Stimulated bremsstrahlung effect in the scattering of electrons by atoms subject to the polarization of a target

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(Submitted 8 July 1987)
Zh. Eksp. Teor. Fiz. **94**, 87–94 (July 1988)

A calculation is made of the cross section for the stimulated bremsstrahlung effect in the scattering of nonrelativistic electrons by atoms. In addition to the usual static part found in the approximation of screening of the nucleus of an atom, an allowance is made for the polarization of the atom by an external electromagnetic field. An effective potential allowing for the interaction of the incident electron with oscillations of the electron cloud of the atom is constructed. The interaction of the scattered electron with an external field is allowed for exactly and the effective potential is included in the first Born approximation. A relationship is established between the dynamic part of the bremsstrahlung effect and nonlinear susceptibilities of an atom. It is shown that in a certain range of external field frequencies and polarizations the scattering cross is considerably greater than the cross section for the static bremsstrahlung effect.

1. A multiphoton stimulated bremsstrahlung effect (SBE) in a strong optical field has already been investigated extensively both theoretically using the static approximation for the scattering potential[1,2] and experimentally.[3] On the other hand, it has been shown in a number of reports[4–9] that in calculations dealing with the one-photon bremsstrahlung effect it is necessary to allow for the virtual excitation of a target in the course of the process when collisions of electrons with atoms and ions are considered. The static approximation is valid in the case of a weak polarization of the target in an external field and also in the limits of high and low frequencies.

The multiphoton SBE in the presence of a one-photon resonance of an external field with a two-level system in a target was considered in Ref. 10. It has also been shown[8,9,11] that the influence of the dynamics of the target in nonresonant one-photon bremsstrahlung of fast and slow electrons can be described by a single atomic characteristic in the form of the dynamic polarizability. In the multiphoton SBE we have to allow for the nonlinear polarization of a target in an external field and the appearance of an alternating dipole moment of this target, which oscillates not only at the frequency of the external field ω but also at frequencies $n\omega$ which are multiples of that frequency.

We shall consider the role of the interaction of an incident electron with induced oscillations in the SBE and the relationship of the SBE cross section to nonlinear susceptibilities of the system.

2. We shall assume that the incident and scattered electrons are fast, so that we can use the Born approximation and ignore the exchange effects. Subject to this condition the potential of the interaction of the scattered electron with the target can be described by the relationship

$$V(\mathbf{r},t)=-\frac{Z}{r}+\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}',\qquad(1)$$

where Z is the nuclear charge and ρ is the electron density.

The Schrödinger equation for the wave function ψ of an electron, scattered by an atom in the laser radiation field, is of the form

$$i\frac{\partial\psi}{\partial t}=\left[-i\nabla-\frac{1}{c}\mathbf{A}(t)\right]^2\frac{\psi}{2}+\left[-\frac{Z}{r}+\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right]\psi,\qquad(2)$$

$\mathbf{A}=-(c\boldsymbol{\varepsilon}_0/\omega)\cos\omega t$ is the vector potential corresponding to an electric field of amplitude $\boldsymbol{\varepsilon}_0$ and of frequency ω; c is the velocity of light in vacuum.

In the absence of an external field the electron density is independent of time $\rho(\mathbf{r},t)=\rho_0(\mathbf{r})$, whereas Eq. (2) reduces to the stationary Hartree equation for the electron being scattered.

At low incident electron energies the greatest contribution to the amplitude of a transition comes from a region which is much larger than the atomic radius r_a and in this region the Coulomb interaction of the incident electron with the target electrons can be expanded as a series in terms of a small parameter $r_a/r\ll1$, retaining only the longest-range dipole term $|\mathbf{r}-\mathbf{r}'|^{-1}\approx r^{-1}+\mathbf{r}\mathbf{r}'/r^3$. Consequently, the additional transient interaction of an electron with an atom is described by the expression

$$U(t)=\mathbf{P}(t)\mathbf{r}/r^3,\qquad(3)$$

where

$$\mathbf{P}(t)=\int\mathbf{r}'\rho(\mathbf{r}',t)\,d\mathbf{r}'=\sum_{n=-\infty}^{\infty}\mathbf{P}_n e^{in\omega t}\qquad(4)$$

is the polarization vector with the components $\mathbf{P}_n$ at frequencies $n\omega$ of the harmonics of the external field.[12] The asymptotic form of the interaction (3) may be used in the calculation of the SBE of slow electrons scattered by atoms. This form was used in Ref. 11 on the assumption that $|n|=1$.

We shall now calculate the cross section for the scattering of electrons by atoms in the presence of a strong laser field. We shall use the first Born approximation. The wave function of an electron with a momentum $\mathbf{p}$ in a strong optical field is

$$\psi_\mathbf{p}=(2\pi)^{-3/2}\exp\left\{i\left[\mathbf{p}\mathbf{r}-\int_0^t\left(\mathbf{p}-\frac{1}{c}\mathbf{A}(\tau)\right)^2\frac{d\tau}{2}\right]\right\}.\qquad(5)$$

0038-5646/88/071346-05$04.00

The amplitude of the probability of a transition to a state $\psi_{\mathbf{p}'}$, of an electron which at a moment $t=0$ is in a state $\psi_{\mathbf{p}}$ is given by the expression[1]

$$C_{\mathbf{pp}'}=-i\int_0^t \exp\left\{i\left[\Delta\varepsilon t/2+\frac{\mathbf{q}}{c}\int_0^{t'}\mathbf{A}(\tau)\,d\tau\right]\right\}V(\mathbf{q},t')\,dt', \quad (6)$$

where

$$V(\mathbf{q}',t)=\int V(\mathbf{r}',t)e^{i\mathbf{qr}'}\,d\mathbf{r}', \quad (7)$$

$\mathbf{q}=\mathbf{p}-\mathbf{p}'$ is the transferred momentum, and $\Delta\varepsilon=(p^2-p'^2)/2$ is the collision-induced change in the average energy. We shall expand the Fourier component $V(\mathbf{q},t)$ of the effective potential as a series in terms of the time harmonics

$$V(\mathbf{q},t)=\sum_{n=-\infty}^{\infty}V_n(\mathbf{q})\exp(in\omega t), \quad (8)$$

using also the familiar relationship[13] for the Bessel functions

$$\exp(i\lambda\sin\omega t)=\sum_{\nu=-\infty}^{\infty}J_\nu(\lambda)\exp(i\nu\omega t). \quad (9)$$

It follows from Eqs. (6), (8), and (9) and from the known relationship between the amplitude of a process and its cross section that the cross section for the emission (absorption) of m photons is

$$\frac{d\sigma^{(m)}}{d\Omega_{\mathbf{p}'}}=\frac{p'}{(2\pi)^2p}\left|\sum_{n+\nu=m}V_n(\mathbf{q})J_\nu\left(\frac{\mathbf{q}\varepsilon_0}{\omega^2}\right)\right|^2. \quad (10)$$

The amplitude of the SBE probability in Eq. (10) is a sum of the amplitudes of the processes involving the absorption (emission) of ν photons by an electron and m photons by the target, so that the total number of absorbed (emitted) photons is $m=n+\nu$.

The time-dependent electron density can be represented by a series

$$\rho(\mathbf{r},t)=\sum_{n=-\infty}^{\infty}\rho_n(\mathbf{r})e^{in\omega t}. \quad (11)$$

Turning back to Eq. (2), we readily see that the time harmonics of the effective potential can be expressed in terms of the corresponding harmonics of the electron density, so that

$$V_n(\mathbf{q})=\frac{4\pi}{q^2}(-Z\delta_{n0}+F_n(\mathbf{q})), \quad (12)$$

where

$$F_n(\mathbf{q})=\int\rho_n(\mathbf{r}')e^{i\mathbf{qr}'}\,d\mathbf{r}'$$

is the form factor of the harmonics of oscillations of the electron density. The corresponding differential SBE cross section can be written in the form

$$\frac{d\sigma^{(m)}}{d\Omega_{\mathbf{p}'}}=\frac{4p'}{pq^4}\left|\sum_{n+\nu=m}[-Z\delta_{n0}+F_n(\mathbf{q})]J_\nu\left(\frac{\mathbf{q}\varepsilon_0}{\omega^2}\right)\right|^2. \quad (13)$$

We shall now assume that the laser field is such that the parameter $\alpha=\mathbf{q}\varepsilon_0/\omega^2$ is small ($\alpha\ll 1$) and the Bessel functions can be used in their asymptotic form[13]: $J_\nu(x)=x^\nu/\nu!$. In the case of low values of the transferred momentum we have $\exp(i\mathbf{qr})\approx 1+i\mathbf{qr}$ and the differential scattering cross section becomes

$$\frac{d\sigma^{(m)}}{d\Omega_{\mathbf{p}'}}=\frac{4p'}{pq^4}\left|\sum_{n+\nu=m}\frac{-Z+F_0(\mathbf{q})+i\mathbf{qP}_n}{|\nu|!}\left(\frac{\varepsilon_0\mathbf{q}\,\mathrm{sign}(\nu)}{\omega^2}\right)^{|\nu|}\right|^2 \quad (14)$$

with the target polarization vector at a frequency $n\omega$

$$\mathbf{P}_n=\int\mathbf{r}'\rho_n(\mathbf{r}')\,d\mathbf{r}',\quad \mathrm{sign}(\nu)=\begin{cases}1, & \nu\geqslant 0\\ -1, & \nu<0\end{cases}.$$

In the ground state of an atom the quantities $\mathbf{P}_n$ differ from zero only for odd values $n=1,3,\ldots$.

We shall analyze the relative contribution of the polarizations $\mathbf{P}_n$ at different frequencies. In the case of scattering by atoms the static part of the form factor is $F_0(\mathbf{q})-\mathbf{Z}\to 0$ in the limit $\mathbf{q}\to 0$. Therefore, at low transferred momenta the main contribution to the SBE comes from scattering by oscillations of the electron cloud. The square of the transferred momentum $\mathbf{q}$ is $p^2+p'^2-2pp'\cos\theta$, where θ is the angle between the initial momentum $\mathbf{p}$ and the final momentum $\mathbf{p}'$, and the differential of the solid angle of scattering is $d\Omega_{\mathbf{p}'}=qd\,qd\,\varphi/pp'$. In the case of the SBE accompanied by the absorption (emission) of n photons by an atom, we can integrate over the directions of motion of the scattered electron and this gives

$$\sigma_n^{(m)}\propto\frac{P_n^2}{\eta^2}\begin{cases}\dfrac{1}{|\nu|!^2}\left(\dfrac{\varepsilon_0}{\omega^2}\right)^{2\nu}\dfrac{\varkappa^{2|\nu|}}{2|\nu|}, & \nu\neq 0\\[2ex] \ln\dfrac{\varkappa}{|p-p'|}, & \nu=0\end{cases}, \quad (15)$$

where $\varkappa=(2|E_0|)^{1/2}$ is the characteristic momentum of the valence electrons of energy E_0.

It is known[12] that a harmonic of the polarization vector of $\mathbf{P}_n$ considered using the perturbation theory framework can be expressed in terms of the nonlinear susceptibility tensor $\hat{\chi}^{(n)}$:

$$\mathbf{P}_n=\hat{\chi}^{(n)}\varepsilon_0{}^n. \quad (16)$$

The nonlinear resonance value of $\chi^{(n)}$ can be estimated to be $\chi^{(n)}\propto(d^{(n+1)}/\Delta E^n$, where d is the characteristic size of an atom and ΔE is the characteristic excitation energy; in this case we have $d\propto 1/\varkappa$, $\Delta E\propto\varkappa^2$. Using these results, we can readily find the ratio of the SBE cross sections for the cases of absorption (emission) of ν and $\nu-1$ photons by an atom: $\beta=(\Delta E/\omega\nu^{1/2})^4$. In the limit of low frequencies this quantity is considerably greater than unity for any finite ν and in calculations it is sufficient to consider only the dipole polarizability. This is in agreement with the observation that in multiphoton detachment of electrons from negative ions the greatest contribution to the process is again made by the polarizability of an atom.[14,15] The dipole polarizability is sufficient in the region of a one-photon resonance.

In the opposite case of high frequencies and large numbers of photons absorbed (emitted) by an electron, we have $\nu\gg 1$, $\beta\ll 1$ and the SBE processes involving multiphoton excitation of an atom may be more important. On the other hand, since at low angles the differential scattering cross section is dominated by the polarization term $q\lesssim\varkappa$, at high angles the main contribution comes from the amplitude of the static SBE.[16] We shall calculate the total cross section of the SBE. The SBE cross section allowing for the emission of m photons via excitation of an atom diverges logarithmically in the limit $q \to 0$. However, the static SBE cross section for the scattering by a neutral atom vanishes in the limit $q \to 0$. On the other hand, if $q > \varkappa$, the contribution of virtual excitation of the target in the SBE becomes exponentially small because of rapid oscillations of the factor $\exp(i\mathbf{q}\cdot\mathbf{r})$ in the integrand of the expression for the cross sections. In this range of the transferred momenta the greatest contribution comes from the static scattering, because an increase in the transferred momentum gradually lifts the screening of the nucleus and the Coulomb SBE cross section decreases logarithmically on increase in q. Therefore, the static and polarization amplitudes interfere weakly and the total cross section $\sigma^{(m)}$ can be represented by a sum of the direct static $\sigma_0^{(m)}$ and polarization $\sigma_{\rm tot}^{(m)}$ cross sections:

$$\sigma^{(m)} = \sigma_0^{(m)} + \sigma_{\rm tot}^{(m)}. \tag{17}$$

In the case under discussion we have

$$\sigma_{\rm tot}^{(m)} = \frac{4\pi P_m^2 \sin^2\gamma}{p^2} \ln \frac{\varkappa}{|p-p'|}, \tag{18}$$

where γ is the angle between the vectors $\mathbf{p}$ and $\mathbf{P}_m$. A similar result is obtained in Ref. 11 for the one-photon bremsstrahlung effect ($m = \pm 1$). We can see that if $m > 1$ and the condition $\beta \ll 1$ is satisfied, the polarization SBE cross section can be expressed in terms of a single atomic characteristic which is the nonlinear susceptibility of an atom $\hat{\chi}^{(n)}$.

3. In the calculation of the static part of the cross section we have to find $V_0(\mathbf{q})$. We shall consider only an external field which varies slowly with the potential and in the case of heavy atoms we shall employ the Titus approximation[17] for the potential of an atom in the Thomas-Fermi model:

$$V_0(r) = -Z/[r(1+\mu r)^2], \quad \mu = (8\cdot 2^{1/2}/3\pi)^{2/3} Z^{1/3}. \tag{19}$$

The Fourier transform of the potential (19) can be expressed in terms of the function[18]

$$g(u) = \int_0^\infty \frac{\cos t\, dt}{u+t}, \quad u>0$$

in the form

$$V_0(q) = -\frac{4\pi Z}{\mu^2} g\left(\frac{q}{\mu}\right). \tag{20}$$

Bearing this point in mind, we find that

$$\sigma_0^{(m)} = \frac{4p'}{\mu^2 p} \int g^2\left(\frac{q}{\mu}\right) J_m^2\left(\frac{\mathbf{q}\boldsymbol{\varepsilon}_0}{\omega^2}\right) d\Omega_{\mathbf{p}'}. \tag{21}$$

A calculation of the dynamic part of the cross section $\sigma_{\rm tot}^{(m)}$ is very difficult in the general case of arbitrary intensities of the external field. We shall confine ourselves to the case of moderately strong fields when we can still use perturbation theory to describe the interaction of an atom with the external field ($d\varepsilon_0/\Delta E \ll 1$). We shall regard the interaction as dipolar, i.e., we shall assume that the wavelength of the external field is much greater than the dimensions of an atom:

$$W = -\mathbf{d}\boldsymbol{\varepsilon}(t) = -Fe^{i\omega t} + \text{c.c.} \tag{22}$$

In the first order of perturbation theory,[19] we obtain

$$\rho = \rho_0 + (\rho_1 e^{i\omega t} + \text{c.c.}),$$
$$\rho_1(\mathbf{q}) = \sum_k \left[\frac{\langle 0|e^{i\mathbf{q}\mathbf{r}}|k\rangle F_{k0}}{\omega_{k0}+\omega} + \frac{F_{0k}\langle k|e^{i\mathbf{q}\mathbf{r}}|0\rangle}{\omega_{k0}-\omega} \right]. \tag{23}$$

The value of the quantity $\rho_1(\mathbf{q})$ makes it possible to calculate the SBE cross section for $\beta \gg 1$, and also for any one- and two-photon SBE. A method for calculating this quantity for the ground state of a hydrogen-like system is given in Ref. 5. Since in the present investigation the SBE is considered without allowance for the change in the state of an atom, the selection rules and the angular coefficient are identical with those for the dipole polarizability[20] and in the case of $\exp(i\mathbf{q}\cdot\mathbf{r})$ in the matrix element only the dipole term of the expansion remains in a series in terms of spherical harmonics: $4\pi i j_1(qr) Y_{1\mu}(\mathbf{q}/q)$. The Z axis is selected along the direction of an external electric field ε_0. Therefore, the dependence of the amplitude $\rho_1(\mathbf{q})$ on the direction of the transferred momentum is characterized by a function $Y_{1\mu}(\mathbf{q}/q)$, in agreement with the results of a more specific analysis.[5]

In the optical range of frequencies the greatest contribution to the change in the electron density comes from external optical electrons. A calculation of the radial matrix element for this case is given in the Appendix.

By way of example we calculated, using the Born approximation, the cross section for two-photon emission as a result of scattering of electrons of energy 100 eV by an atom of xenon in the range of laser radiation field intensities $\varepsilon_0 \ll \omega^2/\varkappa$, where the dependence of the cross section on the field intensity is described by a power law. The field polarization is assumed to be orthogonal to the direction of the momentum of the incident electrons. The results of the calculation are presented in Fig. 1 in the form of the dependence of $\ln(\sigma^2/\varepsilon_0^4)$ on the frequency ω. The range of variation of the frequency of the external field is selected to be close to the $5p^{6\,1}S_0 - 6p[3/2]_1$ resonance. It is clear from Fig. 1 that

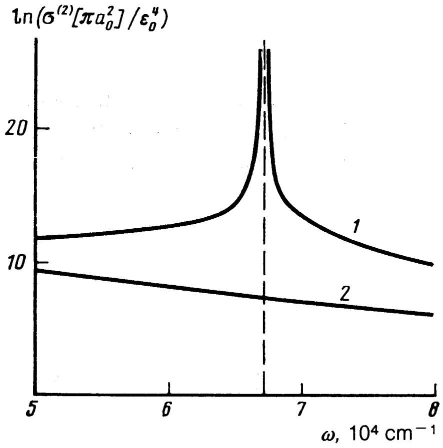

FIG. 1. Dependence of the logarithm of the cross section for two-photon bremsstrahlung emission of radiation by electrons $\ln(\sigma^{(2)}/\varepsilon_0^4)$ on the frequency near a resonance transition in a xenon atom. Curve 1 is plotted allowing for the polarization of an atom by an external field and curve 2 represents calculations of cross sections in the static approximation.

the polarization contribution is greatest near the resonance and it decreases rapidly away from it.

4. Our calculations of the SBE cross section demonstrate the importance of an allowance for the target polarization. The results obtained apply to the case of when an atom is not excited in its final state. The approach developed here can be generalized also to the case when the final state of an atom is not identical with the initial state. This can be done conveniently using the formalism of quasienergy states.[20] This problem was solved in Ref. 21 using perturbation theory for the external field. The SBE amplitude then reduces to radial matrix elements and the calculation of these elements in the method of the model potential is equivalent to a numerical summation of the series. In the case of the one-photon SBE the amplitude of the process accompanied by virtual excitation of an atom is identical with the matrix element $\rho_1(\mathbf{q})$, the radial part of which is given in the Appendix. The matrix elements corresponding to the emission (absorption) of a large number of photons are calculated very similarly, but their explicit form will not be given because the expressions obtained are cumbersome.

In the case of an exact resonance our perturbation theory formulas are no longer valid. An analysis of SBE in the case of a one-photon resonance carried out in Ref. 10 gives, by analogy with Eq. (13), an expansion of the differential cross section in terms of Bessel functions with $\nu = m \pm 1$. The coefficient of this expansion can be expressed in terms of the matrix elements of transitions between the resonating levels.

The relativistic generalization of the problem of the one-photon SBE allowing for the target polarization can be found in Refs. 22 and 23. The multiphoton SBE can be analyzed by a method described above if the wave function of the incident electron is described by the Volkov solutions.[24]

The author is grateful to B.A. Zon for his interest in this problem.

APPENDIX

The radial matrix element $\langle 0|j_1(qr)g_l(r,r',E)r'|0\rangle$ in the expression for $\rho_1(\mathbf{q})$ can be calculated for the optical range of frequencies using the Sturm expansion of the Green function for the model potential[25]:

$$V_\mu(r) = -\frac{z}{r} + \sum_l \frac{B_l \hat{P}_l}{r^2}. \tag{A.1}$$

The operator $\hat{P}_l$ projects the wave function onto a state with a given orbital quantum number l; z is the effective charge of an optical electron. The Sturm expansion for a Green function is

$$g_l(r,r',E) = \frac{4z}{\nu}\sum_{k=0}^{\infty} \frac{F_{kl}(\alpha,r)F_{kl}(\alpha r')}{k+\lambda+1-\nu}, \tag{A.2}$$

where

$$F_{kl}(x) = (k!/\Gamma(k+2\lambda+2))^{1/2} x^\lambda e^{-x/2} L_k^{2\lambda+1}(x) \tag{A.3}$$

and

$$\alpha = 2z/\nu, \quad \nu = (-2E)^{1/2}, \quad \lambda = z/(-E_l)^{1/2} - 1.$$

The radial wave functions can be expressed in terms of functions $F_h(x)$ with $\nu_0 = z/(-2E_{k0l0})^{1/2}$:

$$R_{k_0l_0}(r) = (2z^{3/2}/\nu_0^2) F_{k_0l_0}(\alpha_0 r). \tag{A.4}$$

The results of a calculation of the radial matrix element can be expressed in terms of a sum of products of hypergeometric functions. We can however, obtain simpler and in practice a more convenient expression employing the explicit equation for the Laguerre polynomials[19]

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

and for a Bessel function

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

Integrating with respect to the radial variable, we obtain

$$\langle 0|j_1(qr)g_l(r,r',E)r'|0\rangle = \frac{16}{\nu}\left(\frac{z}{\nu_0}\right)^4 \frac{k_0!}{\Gamma(k_0+2\lambda_0+2)} \times \sum_{k=0}^{\infty} \frac{k!\,A(\lambda_0\lambda, k_0 k)\,B(\lambda_0,\lambda,k_0,k)}{\Gamma(k+2\lambda+2)\,(k+\lambda+1-\nu)}. \tag{A.5}$$

The quantity A appears as a result of integration of the matrix element of the Bessel function[26]

$$A(\lambda_0,\lambda,k_0,k) = \sum_{m_0,m=0}^{k_0,k} (-1)^{m_0+m} \binom{k_0+2\lambda+1}{k_0-m_0}\binom{k+2\lambda+1}{k-m} \frac{\alpha_0^{\lambda_0+m_0}\alpha^{\lambda+m}}{m_0!\,m!} \times\left[\frac{1}{q^2}\frac{\Gamma(\beta)\sin C}{(q^2+\xi^2)^{\beta/2}} - \frac{1}{q}\frac{\Gamma(\beta_1)\cos C_1}{(q^2+\xi^2)^{\beta_1/2}}\right], \tag{A6}$$

$$\beta = \lambda_0+\lambda+m_0+m+1, \quad \xi = (\alpha_0+\alpha)/2,$$
$$C = \beta\,\mathrm{arctg}(q/\xi), \quad \beta_1 = \beta+1, \quad C_1 = C\beta_1/\beta.$$

The quantity B is the radial matrix element of the dipole moment and, as A, can be expressed in terms of a double series:

$$B(\lambda_0,\lambda,k_0,k) = \sum_{m_0,m=0}^{k_0 k} (-1)^{m_0+m} \binom{k_0+2\lambda_0+1}{k_0-m_0}\binom{k+2\lambda+1}{k-m} \frac{\alpha_0^{\lambda_0+m_0}\alpha^{\lambda+m}}{m_0!\,m!} \cdot \Gamma(\beta_1+2)\,\xi^{-(\beta_1+2)}. \tag{A.7}$$

Equations (A.6) and (A.7) are obtained in the case when $E<0$. If $E>0$, the matrix elements can be found by analytic continuation of the results obtained at negative energies.

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Translated by A. Tybulewicz