Quantum theory of gauge-invariant cosmological perturbations

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Metric perturbations of longitudinal type in an isotropic universe filled with a scalar field are considered. The action for the perturbations is obtained, and this action is expressed in terms of a gauge-invariant variable which completely characterizes the perturbations. A consistent quantum theory of such perturbations is constructed. The spectrum of inhomogeneities in inflationary models of the evolution of the universe is calculated.

I. INTRODUCTION

Many studies (see, for example, Refs. 5–13) have been devoted to the spectrum of inhomogeneities generated in inflationary models of the evolution of the universe.[1-4] However a consistent quantum theory of inhomogeneities in inflationary models with scalar field has still not yet been constructed. Initially the perturbations in these models at the termination of the inflationary stage were estimated by qualitative methods.[7] Subsequently quantitative methods for calculating the perturbations were developed,[8-10] but all of them, strictly speaking, related to investigation of the behavior of classical perturbations. To find the final spectrum of the inhomogeneities, it is necessary to specify certain initial (primordial) perturbations. These are usually taken to be the minimal quantum fluctuations unavoidably present in the universe. Their amplitude is estimated[7-10] by quantizing the perturbations of the scalar field without allowance for the perturbations of the metric. However, the corrections to the equations of the scalar field, which are due to the metric perturbations that are ignored on quantization, are of the same order as the terms retained in the equations. In addition, the perturbations $\delta\varphi$ of the scalar field are not gauge-invariant quantities and depend (particularly strongly for long-wave perturbations) on the choice of the coordinate system. Therefore, strictly speaking such theories are not consistent, and $\delta\varphi$ cannot be regarded as a variable that must be quantized. We note also that in the majority of the quoted papers the behavior of the perturbations is analyzed in particular gauges.

The aim of the present paper is to construct a gauge-invariant quantum theory of perturbations in an isotropic universe filled with a scalar field. The remainder of the paper is arranged as follows. In Sec. 2 we review the necessary results on the background cosmological model; Sec. 3 is devoted to a gauge-invariant perturbation theory; in Sec. 4, expanding the action for the gravitational and scalar fields to the second order in the perturbations, we find the action for the perturbations and express it in terms of a gauge-invariant variable that completely characterizes the perturbations; these are quantized in Sec. 5, and in Sec. 6 we calculate the inhomogeneity spectrum in inflationary models of the evolution of the universe.

We shall consider the theory with total action

$$S=-\frac{1}{16\pi G}\int R(-g)^{1/2}\,d^4x+\int\left(\frac{1}{2}\varphi_{;i}\varphi^{;i}-V(\varphi)\right)(-g)^{1/2}\,d^4x, \qquad (1)$$

where the first term corresponds to the usual Einstein action, and the second to the action for the scalar field φ with potential $V(\varphi)$. Here and in what follows, we have used units in which $c=\hbar=1$, and the signature is $(+,-,-,-)$.

2. BACKGROUND MODEL

As the background model, we consider a homogeneous isotropic Friedmann universe with zero spatial curvature. Its metric has the form

$$ds^2=a^2(\eta)(d\eta^2-\delta_{\alpha\beta}dx^\alpha dx^\beta), \qquad (2)$$

where the Greek indices take values from 1 to 3. The time evolution of the background model is completely characterized by the dependence of the scale factor a on the conformal time η.

The Einstein equations governing this dependence have the form

$$\alpha^2=l_{pl}^2\left(\frac{1}{2}\varphi_0'^2+V(\varphi_0)a^2\right), \qquad (3)$$

$$2\alpha'+\alpha^2=3l_{pl}^2\left(-\frac{1}{2}\varphi_0'^2+V(\varphi_0)a^2\right), \qquad (4)$$

where the prime denotes the differentiation with respect to η, $\alpha\equiv a'/a$, and $l_{pl}^2=8\pi G/3$. From (3) and (4) we obtain the useful relation

$$\alpha^2-\alpha'={}^3/_2 l_{pl}^2\varphi_0'^2 \qquad (5)$$

and the equation for the homogeneous field

$$\varphi_0''+2\alpha\varphi_0'+V_{,\varphi}a^2=0,\quad V_{,\varphi}=dV/d\varphi. \qquad (6)$$

The system of equations (3)–(6) admits complete investigation on the phase plane.[14] For a large class of potentials $V(\varphi)$ there exists for the solutions of these equations an intermediate asymptote behavior corresponding to an inflationary (quasi-de-Sitter) regime of evolution of the universe.[3]

$$a\propto\exp\left(\int H(t)\,dt\right),\qquad |\dot H|\ll H^2. \qquad (7)$$

For example for the potential

$$V(\varphi)=\frac{1}{2}m^2\varphi^2,\qquad H=\frac{m^2}{3}(t_0-t),$$

where $t=\int a(\eta)d\eta$ is the physical time. The quasi-de-Sitter stage is followed by a stage in which the scalar field oscillates and is damped. In the theory with $V=1/2m^2\varphi^2$ the scale factor during this stage behaves as

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$$a(t)\propto(t-t_0)^{2/3}\left(1+\frac{\sin[2m(t-t_0)]}{6[m(t-t_0)]^2}\right), \tag{8}$$

i.e., apart from small oscillations it behaves in the same way as the scale factor in a universe filled with dust.

3. GAUGE-INVARIANT PERTURBATIONS

In the case of perturbations of scalar type, the total metric of the most general form can be written as[13,15]

$$ds^2=a^2(\eta)\{(1+2\phi)d\eta^2-2B_{,\alpha}dx^\alpha d\eta -[(1-2\psi)\delta_{\alpha\beta}+2H_{,\alpha\beta}]dx^\alpha dx^\beta\}, \tag{9}$$

where ϕ, ψ, B and H characterize the perturbations of the metric δg_{ik} (the Latin indices take values from 0 to 3). We consider the diffeomorphism generated by the displacement of the space–time points x^i by the vector $\Delta\xi^i$ $(x^i\to x^i+\Delta\xi^i)$. The change in a quantity f (f may be a scalar, vector, tensor, etc.) as a result of such a transformation has the form

$$\Delta f=-L_{\Delta\xi}f, \tag{10}$$

where $L_{\Delta\xi}$ denotes the Lee derivative. The transformation (10) is called a gauge transformation, and the corresponding group of diffeomorphisms the gauge group of gravitation (see Ref. 16). The most general diffeomorphism associated with the scalar perturbations (9) can be expressed in terms of two arbitrary functions $\Delta\xi^0$ and $\Delta\xi$:

$$\eta\to\eta+\Delta\xi^0(\eta,x^\alpha),\quad x^\alpha\to x^\alpha+\Delta\xi_{,\alpha}(\eta,x^\beta), \tag{11}$$

where the subscript following a comma denotes the derivative with respect to the corresponding coordinate. It is clear that the metric perturbations δg_{ik} by themselves, and accordingly ϕ,ψ,B,H, are not gauge-invariant quantities and under the transformation (11) change in accordance with (10) $(\delta g_i\to\delta g_{ik}+\Delta g_{ik})$.[1)] Nevertheless, from the metric perturbations one can construct gauge-invariant quantities that characterize them completely[15]:

$$\Phi=\phi+\frac{1}{a}[(B-H')a]', \tag{12}$$

$$\Psi=\psi-\alpha(B-H'). \tag{13}$$

The perturbations $\delta\varphi$ $(\varphi(x^\alpha,\eta)=\varphi_0(\eta)+\delta\varphi(x^\alpha,\eta))$ of the scalar field are also not gauge invariant, since

$$\delta\varphi\to\delta\varphi-\varphi_0'\Delta\xi^0 \tag{14}$$

under the transformation (11). The gauge-invariant quantity corresponding to them has the form[13]

$$\delta\tilde\varphi=\delta\varphi+\varphi_0'(B-H'). \tag{15}$$

The physical meaning of Φ and Ψ will become clear if we choose a definite gauge: $B=H=0$. Then the metric (9) becomes

$$ds^2=a^2(\eta)[(1+2\phi)d\eta^2-(1-2\psi)\delta_{\alpha\beta}dx^\alpha dx^\beta],$$

and $\Phi=\phi$ and $\Psi=\psi$, i.e., the gauge-invariant quantities Φ and Ψ are identical to the metric perturbations in a conformally Newtonian coordinate system.[17]

4. VARIATIONAL PRINCIPLE FOR THE PERTURBATIONS

To obtain the action for small perturbations, we expand the total action (1) to terms of second order in the perturbations. For this, it is convenient to represent the part of the action associated with the gravitational field in the form

$$S_{gr}=\frac{1}{16\pi G}\int R(-g)^{1/2}d^4x=\frac{1}{16\pi G}\int\Big\{N\gamma^{1/2}(K_\beta{}^\alpha K_\alpha{}^\beta-K^2) +\frac{1}{2}(\gamma^{1/2}\gamma^{\alpha\beta}N)_{,\alpha}(\ln\gamma)_{,\beta}+N_\alpha(\gamma^{1/2}\gamma^{\alpha\beta})_{,\beta} -\frac{1}{2}N\gamma^{1/2}\,{}^{(3)}\Gamma_{\alpha\beta}{}^{\nu}\gamma^{\alpha\beta}{}_{,\nu}+D_1\Big\}d^4x, \tag{16}$$

where the meaning of the coefficients N, N_α, $\gamma_{\alpha\beta}$ is determined by the form of the metric in the (3 + 1) formalism:

$$ds^2=(N^2-N_\alpha N^\alpha)d\eta^2-2N_\alpha dx^\alpha d\eta-\gamma_{\alpha\beta}dx^\alpha dx^\beta. \tag{17}$$

The action (16) is simply the Arnowitt-Deser-Misner action,[18] in which for the three-dimensional curvature ${}^{(3)}R$ we have used the expression obtained by Fock [see Eqs. (B49) and (B50) in Ref. 19)] In this action, K^α_β is the tensor of the extrinsic curvature and

$$D_1=-2(\gamma^{1/2}K)'+[2\gamma^{1/2}(KN^\alpha-\gamma^{\alpha\beta}N_{,\beta}) -N(\gamma^{\alpha\beta}(\gamma^{1/2})_{,\beta}+(\gamma^{1/2}\gamma^{\alpha\beta})_{,\beta})]_{,\alpha} \tag{18}$$

is a term of divergence type that does not contribute to the equations of motion. Comparing (17) with (9), we express N, N_α, etc., in terms of ϕ, ψ, B, H. Further, calculating the terms quadratic in the perturbations in the action (16) and in the action for the scalar field, and also using Eqs. (3)–(6) for the background model, we obtain the action for the cosmological perturbations of longitudinal type:

$$\delta_2S=\delta_2S_{gr}+\delta_2S_\varphi =\frac{1}{16\pi G}\int\{a^2[-6\psi'^2-12\alpha\phi\psi'-2\psi_{,\alpha}(2\phi_{,\alpha}-\psi_{,\alpha}) -2(\alpha'+2\alpha^2)\phi^2+8\pi G(\delta\varphi'^2-\delta\varphi_{,\alpha}\delta\varphi_{,\alpha}-V_{,\varphi\varphi}a^2\delta\varphi^2) +16\pi G(\varphi_0'(\phi+3\psi)'\delta\varphi-2V_{,\varphi}a^2\phi\,\delta\varphi) +4(B-H)_{,\alpha\alpha}(4\pi G\varphi_0'\delta\varphi -\psi'-\alpha\phi)]+D_1+D_2\}d^4x, \tag{19}$$

where

$$D_2=[16\pi Ga^2\varphi_0'\delta\varphi'(H_{,\alpha\alpha}-\phi-3\psi)+2\alpha a^2(H_{,\alpha\alpha}{}^2 +2\psi H_{,\alpha\alpha}-3\psi^2)]'+\{a^2[6\alpha^2(H_{,\alpha\delta}H_{,\delta}-H_{,\delta\delta}H_{,\alpha}) -4\alpha(H_{,\eta\eta}B_{,\alpha}+2H_{,\alpha\delta}(B-H')_{,\delta}-2H_{,\delta\delta}(B-H')_{,\alpha}) +(B-H')_{,\alpha\delta}(B-H')_{,\delta}-(B-H')_{,\delta\delta}(B-H')_{,\alpha}+H_{,\alpha\beta\nu}H_{,\beta\nu} -H_{,\nu\nu\beta}H_{,\alpha\beta}+2(2\alpha'+\alpha^2)(H_{,\beta\alpha}H_{,\beta}-H_{,\beta\beta}H_{,\alpha}) -4\alpha\psi B_{,\alpha}-16\pi G\varphi_0'B_{,\alpha}\delta\varphi]\}_{,\alpha} \tag{20}$$

is a term of divergence type. Variation of the action (19) with respect to $B-H'$ leads to the constraint

$$\psi'+\alpha\phi=4\pi G\varphi_0'\delta\varphi. \tag{21}$$

It was found that the action for the perturbations can be rewritten in such a way that essentially it contains only the gauge-invariant quantity[2)]

$$v=a\left(\delta\varphi+\frac{\varphi_0'}{\alpha}\psi\right)=a\left(\delta\tilde\varphi+\frac{\varphi_0'}{\alpha}\Psi\right). \tag{22}$$

Indeed, using (21) and (22) to express $\delta\varphi$ and ψ' in the action (19) in terms of v, ψ, and ϕ, ψ, B, H, and making simple but rather lengthy calculations, we obtain

$$\delta_2 S=\int\left(L+\frac{1}{16\pi G}(D_1+D_2+D_3)\right)d^4x$$
$$=\frac{1}{2}\int\left\{v'^2-v_{,\alpha}v_{,\alpha}+\frac{z''}{z}v^2+\frac{1}{8\pi G}\sum_1^3 D_n\right\}d^4x, \qquad (23)$$

where $z=a\varphi'_0/\alpha$ and

$$D_3=-8\pi G\left[2\frac{a^2}{\alpha}\left(\frac{\varphi'}{a}\right)' v\psi + \frac{4\pi G\varphi_0'^2}{\alpha}v^2-2a^2\varphi_0' v\phi-\frac{a^2}{4\pi G\alpha}\psi_{,\alpha}\psi_{,\alpha} + \frac{2\varphi_0'^2}{\alpha}a^2\phi\psi+\frac{1}{2}\left(\frac{\varphi_0'}{a}\right)^{2'}\frac{a^4}{\alpha}\psi^2+\alpha v^2\right]' \qquad (24)$$

is a divergence term.

5. QUANTIZATION

The quantization of the dynamical system with the action (23) is analogous to quantization of an ordinary scalar field v with time-dependent mass $m^2=-z''/z$. We note first of all that the divergence term in (23) can be omitted. In quantum theory this corresponds to a certain renormalization.[20] If the momenuum $\pi(\eta, x^\alpha)$ canonically conjugate to the field variable v is determined by the relation

$$\pi=\partial L/\partial v'=v', \qquad (25)$$

then the corresponding Hamiltonian will be

$$\mathscr{H}=\int(v'\pi-L)d^3x=\frac{1}{2}\int\left\{\pi^2+v_{,\alpha}v_{,\alpha}-\frac{z''}{z}v^2\right\}d^3x; \qquad (26)$$

the following quantization procedure is standard.[20] We replace v and π by corresponding operators $\hat{v}$ and $\hat{\pi}$ and specify on the hypersurface $\eta=\eta_0$ the commutation relations

$$[\hat{v}(\eta_0,x^\alpha),\hat{v}(\eta_0,x^{\alpha'})]=[\hat{\pi}(\eta_0,x^\alpha),\hat{\pi}(\eta_0,x^{\alpha'})]=0,$$
$$[\hat{v}(\eta_0,x^\alpha),\hat{\pi}(\eta_0,x^{\alpha'})]=i\delta(x^\alpha-x^{\alpha'}). \qquad (27)$$

Varying the operator analog of the action (23) with respect to $\hat{v}$, we obtain the field equation for the operator $\hat{v}$:

$$\hat{v}''-\Delta\hat{v}-\frac{z''}{z}\hat{v}=0. \qquad (28)$$

We note that it is equivalent to the Heisenberg equations

$$i\hat{v}'=[\hat{v},\mathscr{H}],\quad i\hat{\pi}'=[\hat{\pi},\mathscr{H}].$$

We introduce the operators $\hat{a}_k^+$ and $\hat{a}_k^-$ of creation and annihilation of quanta of the field $\hat{v}$; they satisfy the standard commutation relations

$$[\hat{a}_k^-,\hat{a}_{k'}^-]=[\hat{a}_k^+,\hat{a}_{k'}^+]=0,$$
$$[\hat{a}_k^-,\hat{a}_{k'}^+]=\delta_{kk'}, \qquad (29)$$

so that

$$\hat{v}=\frac{1}{2^{1/2}}\int\frac{d^3k}{(2\pi)^{3/2}}\{v_k^*(\eta)e^{i\mathbf{kx}}\hat{a}_k^-+v_k e^{-i\mathbf{kx}}\hat{a}_k^+\}. \qquad (30)$$

Substituting the expansion (30) in (28), we obtain an equation for the complex amplitude $v_k(\eta)$:

$$v_k''(\eta)+Ev_k(\eta)=0,\quad E=k^2-z''/z. \qquad (31)$$

If the commutation relations (27) and (29) are to be consistent with each other, the function $v_k(\eta)$ must satisfy the normalization condition

$$v_k'v_k^*-v_k^{*\prime}v_k=2i. \qquad (32)$$

Equation (31) is analogous to a Schrödinger equation, and in a certain sense our problem reduces to studying the penetration of a wave through the potential barrier $U=z''z$. We define the vacuum state vector $|0\rangle$ at $\eta=\eta_0$ as an eigenvector of the operators $\hat{a}_k^-$ corresponding to zero eigenvalues. However, this vector is not defined until we have specified v_k and v_k' at the time η_0. The normalization condition (32) is not sufficient for the unique fixing of the values of $v_k(\eta_0)$ and $v_k'(\eta_0)$. Additional physical arguments are usually employed. Therefore, different physically inequivalent definitions of the vacuum are possible. For example, the requirement of diagonality of the Hamiltonian (26) with respect to the operators $\hat{a}_k^-$ and $\hat{a}_k^+$ with allowance for (32) gives[20]

$$v_k(\eta_0)=E^{-1/4}(\eta_0),\quad v_k'(\eta_0)=iE^{1/4}(\eta_0). \qquad (33)$$

These conditions can be used (and, accordingly, the vacuum defined) only if $z''/z\leqslant 0$. In the case of greatest interest for us in the inflationary stage of evolution of the universe $z''/z\approx a''/a>0$, and therefore the above definition of the vacuum is invalid. In the de Sitter universe it is frequently convenient to introduce the de Sitter-invariant vacuum state[21] defined at the time η_0 by the conditions

$$v_k(\eta_0)=\frac{1}{k^{3/2}}(\alpha+ik)e^{ik\eta_0},\qquad v_k'(\eta_0)=\frac{i}{k^{1/2}}\left(\alpha+ik-\frac{\alpha'}{k}\right)e^{ik\eta_0}. \qquad (34)$$

It is here important to note that the results for the spectrum ot perturbations generated in the inflationary stage will in fact be almost independent of the particular choice of the vacuum. Indeed, as we shall see, the perturbation spectrum in the region of scales in which we are interested is determined by the short-wave part of the initial vacuum spectrum, and $|v_k|\to k^{-1/2}$, $|v_k'|\to k^{1/2}$ as $k\to\infty$ for any definition of the vacuum. Therefore, as the most general initial conditions for $v_k(\eta_0)$ we shall use

$$v_k(\eta_0)=k^{-1/2}M(k\eta_0),\quad v_k'(\eta_0)=ik^{1/2}N(k\eta_0), \qquad (35)$$

where the functions M and N are such that the normalization condition (32) is satisfied and, in addition, $|M(k\eta_0)|$, $|N(k\eta_0)|\to 1$ for $k\eta_0\gg 1$.

6. SPECTRUM OF INHOMOGENEITIES

To calculate the spectrum of the metric perturbations, we must express the gauge-invariant quantities Φ and Ψ, which are the amplitudes of the metric perturbations in the conformally Newtonian coordinate system, in terms of v. We use for this the Einstein equations. We obtain these equations from the variational principle. Expressing ϕ, ψ, $\delta\varphi$ in the action (19) in terms of Φ, Ψ, $\delta\tilde{\varphi}$, B, and H [see (12)–(14)] and omitting the divergence terms, we find after straightforward calculations an action that in fact contains only gauge-invariant quantities:

$$\delta_2 S=\frac{1}{16\pi G}\int \{a^2[-6\Psi'^2-12\alpha\Phi\Psi'-2\Psi_{,\alpha}(2\Phi_{,\alpha}-\Psi_{,\alpha})$$
$$-2(\alpha'+2\alpha^2)\Phi^2]+8\pi G(\delta\bar\varphi'^2-\delta\bar\varphi_{,\alpha}\delta\bar\varphi_{,\alpha}-V_{,\varphi\varphi}a^2\delta\bar\varphi^2)$$
$$+16\pi G(\varphi_0'(\Phi+3\Psi)'\delta\bar\varphi-2V_{,\varphi}a^2\Phi\delta\bar\varphi)\}d^4x. \quad (36)$$

Varying the action (36) with respect to Φ and Ψ, we obtain the equations

$$\Delta\Psi-3\alpha\Psi'-(\alpha'+2\alpha^2)\Phi=4\pi G(\varphi_0'\delta\bar\varphi'+V_{,\varphi}a^2\delta\bar\varphi), \quad (37)$$

$$^1/_3\Delta(\Phi-\Psi)+\Psi''+\alpha\Phi'+2\alpha\Psi'$$
$$+(\alpha'+2\alpha^2)\Phi=4\pi G(\varphi_0'\delta\bar\varphi'-V_{,\varphi}a^2\delta\bar\varphi). \quad (38)$$

Instead of the equation for $\delta\bar\varphi$, it is convenient to use the constraint (21), which in terms of the gauge-invariant quantities takes the form

$$\Psi'+\alpha\Phi=4\pi G\varphi_0'\delta\bar\varphi. \quad (39)$$

Substituting $\delta\bar\varphi$ from (39) in (38), we obtain

$$\Phi=\Psi. \quad (40)$$

Further, substitution of (39) in (37) with allowance for (40) leads to an equation for Φ:

$$\Phi''+2\left(\frac{\varphi_0'}{a}\right)\left(\frac{a}{\varphi_0'}\right)'\Phi'-\Delta\Phi+2\varphi_0'\left(\frac{\alpha}{\varphi_0'}\right)'\Phi=0. \quad (41)$$

In the quantum theory, Φ, Ψ, etc., acquire the status of operators $\hat\Phi$ and $\hat\Psi$, and Eqs. (37)–(41) become the corresponding equations for the operators.

We represent the operator Φ in the form

$$\hat\Phi(\eta,x^\alpha)=\frac{1}{2^{1/2}}\frac{\varphi_0'}{a}\int\frac{d^3k}{(2\pi)^{3/2}}[u_k^*(\eta)e^{ikx}\hat a_k^-+u_k(\eta)e^{-ikx}\hat a_k^+]. \quad (42)$$

Substituting the expansion (42) in Eq. (41), we find that the functions $u_k(\eta)$ must satisfy the equation

$$u_k''(\eta)+\left(k^2-\left(\frac{1}{z}\right)''\Big/\left(\frac{1}{z}\right)\right)u_k(\eta)=0. \quad (43)$$

We now find the constraint between $\hat\Phi=\hat\Psi$ and $\hat v$. Expressing $\delta\hat{\bar\varphi}$ and $\hat\Psi$ in Eq. (37) in terms of $\hat v$ and $\hat\Psi$ by means of (39), we obtain

$$\Delta\hat\Phi=+4\pi G\frac{\varphi_0'^2}{\alpha}\left(\frac{\hat v}{z}\right)'. \quad (44)$$

Substituting the expansions of the operators $\hat v$ and $\hat\Phi$ [see (30) and (42)] in (44), we find a constraint between v_k and u_k:

$$u_k=-4\pi G\frac{z}{k^2}\left(\frac{v_k}{z}\right)'. \quad (45)$$

The initial conditions for $u_k(\eta_0)$ and $u_k'(\eta_0)$ under which the eigenstate of the operators $\hat a_k^-$ corresponding to zero eigenvalue can be interpreted as the vacuum at time η_0 correspond to the conditions (35) for $v_k(\eta)$, and they can be readily obtained on the basis of Eq. (45):

$$^0u_k=u_k(\eta_0)=-4\pi G\left(\frac{i}{k^{3/2}}N(k\eta_0)-\frac{z_0'}{z_0}\frac{1}{k^{5/2}}M(k\eta_0)\right),$$
$$^0u_k'=u_k'(\eta_0)=-4\pi G\left(\frac{1}{k^{1/2}}M(k\eta_0)+3\frac{z_0'}{z_0}\left(\frac{i}{k^{3/2}}N(k\eta_0)-\frac{z_0'}{z_0}\frac{1}{k^{5/2}}M(k\eta_0)\right)\right), \quad (46)$$

where here and in what follows the superscript 0 denotes the values of the corresponding quantities at the time $\eta=\eta_0$.

The correlation function of the metric fluctuations in the conformally Newtonian coordinate system, which are equal to the gauge-invariant quantities $\Phi=\Psi$, for the initial vacuum state (specified at $\eta=\eta_0$) is equal at an arbitrary time η to

$$\langle 0|\hat\Phi(\eta,x^\alpha)\hat\Phi(\eta,x^\alpha+r^\alpha)|0\rangle=\int|\delta_k|^2\frac{\sin kr}{kr}\frac{dk}{k}, \quad (47)$$

where

$$|\delta_k(\eta)|^2=\frac{1}{4\pi^2}\frac{\varphi_0'^2}{a^2}|u_k(\eta)|^2k^3 \quad (48)$$

characterizes the square of the amplitude of the perturbations on scales $\sim 1/k$. To find this amplitude, it is necessary to solve Eq. (43) for $u_k(\eta)$ with the initial conditions (46).

As an illustration, we calculate the spectrum of the perturbations in a universe that passes through the inflationary stage (7) during its evolution. Specifying at some initial time $\eta=\eta_0$ the vacuum spectrum, we find how it is deformed with the passage of time.

Solutions of Eq. (43) can be readily found in the asymptotic regions. For short-wave perturbations with $k^2\gg(1/z)''/(1/z)$

$$u_k(\eta)={}^0u_k\cos(k(\eta-\eta_0))+\frac{{}^0u_k'}{k}\sin(k(\eta-\eta_0)), \quad (49)$$

where ${}^0u_k=u_k(\eta_0)$, ${}^0u_k'=u_k'(\eta_0)$ in the long-wave region $[k^2\ll(1/z)''/(1/z)]$

$$u_k(\eta)=A_k\frac{1}{\varphi_0'}\left(\frac{1}{a}\int a^2\,d\eta\right)'. \quad (50)$$

In what follows, we shall be interested in only the part of the spectrum for which at the initial time the corresponding physical wavelengths (${}^0\lambda_{\rm ph}\sim a_0/k$) lie within the Hubble horizon $\sim 1/H_0$, where $H=\alpha/a$ is the Hubble constant. This is due to the fact that for sufficient duration of the inflationary stage large scales, ${}^0\lambda_{\rm ph}>1/H_0$, will at the present time be far outside the visible horizon. Therefore, the problem associated with the ambiguity in the choice of the vacuum state, which is important only for perturbations with ${}^0\lambda_{\rm ph}>1/H_0$ (on account of the ambiguity of the coefficients M and N in the expression (46) for the corresponding $k\eta_0$), can be ignored in the given case. As the universe expands, a perturbation with given comoving wave vector k passes through the horizon and then passes from the short-wave to the long-wave region. For such perturbations, the asymptotes (49) and (50) must be matched.

Comparing the asymptotic solutions of Eqs. (31) and (43) by means of (45), we see that in the interval $z''/z>k^2>(1/z)''/(1/z)$ (if it exists) both regimes (49) and (50) must be simultaneously valid. For the inflationary stage, this region is nonvanishing: $Ha\gg k\gg V_{,\varphi\varphi}^{1/2}a$ $(H\gg V_{,\varphi\varphi}^{1/2})$. Hence, for perturbations with wave vector k satisfying the condition

$$H(\eta)a(\eta)>k>H_0a_0,$$

we obtain

$$u_k(\eta)= \left({}^0u_k \cos(k\eta_0) - \frac{{}^0u_k'}{k}\sin(k\eta_0) \right) \left[\frac{1}{\varphi_0'}\left(\frac{1}{a}\int a^2\,d\eta\right)'\right]^{-1}_{k\sim Ha} \times \frac{1}{\varphi_0'}\left(\frac{1}{a}\int a^2\,d\eta\right)', \qquad (51)$$

where the subscript of the bracket means that its value is estimated at the time when the perturbation crosses the horizon.[3)] For the initial vacuum state, the coefficients ${}^0u_k$ and ${}^0u_k'$ in Eqs. (49) and (51) are given by the expressions (46). Substituting (49) and (51) in (48), estimating the integral in the brackets in the inflationary stage (7), and (5) taking into account, we find the perturbation spectrum at the time t:

$$|\delta_k| \approx \begin{cases} 2G\varphi_0(t), & k_{ph} > H(t) \\ 2G\left(\dfrac{\varphi_0 H^2}{\dot H}\right)_{k\sim Ha}\left(\dfrac{1}{a}\displaystyle\int a\,dt\right)^{\cdot}, & H(t) > k_{ph} > H_0\dfrac{a_0}{a(t)}, \end{cases} \qquad (52)$$

where the subscript $k\sim Ha$ means that the quantity in the corresponding bracket is estimated at the time $t(k_{\rm ph})$, when the perturbation corresponding to the physical scale $k_{\rm ph}\sim a(t)/k$ crosses the horizon $(H(t(k_{\rm ph}))\sim k_{\rm ph}a(t)/a(t(k_{\rm ph})))$; the dot denotes differentiation with respect to the physical time $t=\int a d\eta$. In deriving (52) we have ignored all terms of higher orders in k and noted that $|M|\to|N|\to 1$ for $k\eta_0 \gg 1$, and we have also used the normalization condition (32).

The inflationary stage ends at the time t_f, when $H^2(t_f) \sim V_{,\varphi\varphi}$ If the evolution of the universe after this is described by a law $a\propto t^n$ (in the model with $V=\frac{1}{2}m^2\varphi^2$, $n=2/3$), then, as follows from (52), in the interval of physical scales

$$V^{1/2}_{,\varphi\varphi}\frac{a(t_f)}{a(t)} > k_{ph} > H_0\frac{a_0}{a(t)}$$

(or corresponding comoving $V^{1/2}_{,\varphi\varphi}a(t_f) > k > H_0 a_0$) the amplitude at times $t > t_f$ is

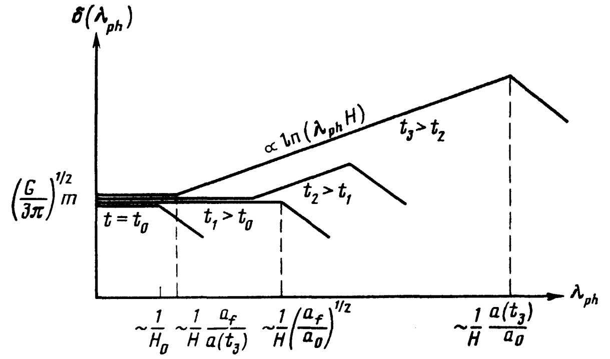

FIG. 1. Amplitude $\delta(\lambda_{\rm ph})$ of metric perturbations as a function of the physical wavelength $\lambda_{\rm ph}\sim a(t)/k$ at different times t during the quasi-de-Sitter stage of evolution of the universe in the model with potential $V=1/2m^2\varphi^2$. Here, t_0 is the initial time at which the vacuum spectrum of the perturbations is specified. Up to the time $t_1 > t_0$, at which $a(t_1)\sim(a_f a_0)^{1/2}$, where a_0 and a_f are, respectively, the scale factor at the beginning and end of the quasi-de-Sitter expansion, the spectrum remains flat to scales $\sim{}_a2(t)/Ha_0$. For $t > t_1$ a section appears in which the perturbation spectrum increases logarithmically into the region of large wavelengths $\lambda_{\rm ph}$; the length of this section increases exponentially. The analytic dependence of the amplitude $\delta(\lambda_{\rm ph})$ on the time t and the wavelength $\lambda_{\rm ph}$ in the stage of exponential expansion in the model has the form $\delta\approx(G/3\pi)^{1/2}m\left(1+\left(1+\ln\frac{a_f}{a(t)}\right)^{-1}\ln\lambda_{\rm ph}H\right)$ for $a(t)/Ha_0 > \lambda_{\rm ph} > 1/H$.

$$|\delta| \approx \frac{2G}{n+1}\left(\frac{\varphi H^2}{\dot H}\right)_{k\sim Ha}. \qquad (53)$$

In the given comoving scale $1/k$, the amplitude of the perturbations after inflation remains practically constant (it changes by a numerical coefficient of order unity only when there are changes of the effective equation of state). The evolution of the spectrum in the evolutionary stage is shown qualitatively in Fig. 1. The spectrum increases slightly in the region of small k. This is due to the fact that the smaller k the earlier the perturbation crosses the horizon and, accordingly, the value of $(\varphi H^2/\dot H)_{k\sim Ha}$ for it is larger (for example, in the model with $V=1/2m^2\varphi^2$ the spectrum increases logarithmically ($\propto \ln k$) in the region of small k).

The resulting spectrum at the time at which the inflationary stage ends [see (53)] agrees with the spectrum obtained earlier (apart from numerical coefficients).[7-10] Therefore, all the estimates for theories of the formation of galaxies remain practically unchanged. In particular, in the theory with $V=\frac{1}{2}m^2\varphi^2$ the amplitude of the metric perturbations on the scales of galaxies for $m\sim 10^{13}$ GeV is $\sim 10^{-4}$, a value that is quite sufficient for the formation of the structure in the universe.

On the basis of the formulas given above, it is readily seen that in models of the evolution of the universe without a prolonged inflationary stage it is not possible to obtain an amplitude of the metric perturbations on galactic scales sufficient for galaxy formation.

7. CONCLUSIONS

On the basis of the action for the scalar field and gravitation, we have constructed a Lagrangian formalism for small metric perturbations of longitudinal type and shown that their behavior can be completely described by the unique gauge-invariant variable $v=a(\delta\varphi+\varphi_0'\psi/\alpha)$. The action for the perturbations can be expressed completely in terms of this quantity, and it is therefore the natural degree of freedom to quantize. By themselves, the perturbations $\delta\varphi$ of the scalar field are not gauge invariant but depend on the choice of the coordinate system in which they are described. In addition, the total action cannot be expressed solely in terms of $\delta\varphi$. Therefore, the method usually employed to treat the quantum fluctuations as perturbations of only $\delta\varphi$ is not in general correct. Nevertheless, in the short-wave region, which is responsible for the formation of the spectrum in the inflationary stage, we have in the case of quasiexponential expansion of the universe $\varphi_0' \psi / a \ll \delta\varphi$. This partly explains why the correct result was "guessed" in the early studies. The qualitative picture of the formation of the spectrum does not agree with the ideas of some authors. Perturbations of the metric do not appear after the decay of the inflationary stage but exist always. In the process of inflation their amplitude grows and simultaneously, because of the exponential expansion, a broad spectral region in which the amplitude is effectively constant is formed. As we have shown above, the result for the flat part of the spectrum does not depend on the manner in which the vacuum of the field perturbations is defined.

[1)] Note that the background Friedmann model (2) is fixed and we consider small perturbations, for which the quadratic terms of the type $\delta g \Delta \xi$, etc., can be ignored.

[2)] This quantity is the analog of the variable that must be quantized in the case of hydrodynamic perturbations.[5,22]

[3)] In the time during which $\lambda_{\rm ph}(k, h)$ varies from $1/H$ to $1/M$, the change in the quantity in the square brackets in (51) during the inflationary stage is much less than its value.

[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[2] A. D. Linde, Phys. Lett. **B108**, 384 (1982).

[3] A. D. Linde, Phys. Lett. **B129**, 177 (1983).

A. A. Starobinsky, Phys. Lett. **B91**, 99 (1980).

[5] G. V. Chibisov and V. F. Mukhanov, Preprint No. 162 of P. N. Lebedev Inst. (1980); Mon. Not. R. Astron. Soc. **200**, 535 (1982).

[6] V. F. Mukhanov and G. V. Chibisov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 31 (1981) [JETP Lett. **33**, 28 (1981)]; Zh. Eksp. Teor. Fiz. **83**, 475 (1982) Sov. Phys. JETP **56**, 258 (1982)].

[7] A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); A. A. Starobinsky, Phys. Lett. **B117**, 175 (1982); S. W. Hawking, Phys. Lett. **B115**, 295 (1982).

[8] R. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).

[9] A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 496 (1984) [JETP Lett. **40**, 1333 (1984)].

[10] L. A. Kofman adn V. F. Mukhanov, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 481 (1986) [JETP Lett. **44**, 619 (1986)].

[11] L. A. Kofman, V. F. Mukhanov, and D. Yu. Pogosyan, Zh. Eksp. Teor. Fiz. **93**, 769 (1987) [Sov. Phys. JETP **66**, 433 (1987)].

[12] V. F Mukhanov L. A. Kofman, and D. U. Pogosyan Phys. Lett. **193**, 427 (1987).

[13] M. Sasaki Prog. Theor. Phys. **76**, 1036 (1986).

[14] V. A. Belinskii, L. P. Grishchuk, I. M. Khalatnikov, and Ya. B. Zel'dovich Zh. Eksp. Teor. Fiz. **89**, 346 (1985) [Sov. Phys. JETP **62**, 195 (1985)].

[15] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[16] B. S. DeWitt, in: *General Relativity*, edited by S. W. Hawking and W. Israel (Cambridge University Press, 1964).

[17] G. V. Chibisov and V. F. Mukhanov, Preprint No. 154 of P. N. Lebedev Inst. (1983).

[18] R. Arnowitt, S. Deser, and C. W. Misner, in: *Gravitation, an Introduction to Current Research*, edited by L. Whitten (New York-London, 1963).

[19] V. A. Fock, *The Theory of Space, Time, and Gravitation* (Oxford, 1964) [Russ. transl., Gostekhizdat, Moscow, 1955].

[20] A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, *Quantum Effects in Strong External Fields* [in Russian] (Atomizdat, Moscow, 1980).

[21] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. Ser. A: **360**, 117 (1978).

[22] V. N. Lukash, Zh. Eksp. Teor. Fiz. **79**, 1601 (1980) [Sov. Phys. JETP **52**, 807 (1980)].

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