Electron stochastization by a longitudinal sound wave in a metal

V. A. Burdov and V. Ya. Demikhovskiĭ

Scientific Physicotechnical Institute of the Gorky State University
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The damping of a longitudinal sound wave propagating in a metal in an oblique magnetic field under nonlinear conditions is investigated. If the resonant-particle groups defined by a specified condition do not overlap on the Fermi surface, the mechanism of nonlinear Landau damping by isolated resonances is realized. If the widths of the resonances exceed the distance between them, the electron motion becomes random and the distribution function is set by the diffusion and by the relaxation on the impurities. The transport equation is solved for either case, and the sound damping coefficient is obtained. It is shown that in the regime of isolated resonances the coefficient of sound damping on the l th resonance is proportional to a small parameter having the meaning of the ratio of the collision frequency to the oscillation frequency of the trapped particles. In the global stochastization regime, the parameter that determines the decrease of the damping is ratio of the size of the stochastization region to the diffusion momentum $P_0 = (D\tau_r)^{1/2}$, where D is the diffusion coefficient and τ_r is the relaxation time.

I. INTRODUCTION

Numerous oscillatory phenomena observed when sound and electromagnetic waves propagate in metals placed in a magnetic field are due to resonant interaction of the wave with the electrons. The particles at resonance with the waves are those whose velocity projection on the direction of the magnetic field $\mathbf{H}\|z$ satisfies the condition

$$k_z v_{zl} - \omega = l\Omega, \quad l = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

and the particles most strongly interacting with the wave are those whose Larmor orbits are multiples of $\pi/k_\perp$. Here k_z and $k_\perp$ are the wave-vector components, ω the wave frequency, and Ω the cyclotron frequency.

In the linear theory that describes the damping of waves and the renormalization of their spectrum it is assumed that the charged-particle trajectories in the wave fields are insignificantly distorted during the time between electron collisions. With increase of amplitude, the most changed are the trajectories of those electrons for which condition (1) is met. The velocity of these particles in the wave field oscillates about the resonant values determined by (1), and the oscillation frequency at the l th resonance turns out to be

$$\tilde{\omega}_l = k_z [\Phi_0 J_l(k_\perp \rho)/m^*]^{1/2}, \tag{2}$$

where Φ_0 is the amplitude of the electron potential energy in the sound-wave field, m^* is the effective mass, J_1 is a Bessel function, and ρ is the Larmor radius. This result can be obtained with the aid of the so-called resonant perturbation theory. We shall not derive Eq. (2) here, and refer the reader to the known monograph.[1]

If the particle oscillation period is less than the electron relaxation time, one can say that the particle is trapped in the l th resonance. Of cource, trapping at $l = 0$ differs from that at nonzero l. We consider first the trajectories of particles trapped at zero resonance ($l = 0$). The trapped particles neither overtake nor lag the wave and move on the average with it, at a velocity in the interval

$$|v_z - v_{z0}| < \Delta v_0.$$

Here Δv_0 is the resonance broadening due to the finite amplitude of the wave. The entire Larmor orbit of the trapped particles oscillates here, as it were, in the wave field with a frequency $\tilde{\omega}_0$. This is also the oscillation frequency of the velocity v_z and the resonant phase α, equal to $k_z z - \omega t$. It must be emphasized that in trapping by a zeroth resonance the Larmor orbit may also not be subtended by a single potential well. What is important for such orbits, however, is that where $\mathbf{k}\cdot\mathbf{v} = \omega$ and where the electron interacts most strongly with the wave the turning point oscillates within the limits of one potential well. As to detrapped particles, their velocity is outside the trapping region; they either overtake the wave or lag it.

In the l th resonance, as a result of trapping, particles having a velocity

$$|v_z - v_{zl}| < \Delta v_l$$

negotiate on the average l wavelengths per period. The oscillations here hardly differ from those corresponding to $l = 0$. The only difference is that after the revolution the particle will no longer land in the potential well from which the magnetic field extracted it. The position of the point $\mathbf{k}\cdot\mathbf{v} = \omega$ in the new potential well will be shifted relative to the preceding one by the sound wave. Thus, the particle motion is weakly modulated in the magnetic-field direction and the modulation frequency is $\tilde{\omega}_0$.

This pattern of capture by resonances with different l obtains only in the case when the angle ϑ betwen the wave vector $\mathbf{k}$ and the vector $\mathbf{H}$ is not close to $\pi/2$. In the opposite case there are no electrons with resonant values of v_z, but modulation of the motion in a plane perpendicular to the magnetic field may turn out to be substantial. This case calls for a separate analysis and will not be discussed here.

According to contemporary notions, motion is regular in multidimensional dynamic systems only when the resonances do not overlap. Thus, the resonances (1) do not overlap if the distance between them along v_z, which is equal to Ω/k_z, exceeds the combined broadening of $\Delta v_{l+1} + \Delta v_l$ of two nearest resonances; the broadening can be determined with the aid of (2) to be

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$\Delta v_l = 2\tilde{\omega}_l/k_z$.

If the inverse condition holds, i.e., the resonances overlap, the motion acquires a stochastic character. The overlap condition (the Chirikov criterion) takes in the case considered the form[1)] (Ref. 2)

$$K = 2(\tilde{\omega}_{l+1} + \tilde{\omega}_l)/\Omega > 1. \tag{3}$$

We investigate in the present paper the damping coefficient of a longitudinal acoustic wave propagating in a metal in an oblique magnetic field.

It is known that in the linear regime the damping coefficient of such a wave undergoes magnetoacoustic oscillations when the value of **H** is changed.[3] We consider damping in the regime of capture by isolated resonances and in the regime of global stochasticity. We show that with contemporary experimental techniques a wave can randomize an appreciable part of the electron Fermi surface.

In Sec. 2 we consider sound absorption by isolated resonances. The Boltzmann kinetic equation is written in terms of the resonance action and angle variables; one of the phases (the slow one) describes oscillation at resonance, and the other (fast) describes cyclotron rotation. After averaging the kinetic equation over the fast phase, we obtain the distribution function in the l th resonance and determine the wave damping. Resonant particle groups corresponding to different l make an additive contribution to the wave absorption, and the contribution from each resonance decreases by an approximate factor $2a_l$ compared with the linear regime, where $a_l = (\tilde{\omega}_l \tau_r)^{-1}$.

The third section is devoted to sound damping under conditions of global stochasticity of the electron trajectories, in a regime in which several resonances overlap. The distribution function is established in this case through diffusion of the electrons in momentum space (the diffusion is a consequence of stochastic mixing of the phase trajectories), and as a result of relaxation to a localequilibrium Fermi distribution function. In this case the kinetic energy is reduced, after averaging over all phases, to an equation of the diffusion type, which is then solved. In addition, linear corrections are obtained for the averaged distribution function and determine the sound damping. The dpendence of the damping coefficient on the magnetic field and on the wave amplitude is obtained.

In the concluding Sec. 4 we discuss the results of an analytic and a numerical calculation of the damping coefficient in the regime of isolated resonances and in the regime of global stochasticity.

2. ABSORPTION BY ISOLATED RESONANCES

We calculate the absorption coefficient of longitudinal sound propagating at an angle to the magnetic-field directions, under conditions of isolated resonant electron groups. Following Ref. 4, we express the Hamiltonian of an electron in a uniform magnetic field **H** and in the field of a sound wave in terms of the canonical variables $\mathbf{q}(z,\varphi,Y)$ and $\mathbf{p}(p_z, p_\varphi, m^*\Omega X)$, defined as follows:

$$\begin{aligned} &\varphi = \operatorname{arctg}\,[(p_x + m^*\Omega y)/p_y], \\ &P_\varphi = [(p_x + m^*\Omega y)^2 + p_y^2]/2m^*\Omega, \\ &Y = -p_x/m^*\Omega, \quad X = x + p_y/m^*\Omega, \end{aligned} \tag{4}$$

where φ is the gyrophase, X and Y are the coordinates of the rotation center, and the generalized momentum P_φ is connected with the gyro-radius ρ and with the magnetic moment by

$$P_\varphi = \frac{m^*\Omega\rho^2}{2} = \frac{m^*c}{e}\mu. \tag{5}$$

In the new variables, the Hamiltonian takes the form

$$H = p_z^2/2m^* + \Omega P_\varphi - \Phi_0 \cos(k_z z - k_\perp \rho \sin\varphi - k_\perp Y - \omega t). \tag{6}$$

According to (6), the total Hamiltonian H is independent of the generalized momentum $m^*\Omega X$, so that $Y = \text{const}$. Next, transforming to a coordinate frame moving with velocity $v_z = \omega/k_z$ in the direction of the field **H**, and introducing the variables $\psi = k_z z - \omega t$ and the conjugate momentum $P = p_z/k_z$, and using the generating function

$$F = (k_\perp Y + k_z z - \omega t)P + P_\varphi \varphi$$

we obtain

$$\mathcal{H} = \frac{k_z^2 P^2}{2m^*} - \omega P + \Omega P_\varphi - \Phi_0 \cos(\psi - k_\perp \rho \sin\varphi). \tag{7}$$

Finally, using the expansion

$$\exp(ia\sin\varphi) = \sum_n J_n(a)\exp(in\varphi),$$

we get ultimately

$$\mathcal{H} = \frac{k_z^2 P^2}{2m^*} - \omega P + \Omega P_\varphi - \Phi_0 \sum_n J_n(k_\perp\rho)\cos(\psi - n\varphi). \tag{8}$$

The new variables introduced here, of the action-angle type, are convenient also for the solution of the Boltzmann kinetic equation

$$\frac{\partial f}{\partial t} + \dot{P}\frac{\partial f}{\partial P} + \dot{P}_\varphi \frac{\partial f}{\partial P_\varphi} + \dot{\psi}\frac{\partial f}{\partial \psi} + \dot{\varphi}\frac{\partial f}{\partial \varphi} = -\frac{f - f_0}{\tau_r}, \tag{9}$$

where τ_r is the electron relaxation time, and P, P_φ, ψ, and φ satisfy, according to (8), the equations

$$\dot{P} = -\Phi_0 \sum_n J_n(k_\perp\rho)\sin(\psi - n\varphi), \qquad \dot{\psi} = k_z^2 P/m^* - \omega,$$

$$\dot{P}_\varphi = \Phi_0 \sum_n n J_n(k_\perp\rho)\sin(\psi - n\varphi),$$

$$\dot{\varphi} = \Omega - \Phi_0 \sum_n \frac{dJ_n}{dP_\varphi}\cos(\psi - n\varphi). \tag{10}$$

Putting, as usual, $f = f_0(\varepsilon + V) + g$, where $f_0(\varepsilon + V)$ is the Fermi local-equilibrium distbution function ε and the energy we obtain for g the equation

$$\frac{\partial g}{\partial t} + \dot{P}\frac{\partial g}{\partial P} + \dot{P}_\varphi \frac{\partial g}{\partial P_\varphi} + \dot{\psi}\frac{\partial g}{\partial \psi} + \dot{\varphi}\frac{\partial g}{\partial \varphi} = -\frac{g}{\tau_r} - \frac{df_0}{dt}. \tag{11}$$

To solve the kinetic equation in the region of an l th isolated resonance it is convenient to change in (11) and (8), as in resonant perturbation theory, to new resonant variables using the generating function $F_2 = (\psi - l\varphi)P_\alpha + \varphi P_\beta$:

$$\alpha = \psi - l\varphi, \quad P_\alpha = P, \quad \beta = \varphi, \quad P_\beta = P_\varphi + lP. \tag{12}$$

The new phase α is a slow variable in the l th resonance, and β is the fast one. In terms of the resonance variables, the Hamiltonian of our system and the kinetic equation for the function g take the form

$$\mathcal{H}=\frac{k_z^2P_\alpha^2}{2m^*}-(\omega+l\Omega)P_\alpha+\Omega P_\beta-\sum_n V_n\cos[\alpha-(n-l)\beta], \tag{8a}$$

$$\frac{\partial g}{\partial t}+\sum_n (n-l)V_n\sin[\alpha-(n-l)\beta]\frac{\partial g}{\partial P_\beta}$$

$$-\sum_n V_n\sin[\alpha-(n-l)\beta]\frac{\partial g}{\partial P_\alpha}$$

$$+\left\{\Omega-\sum_n V_n^\beta\cos[\alpha-(n-l)\beta]\right\}\frac{\partial g}{\partial\beta}$$

$$+\left\{\frac{k_z^2P_\alpha}{m^*}-\omega-l\Omega-\sum_n V_n^\alpha\cos[\alpha-(n-l)\beta]\right\}\frac{\partial g}{\partial\alpha}+\frac{g}{\tau_r}$$

$$=\omega f_0'\sum_n V_n\sin[\alpha-(n-l)\beta], \tag{13}$$

where

$$V_n=\Phi_0J_n(k_\perp\rho),\ \rho=[2(P_\beta-lP_\alpha)/m^*\Omega]^{1/2},$$
$$V_n^\beta=\partial V_n/\partial P_\beta,\ V_n^\alpha=\partial V_n/\partial P_\alpha,\ f_0'=df_0/d\varepsilon.$$

We represent the distribution function g in the region of the l th resonance by the series

$$g=\sum_n g_n(P_\alpha,P_\beta,\alpha)e^{in\beta}, \tag{14}$$

where the term g_0 is the slow part of g. Substituting the series (14) in the kinetic equation (13) and averaging over the fast phase β, we obtain for g_0 the equation

$$\frac{\partial g_0}{\partial t}+\left(\frac{k_z^2P_\alpha}{m^*}-\omega-l\Omega\right)\frac{\partial g_0}{\partial\alpha}$$

$$-\sum_n\frac{V_n^\alpha}{2}\left(\frac{\partial g_{n-l}}{\partial\alpha}e^{i\alpha}+\frac{\partial g_{l+n}}{\partial\alpha}e^{-i\alpha}\right)$$

$$-\frac{i}{2}\sum_n(n-l)V_n^\beta(g_{n-l}e^{i\alpha}-g_{l-n}e^{-i\alpha})$$

$$+\sum_n(n-l)\frac{V_n}{2i}\left(\frac{\partial g_{n-l}}{\partial P_\beta}e^{i\alpha}-\frac{\partial g_{l-n}}{\partial P_\beta}e^{-i\alpha}\right)$$

$$-\sum_n\frac{V_n}{2i}\left(\frac{\partial g_{n-l}}{\partial P_\alpha}e^{i\alpha}-\frac{\partial g_{l-n}}{\partial P_\alpha}e^{-i\alpha}\right)+\frac{g_0}{\tau_p}=\omega f_0'V_l\sin\alpha. \tag{15}$$

We can similarly obtain an equation for the coefficients g_n ($n\neq0$) which determine the rapid part of the distribution function. Multiplying (13) by $\exp(-in\beta)$ and averaging over β, we have

$$\frac{\partial g_n}{\partial t}+\left(\frac{k_z^2P_\alpha}{m^*}-\omega-l\Omega\right)\frac{\partial g_n}{\partial\alpha}$$

$$-\sum_m\frac{V_m^\alpha}{2}\left(\frac{\partial g_{m-l+n}}{\partial\alpha}e^{i\alpha}+\frac{\partial g_{l-m+n}}{\partial\alpha}e^{-i\alpha}\right)$$

$$+i\Omega ng_n-\frac{i}{2}\sum_m V_m^\beta[(m-l+n)g_{m-l+n}e^{i\alpha}$$

$$+(l-m+n)g_{l-m+n}e^{-i\alpha}]$$

$$+\sum_m(m-l)\frac{V_m}{2i}\left(\frac{\partial g_{m-l+n}}{\partial P_\beta}e^{i\alpha}-\frac{\partial g_{l-m+n}}{\partial P_\beta}e^{-i\alpha}\right)$$

$$-\sum_m\frac{V_m}{2i}\left(\frac{\partial g_{m-l+n}}{\partial P_\alpha}e^{i\alpha}-\frac{\partial g_{l-m+n}}{\partial P_\alpha}e^{-i\alpha}\right)$$

$$+\frac{g_n}{\tau_p}=\frac{\omega f_0'}{2i}(V_{l-n}e^{i\alpha}-V_{l+n}e^{-i\alpha}). \tag{16}$$

It follows from (15) and (16) that in the resonant region the function g_0 is of the order of $\Phi_0^{1/2}$ while g_n is proportional to the wave amplitude. Consequently the main contribution to the absorption is made by the function g_0, the only function we calculate here.[2)] When estimating the order of magnitude of the terms contained in (15) and (16) it can be assumed that the coefficients V_n which are functions of the gyroradius $\rho(P_\alpha,P_\beta)$, are taken at points correspond to resonance values $P_{\alpha l}=m^*(\omega+l\Omega)/k_z^2$. Retaining in the equation for g_0 the terms of order Φ_0 and transforming to the dimensionless momentum $s=(P_\alpha-P_{\alpha l})/\bar{P}_l$ we express (15) in the form

$$s\frac{\partial g_0}{\partial\alpha}-\frac{\partial g_0}{\partial s}\sin\alpha+a_lg_0=f_0'\bar{P}_l\omega\sin\alpha, \tag{17}$$

where we have introduced the dimensionless parameter $a_l=m^*/k_z^2\tau_r\bar{P}_l$, and $\bar{P}_l=(\Phi_0J_lm^*)^{1/2}/k_z$ is the characteristic momentum at the l th resonance.

The equations of the characteristics of the kinetic equation (17) have an integral $\mathcal{H}=(s^2/2)-\cos\alpha$, which is the Hamiltonian that describes the averaged motion at the resonance. The solution of (17) for $a_l\ll1$ in the trapped-particle region is a function of the form (see Ref. 5)

$$g_0^t=\bar{P}_l\omega f_0'[a_l\alpha-s], \tag{18}$$

and in the region of the detrapped particles, the function

$$g_0^{ut}=\bar{P}_l\omega f_0'[a_l(\alpha-\bar{\alpha})-(s-\bar{s})], \tag{19}$$

where

$$\bar{\alpha}=\pi F(\alpha/2,\varkappa)/K(\varkappa),\ \bar{s}=\pi/\varkappa K(\varkappa),$$

$F(\alpha/2,\varkappa)$ is an incomplete elliptic integral of the first kind, $K(\varkappa)$ is a complete elliptic integral of the first kind, $\varkappa$ and is defined by the relation $\varkappa^{-2}=(s^2/4)+\sin^2(\alpha/2)$. Trapped particles correspond to $|\varkappa|>1$ and detrapped ones to $|\varkappa|<1$. It should be noted that an equation for the slow part of the distribution function g_0 can be obtained also with the aid of the Hamiltonian (8a) averaged over the rapid phase β.

Having found the distribution function, it is easy to solve the problem of absorption of a longitudinal sound wave

by resonant electrons. We use for this purpose the energy-balance equation in the wave + particle system:

$$\frac{dS}{d\zeta}=-\left\langle \frac{\partial V}{\partial t}n_p\right\rangle =-2\Gamma S, \tag{20}$$

where Γ is the damping coefficient, $V=\Sigma_n V_n\cos(\psi-n\varphi)$ is the potential energy, the angle brackets denote averaging over the volume, $n_r=[2/(2\pi\hbar)^3]\int g d\mathbf{p}$, is the nonequilibrium density of the resonant particles, S is the average flux density of the sound wave and is equal to $\rho_0 u^2\omega^2 c/2$, c is the speed of sound, ρ_0 is the density of the metal, u is the amplitude of the latice-atom displacement, and ζ is the coordinate in the wave propagation direction. Carrying out the necessary calculations with the aid of (20), and also (18) and (19), we can express the nonlinear damping coefficient of the sound wave by the l th group of resonant particles in the form

$$\Gamma^l=\Gamma_L^l(\gamma_l+\gamma_{ul}), \tag{21}$$

where

$$\Gamma_L^l=(n_0 m^* \pi v_F k/12\rho_0 c\cos\theta)J_l^2(x_l)$$

is the linear damping coefficient in the l th resonance, n_0 the density, v_F the Fermi velocity,

$$x_l=(k_\perp/\Omega)(v_F^2-(\omega+l\Omega)^2/k_z^2)^{1/2},$$

$$\gamma_l+\gamma_{ul}=\frac{128}{9\pi^2}a_l\left(1+\frac{9}{4}\int_0^1\frac{d\varkappa}{\varkappa^4}\left[E(\varkappa)-\frac{\pi^2}{4K(\varkappa)}\right]\right)\approx 2a_l,$$

and $E(\varkappa)$ a complete elliptic integral of the second kind. The total nonlinear absorption coefficient can be written as the sum over all groups of resonant particles, i.e.,

$$\Gamma=\frac{\Gamma_L(\mathbf{H}=0)}{\cos\vartheta}\sum_{n=-N}^{N}J_n^2(x_n)2a_n. \tag{22}$$

Here ϑ is the angle between $\mathbf{H}$ and $\mathbf{k}$, and N is the number of resonances on the Fermi surface and is determined from the inequalities $N\Omega/k_z<v_f<(N+1)\Omega/k_z$.

Equation (22) describes nonlinear sound absorption in isolated resonances. The nolinearity parameters $a_l\ll 1$ determine the ratio of the scattering frequency to the oscillation frequency of the trapped particles in the l th resonance. If $a_l\gtrsim 1$ for some resonance, the corresponding contribution to the absorption is described by the linear theory. Equations for Γ in the transition regime, when $a_l\sim 1$ can be obtained in analogy with Ref. 5. The explicit form of the dependence of Γ on the magnetic field is discussed in Sec. 4.

For electrons in a constant magnetic field and in a wave field, resonances defined by condition (1) are not the only possible ones. In higher orders of resonant perturbation theory it is possible to obtain the so-called fractional resonances, to which are subjected particles with a resonant velocity v_z satisfying the condition

$$k_z v_z-\omega=n\Omega/m, \tag{23}$$

where n and m are integers. In second order (in the amplitude Φ_0) of peturbation theory there occur half-integer resonances ($n=\pm 1,\pm 3\ldots$, $m=2$). The characteristic width of these resonances is $\propto\Phi_0$, whereas for primary resonances it is $\propto\Phi_0^{1/2}$. In n-th order perturbation theory, the width of the corresponding resonance is $\propto\Phi_0^{n/2}$. One can expect the character of sound absorption in half-integer and other fractional resonances to depend on the parameter $a_{nm}=(\tilde\omega_{nm}\tau_r)^{-1}$. Thus, for example, $\tilde\omega_{n2}$ is the frequency of particle oscillations in half-integer resonance. In a metal we actually have $a_{nm}\geqslant 1$, and in this case it can be shown that the additional damping is $\Gamma\sim\Phi_0^2$. It should be noted, however, that fractional resonances can alter the stochastization criterion (3); allowance for them leads to overlap at smaller sound amplitudes.

Besides the main and fractional resonances, it is known that in our system there exist secondary resonances defined by the condition

$$n\tilde\omega_l=m\Omega, \tag{24}$$

where $\tilde\omega_l$ is the oscillation frequency in the l th primary resonance, and the resonances realized for the most part are those with $m=1$ and $n\gg 1$. Secondary resonances form on the phase plane closed garlands of islands inside the primary resonances. The particle oscillation frequencies in a secondary resonance is, according to Ref. 1, of the order of

$$\tilde\omega_s\sim\tilde\omega_l(n!)^{-1/2}, \tag{25}$$

i.e., $\tilde\omega_s\ll\tilde\omega_l$. Consequently, secondary resonances can influence the absorption only at large amplitudes, when the condition $\omega_s\tau_r\gg 1$ is met. Estimates show, however, that in this case the overlap criterion is already met for primary resonances, i.e., the motion is stochastic. We therefore disregard seconary resonances.

3. SOUND DAMPING IN THE GLOBAL STOCHASTICITY REGIME

It follows from the results of the preceding section that when resonances corresponding to different l are isolated, the electron motion has a regular character. The reason is that at low amplitudes there exist in a two-dimensional system two integrals of motion. With increase of amplitude, one of them begins to disintegrate. Inasmuch as invariant surfaces that limit the motion vanish here, the trajectories become substantially more complicated. A detailed description of the motion along the trajectories becomes impossible in this case, since the motion is stochastic. In this regime, as shown by numerous studies of stochastic dynamics of particles (see, e.g., Ref. 6), a function averaged over the phases satisfies an equation of the diffusion type. In our problem the kinetic equation in the stochastic regime also reduces to an equation of the diffusion type.

Let us find the distribution function under conditions when the cyclotron resonances corresponding to different l overlap, i.e. the Chirkikov criterion (3) is met. It becomes then incorrect to calculate the distribution function by the procedure of separating the fast and slow phases in an isolated resonance, since the electron motion acquires a stochastic character and the subdivision of the motion into fast and slow is impossible. For electrons in the stochastic regime it is necessary to retain all the terms of the series defined by (10) in the expressions for $\dot P$ and $\dot P_\varphi$ that enter in the kinetic equation (9). The latter takes then the form

$$\frac{\partial f}{\partial t}+\sum_n\left[\omega_n+\frac{1}{2}\sum_m n\frac{dV_m}{dP_\varphi}(e^{i\vartheta_m}+e^{-i\vartheta_m})\right]\frac{\partial f}{\partial\vartheta_n}$$
$$+\frac{i}{2}\sum_n V_n(e^{i\vartheta_n}-e^{-i\vartheta_n})\frac{\partial f}{\partial P}$$
$$-\frac{i}{2}\sum_n nV_n(e^{i\vartheta_n}-e^{-i\vartheta_n})\frac{\partial f}{\partial P_\varphi}=\frac{f_0-f}{\tau_r},\tag{26}$$

where $\omega_n=(k_z^2P/m^*)-\omega-n\Omega$, the phase $\vartheta_n=\psi-n\varphi$, $f\equiv f(P,P_\varphi,...\vartheta_{-n}...\vartheta_n,...,t)$, and the amplitudes V_n have here the same meaning as in Sec. 2. We seek the solution of (26) in series form:

$$f=\sum_{(\mathbf{m})}f^{(\mathbf{m})}e^{i(\mathbf{m};\vartheta)},\tag{27}$$

$(\mathbf{m};\vartheta)=...+m_{-n}\vartheta_{-n}+...+m_n\vartheta_n+...$ The symbol $(\mathbf{m})$ denotes a vector with components $(...,m_{-n}...,m_n...)$. Under stochastization conditions we retain in the distribution function (27) the zeroth component $f^{(0)}$, which is the distribution function averaged over all the phases, as well as the terms linear in the amplitude V_n. The system of equations for the expansion coefficients $f^{(0)}$ and $f^{(\pm 1n)}$ can be obtained from (26) and (27) by integrating (26) over all the phases ϑ_n. As a result we have

$$\frac{\partial f_0}{\partial t}+\frac{i}{2}\sum_n D_n[V_n(f^{(-1n)}-f^{(1n)})]=-\frac{f^{(0)}-f_0(\varepsilon)}{\tau_r},\tag{28}$$

$$\frac{\partial f^{(1n)}}{\partial t}+i\omega_nf^{(1n)}+\frac{i}{2}V_nD_nf^{(0)}=-\frac{f^{(1n)}+V_nf_0'/2}{\tau_r},\tag{29a}$$

$$\frac{\partial f^{(-1n)}}{\partial t}-i\omega_nf^{(-1n)}-\frac{i}{2}V_nD_nf^{(0)}=-\frac{f^{(-1n)}+V_nf_0'/2}{\tau_r},\tag{29b}$$

where the operator D_n is defined as

$$D_n=\frac{\partial}{\partial P}-n\frac{\partial}{\partial P_\varphi},\qquad f_0'=\frac{df_0}{d\varepsilon}.$$

The vector $(\pm 1n)$ which specifies the function $f^{(\pm 1n)}$ has components $m_l=0$ for $n\neq 1$ and $m_n=\pm 1$. The equation for the averaged distribution function is exact, and only the terms linear in V_n are retained in Eqs. (29a) and (29b). In the stationary regime, the derivatives with respect to time can be left out of (28), (29a) and (29b). The solutions (29a) and (29b) take then the form

$$f^{(\pm 1n)}=-\frac{V_n}{2}\frac{[D_nf^{(0)}\mp(i/\tau_p)f_0']}{\omega_n\mp i/\tau_r}.\tag{30}$$

Equation (28) can now be reduced with the aid of (30) to a two-dimensional diffusion equation in the space P, P_φ:

$$\frac{\pi}{2}\sum_n D_n[V_n^2\delta(\omega_n)D_nf^{(0)}]=\frac{f^{(0)}-f_0(\varepsilon)}{\tau_r}.\tag{31}$$

Since P and P_φ are connected by a conservation law, the diffusion equation (31) can be reduced to one-dimensional form by introducing the new variables

$$P=P,\ E=(k_z^2P^2/2m^*)-\omega P+\Omega P_\varphi.$$

Indeed, since the diffusion is over the equal-energy surface $E=\text{const}$, the derivative with respect to E vanishes and we have in place of (31)

$$\frac{\partial}{\partial P}\left(D\frac{\partial f^{(0)}}{\partial P}\right)=\frac{f^{(0)}-f_0(\varepsilon)}{\tau_r},\tag{32}$$

where

$$D=\frac{\pi}{2}\Phi_0^2\sum_n J_n^2(k_\perp\rho)\delta\left(\frac{k_zP}{m^*}-\omega-n\Omega\right)\tag{32a}$$

is the one-dimensional diffusion coefficient. The solution (32) should satisfy the boundary conditions

$$D(\partial f^{(0)}/\partial P)_{P_1}=D(\partial f^{(0)}/\partial P)_{P_2}=0,\tag{33}$$

which mean that there is no particle flow through the boundaries of the stochastization region. Here P_1 and P_2 are the values of the momenta corresponding to these boundaries. The stochastization region in which the criterion (3) is satisfied can, generally speaking, have a complicated form. The reason is that the widths of the resonances change nomonotonically with increase of the number l. With increase of the amplitude, the regions primarily stochasticized are those near the two so-called boundary points ($P_b=\pm(m^*v_F/k_z)\sin\vartheta$), since the Bessel function that determines according to (2) the resonance width is a maximum here. The reason for the last circumstance is that the argument of the Bessel function at these points is of the order of its number. Outside the boundary points the widths of the resonance curves begin to decrease rapidly, since the argument of the Bessel function becomes smaller than the number of the function. As shown by our numerical calculations (see Sec. 4), at the presently attainable sound-input intensities of in a metal, the entire Fermi-surface region betwen the boundary points can become stochasticized. In a semimetal at the same amplitude, the entire Fermi surface can be stochasticized at the same wave amplitude. In the foregoing calculations, the roles of P_1 and P_2 are assumed by the values $P_b=\pm(m^*v_F/k_z)\sin\vartheta$, in a typical metal and by the values $\pm P_F$ in a semimetal.

We examine now the expression for the diffusion coefficient. The appearance of a δ function in (32a) is known[6] to be due to the use of a linear approximation For a more accurate description of the electron motion under stochastization conditions, it is necessary to replace the δ function by the quantity

$$\Delta_n=\{\pi\tau_c[\tau_c^{-2}+(k_zv_z-\omega-n\Omega)^2]\}^{-1},$$

where τ_c the time in which the memory of the initial conditions is lost (the time of decoupling of the correlations). Since the estimate[6] $\tau_c^{-1}\sim\Omega\ln k$ is valid for the correlation time, the broadening of the function at $K\gg 1$ becomes larger than the distance betwen the resonances, and the diffusion ceofficient becomes a smooth function of the momentum P. It can therefore be averaged over the stochasticity region. The average diffusion coefficient turns out to be

$$D=(\pi\Phi_0^2/2\Omega)I,\tag{32b}$$

where $I=\langle J_l^2(k_\perp\rho)\rangle$, and the angle brackets denote double averaging over P (over the region $2m^*/k_z\tau_c$ of smearing of the δ functions) and over n.

We obtain now the solution of the diffusion equation (32) for a fixed value of E. The region in which the Fermi distribution function $f_0 = \vartheta(\varepsilon_F - E - \omega P)$ is not zero is determied with the aid of the inequalities

$$E \leqslant \varepsilon_F + \omega P, \quad E - \frac{k_z^2 P^2}{2m^*} + \omega P \geqslant 0$$

and is shown in Fig. 1 (thick line). The second inequality follows from the condition $P_\varphi \geqslant 0$. In region I of Fig. 1, the solution of (32) satisfying the boundary condition (33) is identically zero, while in region III the solution coincides with the Fermi function. For fixed E in region II, the right-hand side of (32) is discontinuous at the point $P = P_E = (\varepsilon_F - E)/\omega$ and it is natural to require that the function $f^{(0)}$ and its derivative $\partial f^{(0)}/\partial P$ be continuous at this point. The solution takes then the form

$$f_1^{(0)} = \frac{\mathrm{th}(P_E/P_0) - \mathrm{th}(P_1/P_0)}{\mathrm{th}(P_2/P_0) - \mathrm{th}(P_1/P_0)} \mathrm{ch}\frac{P_E}{P_0}\left(\mathrm{ch}\frac{P}{P_0} - \mathrm{th}\frac{P_2}{P_0}\mathrm{sh}\frac{P}{P_0}\right),$$

$$P > P_E,$$

$$f_2^{(0)} = \frac{\mathrm{th}(P_E/P_0) - \mathrm{th}(P_2/P_0)}{\mathrm{th}(P_2/P_0) - \mathrm{th}(P_1/P_0)} \mathrm{ch}\frac{P_E}{P_0}\left(\mathrm{ch}\frac{P}{P_0} - \mathrm{th}\frac{P_1}{P_0}\mathrm{sh}\frac{P}{P_0}\right) + 1,$$

$$P < P_E, \tag{34}$$

where P_0 is the diffusion momentum. The obtained averaged function (34) determines, according to (30), the functions $f^{(\pm 1n)}$ with the aid of which the sound damping coefficient can be calculated under stochastization conditions. Substituting (34) in (30) and taking (27) into account, we obtain with the aid of the balance equation (20)

$$\Gamma = \frac{\Gamma_L(\mathbf{H}=0)}{\cos\vartheta(\mathrm{th}x_2 - \mathrm{th}x_1)} \times \sum_{n=-N_{max}}^{N_{max}} \Big[(\mathrm{sh}\,x_n - \mathrm{th}\,x_1 \mathrm{ch}\,x_n) \int_{x_n}^{x_2} dx\, J_n^2(k_\perp \rho_n) \times (\mathrm{th}\,x_2 \mathrm{ch}\,x - \mathrm{sh}\,x) + (\mathrm{sh}\,x_n - \mathrm{th}\,x_2 \mathrm{ch}\,x_n) \times \int_{x_1}^{x_n} dx\, J_n^2(k_\perp \rho_n)(\mathrm{th}\,x_1 \mathrm{ch}\,x - \mathrm{sh}\,x) \Big],$$

$$k_\perp \rho = \frac{k_\perp}{\Omega}\left[\frac{2\omega P_0}{m^*}\left(\frac{\varepsilon_F}{\omega P_0} - \frac{k_z^2 P_0 x_n^2}{2m^*\omega} + x_n - x\right)\right]^{1/2}, \quad x_n = \frac{P_n}{P_0},$$

$$x_1 = \frac{P_1}{P_0}, \quad x_2 = \frac{P_2}{P_0}, \tag{35}$$

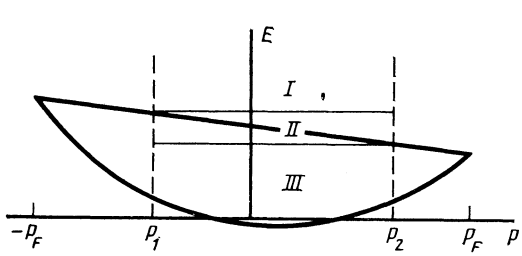

FIG. 1. The electron distribution function in region I is zero, in region II the electrons are randomized, and in region III the distribution function is of the Fermi type.

Here N_{max} is the maximum number of the resonance located in the stochastization region. Expression (35) for the absorption coefficient is quite complicated, but it can be substantially simplified by using the fact that in the integration interval (x_1, x_2) the arguments of the Bessel functions contained in the sum change insignificantly. Therefore, regarding $J_n(k_\perp \rho_n)$ as a constant and integrating with respect to x we obtain

$$\Gamma = \frac{\Gamma_L(\mathbf{H}=0)}{\cos\vartheta} \sum_{n=-N_{max}}^{N_{max}} J_n^2(x_n)\left[1 - \mathrm{ch}\frac{m^*\Omega n}{k_z^2 P_0}\left(\mathrm{ch}\frac{P_2 - P_1}{2P_0}\right)^{-1}\right],$$

$$x_n = \left[v_F^2 - \frac{(\omega + n\Omega)^2}{k_z^2}\right]^{1/2}\frac{k_\perp}{\Omega}, \tag{36}$$

where $P_2 - P_1$ is the width of the stochastization region and $P_0 = (D\tau_r)^{1/2}$ is the characteristic distance in momentum space over which the particles diffuse within a time τ_r.

Equations (35) and (36) take into account the contribution made to the wave damping by only the stochasticized electrons. If the stochastization does not apply to the entire Fermi surface, it is necessary to add to Γ defined by (35) and (36) terms that describe the damping by isolated resonances that do not land in the stochastization region. The corresponding increment is determined by Eq. (22).

The discussion of the damping coefficient in the stochastic regime and the results of numerical calculations in accordance with Eq. (36) are given in the next section.

4. DISCUSSION OF RESULTS

We proceed now to an evaluation of the results. According to the foregoing, sound-absorption both by isolated resonances and by randomized electrons can be realized in the nonlinear regime in an oblique magnetic field. The expressions (22) and (36) for the corresponding absorption coefficients, obtained in Secs. 2 and 3, are relatively simple, and it is easy to trace their connection with the absorption coefficient in the linear theory of magnetoaceoustic oscillations:

$$\Gamma_L(\mathbf{H}) = \frac{\Gamma_L(\mathbf{H}=0)}{\cos\vartheta} \sum_{n=-N_{max}}^{N_{max}} J_n^2(x_n). \tag{37}$$

The nature of the magnetoacoustic oscillations in the linear theory is well known. At angles ϑ not too close to $\pi/2$, a pronounced role is played by electrons whose orbits in momentum space cross the so-called boundary points of the Fermi surface. Corresponding to these electrons in (37) is the term in which the argument of the Bessel function is of the order of the number of the term itself (the corresponding Bessel function is in this case a maximum).

In the regime of capture by isolated cyclotron resonances, when the overlap parameter K and the nonlinearity parameter a_n are less than unity, the pronounced role of the electrons located near the boundary points is preserved. As a result of the capture, however, the absorption here is weaker by a factor $2a_n$ for each resonance, this being typical of nonlinear-damping theory. Since a_n is proportional to $[J_n(x_n)]^{-1/2}$, the contribution from the corresponding resonance is proportional to $[J_n(x_n)]^{3/2}$.

In the stochastization regime, weak oscillations of the absorption coefficient are also preserved. Notwithstanding the predominantly diffuse electron motion is in this case, and

the smoothness of the principal part of the distribution function [see Eq. (34)], there are also small resonant increments [see (30)] with maxima at values of v_z determined by the condition (1). These increments lead to the appearance in (36) of an additional factor

$$\left[1-\mathrm{ch}\frac{m^*\Omega n}{k_z^2 P_0}\left(\mathrm{ch}\frac{P_2-P_1}{2P_0}\right)^{-1}\right], \tag{38}$$

which determines the decrease of the oscillation amplitude. Since the inequality $(m^*\Omega n/k_z^2)\leqslant(P_2-P_1)/2$, holds in (38) for all n, the parameter that determines the decrease of the damping is in this case the ratio

$$b(\mathbf{H})=(P_2-P_1)/2P_0. \tag{39}$$

Figures 2 and 3 show the a damping coefficients numerically calculated from Eqs. (22), (36), and (37) in various regimes for a typical metal and for a semimetal of the bismuth type. The number of terms in the sum of Eq. (36) and the overlap region P_2-P_1 were determined in accordance with the stochastization criterion. Curves 1 in both figures correspond to the linear theory. The absorption in the regime of nonlinear isolated resonances is shown by curves 2. Curves 3 and 4 show the dependences of the damping coefficient on the magnetic field in the stochastization regime. The abscissa in Figs. 2 and 3 is the parameter $u=(k_z v_F/\Omega)\sin\vartheta$. Curves 2, as seen from the figures, lie in the region of u values corresponding to strong magnetic fields, where the parameter K is less than unity. Thus, in a typical metal this region corresponds to fields stronger than 10^3 Oe, for the parameters indicated in the caption of Fig. 2. The dependence of the absorption coefficient on the reciprocal of the magnetic field has in this field interval qualitatively the same form as in the linear theory; the oscillations have the same period in H^{-1}, but the value of Γ is decreased by an approximate factor $2(k_z\tilde{v}\tau_r)^{-1}$ where $\tilde{v}=(\Phi_0/m^*)^{1/2}$. In semimetals, where the effective mass is on the average smaller by two orders, the regime of isolated resonances corresponds to magnetic fields $H>10$ Oe (the parameters of the semimetal and of the sound wave are indicated in the caption of Fig. 3). The $\Gamma(H)$ dependence has here the same character as above.

With decrease of the magnetic field strength, the distance Ω/k_z between the resonances decreases and the isolated-resonances regime goes over gradually into the regime of

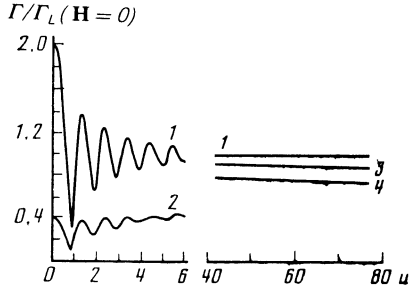

FIG. 2. Coefficient of sound absorption in a metal in the linear (curve) and nonlinear (curves 2, 3, and 4) regimes. The parameters used in the calculations were $k_z=10^3\,\mathrm{cm}^{-1}$, $\vartheta=60°$, $m^*=10^{-27}$ g, $v_F=10^8$ cm/s $\tau_r=10^{-8}$ s (curves 2 and 3). Curve 4 was plotted using the same parameters but for an amplitude $\Phi_0=3\times10^{-15}$ erg.

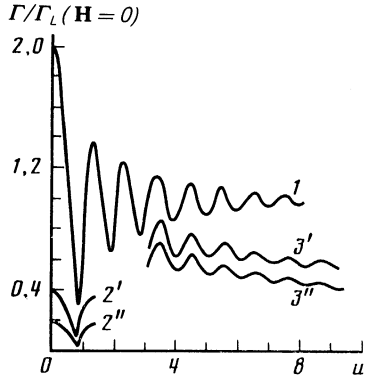

FIG. 3. Coefficient of sound absorption in a semimetal in the linear (curve 1) and nonlinear (curves 2 and 3) regimes. the parameters used are $k_z=2\times10^2\,\mathrm{cm}^{-1}$, $\vartheta=60°$, $m^*=10^{-29}$ g, $v_F=10^8$ cm/s, $\tau_r=5\times10^{-9}$ s (curves 2′ and 3′), while curves 2″ and 3″ were plotted for $\tau_r=10^{-8}$ s.

global stochasticity at a fixed wave amplitude. We put in all the calculations $\Phi_0=\Lambda\mathrm{div}\mathbf{u}\sim10^{-15}$ erg. As seen from Figs. 2 and 3, the stochastic regime sets in at the indicated sound-wave intensity in fields $H<10^2$ Oe for a metal and $H<10$ Oe for semimetals. In a metal, an appreciable part of the Fermi surface located between the "boundary points" ($|v_z|<v_F\sin\vartheta$) has become stochasticized. In the semimetal, the carriers are stochasticized over the entire Fermi surface. In either case, the absorption coefficient is smaller than linear. Curves 3′ and 3″ of Fig. 3 correspond to different values of the parameter $b(\mathbf{H})$ (curve 3″ was obtained at $\tau_r=5\cdot10^{-9}$ s, and curve 3″ at $\tau_r=10^{-8}$ s). The oscillations in the semimetal in the stochastic regime are still observable, although their amplitude is weaker. As seen from Fig. 2 (curves 3 and 4), in the stochasticity region there are practically no socillations in a metal, since they are indistinguishable also in the corrsponding magnetic-field interval in the linear regime. The influence of stochastization in this case is manifested by a general decrease of the absorption compared with the linear regime. In addition, Fig. 2 shows a weak decrease of Γ with decrease of the magnetic field strength.

It must be noted that nonlinear magntoacoustic effects were investigated earlier by Gal'perin and Kozub (see, e.g., Ref. 7). They have assumed that the magnetic field is weak enough, so that the ratio of the Lorentz force to the deformation force, designated $b=m^*\Omega v/k\Phi_0$ in Ref. 7, is less than unity. There exists in this case a group of electrons captured in a potential well of the wave, and their removal by the magnetic field from the wells is quite weak. In addition, it is assumed in Ref. 7 that the angle between the wave propagation and the magnetic field is close to $\pi/2$, so that the projection of the momentum on the magnetic field direction is conserved.

In contrast to Ref. 7, in our case, as is clear from the foregoing, the magnetic field is regarded as strong, the perturbation theory is in terms of the wave amplitude, and here $\Omega>\tilde{\omega}_l$.

Let us ascertain the angles ϑ for which our theory no longer holds. Two variants are possible here: 1) an increase of ϑ causes violation of the nolinearity condition:

$$a_l=\left[k\left(\frac{\Phi_0 J_l(k_\perp\rho)}{m^*}\right)^{1/2}\tau_p\cos\vartheta\right]^{-1}\ll1, \tag{40}$$

and then the nonlinearity is suppressed by collisions and the linear theory of magnetoacoustic oscillations becomes valid; 2) if the relaxation time τ_r, on the other hand, is long enough the condition for the validity of our theory may be violated because of the qualitative change of the electron dynamics. As $\vartheta/2 \to \pi/2$ we can neglect the change of p_z in the equations of motion and take into account only the modulation of the electron transverse momentum by the wave field. As shown in Ref. 1, the regime considered by us is valid under the condition

$$\frac{k_z^2}{m^*} > \Phi_0 \left| \frac{\partial^2 J_l(k_\perp \rho)}{\partial P_\alpha^2} \right|, \tag{41}$$

which reduces to the following simpler inequality:

$$\frac{\pi}{2} - \vartheta > l \frac{(\Phi_0/m^*)^{1/2}}{v_F}.$$

It must be noted that when this inequality is violated the theory does not reduce to that in Ref.7, since the magnetic field remains strong as before. The question of sound absorption in this regime calls for a different analysis.

To determine the conditions under which no electrons will be removed from the potential wells (in the case of ϑ not equal to $\pi/2$), it is necessary (as was done in Refs. 7), to compare the Lorentz force with the deformation force. The criterion for the absence of electron removel is easily seen to be

$$b(\vartheta) = \frac{m^* \Omega v}{k \Phi_0} \sin\vartheta = \left(\frac{\Omega}{\tilde{\omega}}\right)^2 \frac{v \sin\vartheta}{(\Omega/k)} \ll 1,$$

where v is of the order of v_F. In our case the inverse inequality holds, as follows, for example, from the condition $\Omega \gtrsim \tilde{\omega}_l$. Electrons are then removed from the potential wells, but this does not mean suppression of the nonlinearity.

We note in conclusion that our approach can be used also to solve problems in nonlinear damping of waves of other types, propagating in metals and semimetals and interacting with various groups of resonant electrons.

[1)]The criterion (3) overestimates somewhat the critical amplitude of the wave. When fractional resonances and the finite thickness of the stochastic layer near the separatrix are taken into account, the stochastization sets in at $K \approx 0.4$.

[2)]The ratio g_n/g_0 can be easily seen to be of the order of the overlap parameter $K \approx \tilde{\omega}/\Omega$. In the absence of overlap we have $K \ll 1$.

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