does not need large systematic changes in the initial magnetic field.

5. CONCLUSION

The use of the frozen-in coordinate technique enables us to simplify considerably the solution of the MHD equations in the strong field approximation: as a result of the substitution of (25), (26) the problem reduces to solving an ordinary differential equation. For sufficiently simple field configurations one can find the parametrization (22), (23) and Eq. (27) in explicit form as we demonstrated by the example of a dipole field. The application of the method developed here to more complicated three-dimensional fields can facilitate the numerical calculation procedure.

The solutions obtained enable us to study the magnetic bunching mechanism which so far has been studied by the small perturbation method (at initial times)[1,5] or numerically.[6] Such a study revealed a new bunching mechanism caused by small magnetic field fluctuations and leading to the formation of a plasma condensation at the dipole equator.

The mechanism described here may be responsible for the formation of quiet protuberances in active regions on the Sun. A protuberance is formed in regions with a strong magnetic field and has the shape of a stable dense curtain positioned high in the corona along the lines of the inversion of the photospheric magnetic field, i.e., at the vertices of magnetic loops.[11] The matter flows continuously downwards from the protuberance and if it were not replenished it would be lost in the course of days. A protuberance is surrounded by regions with a lower density, called coronal holes. If a protuberance is destroyed for some reason, in two thirds of the cases it appears again after 1–7 days at the same place and most often it has the same shape. In that case the magnetic field configuration does not suffer noticeable changes. The cause of the magnetic field fluctuations may be nonstationary convection on the Sun.

Taking into account magnetic field fluctuations also solves the old and confused problem of the stability of a protuberance.[11,12] The situation here is analogous to the well known effect in the theory of nonlinear oscillations when vibrations change a statically unstable system into a dynamically stable one[10] (e.g., an inverted pendulum with a vibrating point of suspension). In the case when the magnetic field is sufficiently strong to dominate the gravitational forces a protuberance at the magnetic loop vertices is a dynamically stable system.

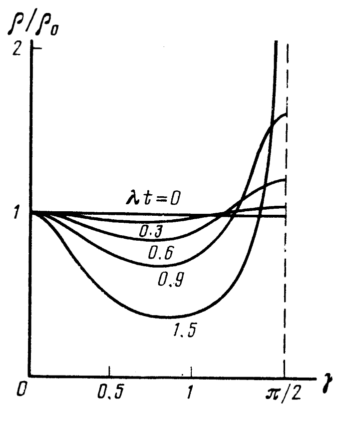

FIG. 2. Density redistribution with time, caused by the fluctuations of a two-dimensional dipole field.

APPENDIX

In the three-dimensional case flows arising when the dipole moment changes do not differ qualitatively from the two-dimensional ones. In the case of an axisymmetric poloidal field ($H_\varphi = 0$) it is convenient to choose as the frozen-in coodinates

$$\xi = \Phi(t, r, \theta) \equiv r \sin\theta A_\varphi(t, r, \theta), \quad \psi = \varphi,$$

where $\Phi(t,r,\theta)$ is the flux function,[1,5] and A_φ the only nonvanishing component of the vector potential. Condition (23) will then be satisfied identically. For a dipole magnetic field

$$\Phi(t, r, \theta) = \frac{m(t)\sin^2\theta}{r}.$$

By analogy with (36), (37) we get

$$\mathbf{R}(t, \alpha, \xi, \psi) = \frac{m(t)\sin^2\theta}{\xi}\{\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta\}, \tag{A1}$$

$$\theta_\alpha \sin^7\theta = \xi^4/\rho_0 m^3(t). \tag{A2}$$

The last equation determines the function $\theta = \theta(t,\alpha,\xi)$. The substitution $\mathbf{r}(t,\alpha,\xi,\psi) = \mathbf{R}(t,\alpha(t,\alpha,\xi),\xi,\psi)$ into (21) leads to the equation

$$\sin\vartheta(3\cos^2\vartheta+1)(m\vartheta_{tt}+2m_t\vartheta_t) + 2m\vartheta_t^2\cos\vartheta(3\cos^2\vartheta-1)+2m_{tt}\sin^2\vartheta\cos\vartheta=0 \tag{A3}$$

with the initial conditions

$$\vartheta(0, \alpha, \xi) = \vartheta_0(\alpha, \xi), \quad m\vartheta_t(3\cos^2\vartheta_0+1)+m_t\sin 2\vartheta=0, \tag{A4}$$

where $\vartheta(t,\alpha,\xi) = \vartheta(t,\sigma(t,\alpha,\xi),\xi)$ [cf. (39)–(46)].

Similar to (48) we get the three-dimensional case an expression for the density

$$\rho = \rho_0\left(\frac{m(0)}{m(t)}\right)^3 \frac{\sin^7\vartheta_0}{\sin^7\vartheta}\frac{(\vartheta_0)_\alpha}{\vartheta_\alpha}. \tag{A5}$$

Close to the dipole ($\vartheta \ll 1$) Eq. (A3) takes the form $(m\vartheta^2)_{tt} = 0$. Its solution with the initial conditions (A4) will be $\vartheta = \vartheta_0[m(0)/m(t)]^{1/2}$ and it follows from (A5) that Eq. (49) remains valid also in the three-dimensional case.

For a linear change of the dipole moment (5) one can integrate Eq. (A3) by means of the substitution

$$\chi=\sin^2\vartheta(3\cos^2\vartheta+1)\vartheta_t^2,$$

bringing (A3) to the form

$$m\chi_t+4m_t\chi=0.$$

The solution has the form

$$\cos\vartheta(3\cos^2\vartheta+1)^{1/2}+\frac{1}{\sqrt{3}}\ln[\sqrt{3}\cos\vartheta+(3\cos^2\vartheta+1)^{1/2}]$$
$$=\frac{bt}{m(t)}\frac{4\sin^2\vartheta\cos\vartheta}{(3\cos^2\vartheta+1)^{1/2}}+\cos\vartheta_0(3\cos^2\vartheta_0+1)^{1/2}$$
$$+\frac{1}{\sqrt{3}}\ln[\sqrt{3}\cos\vartheta_0+(3\cos^2\vartheta_0+1)^{1/2}], \tag{A6}$$

$$\rho=\rho_0\left(\frac{m_0}{m(t)}\right)^3\frac{\sin^7\vartheta_0}{\sin^7\vartheta}\sin\vartheta(3\cos^2\vartheta+1)^{1/2}$$
$$\times\Big[\sin\vartheta_0(3\cos^2\vartheta_0+1)^{1/2}$$
$$-\frac{2bt}{m(t)}\frac{\sin\vartheta_0(12\cos^4\vartheta_0-3\cos^2\vartheta_0-1)}{(3\cos^2\vartheta_0+1)^{3/2}}\Big]^{-1}. \tag{A7}$$

On the dipole equator ($\vartheta=\vartheta_0=\frac{1}{2}\pi$) the density changes as follows

$$\rho=\rho_0\frac{m_0^3}{(m_0+bt)^2(m_0+3bt)}.$$

If the dipole moment fluctuates [see (54)] the averaged equations of the first approximation have the form

$$\gamma_t=\lambda\Omega, \tag{A8}$$

$$\Omega_t=\lambda\left[\frac{3\sin\gamma\cos\gamma}{(3\cos^2\gamma+1)^3}(1-\cos^4\gamma)-\Omega^2\frac{2\cos\gamma(3\cos^2\gamma-1)}{\sin\gamma(3\cos^2\gamma+1)}\right] \tag{A9}$$

with initial conditions $\gamma=\vartheta_0$, $\Omega=0$. Here, as in (61), γ is the smoothly changing part of the exact solution of (A3). On the dipole axis the density has no systematic changes [see (49)], but on the equator we get, similar to (62)

$$\rho=\rho_0/\cos(\sqrt{3}\lambda t). \tag{A10}$$

The nature of the density redistribution with time is the same as in the planar case (see Fig. 2).

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