Anisotropic gravitational instability

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Exact solutions of stability problems are obtained for two anisotropic gravitational systems of different geometries—a layer of finite thickness at rest and a rotating cylinder of finite radius. It is shown that the anisotropic gravitational instability which develops in both cases is of Jeans type. However, in contrast to the classical aperiodic Jeans instability, this instability is oscillatory. The physics of the anisotropic gravitational instability is investigated. It is shown that in a gravitating layer this instability is due, in particular, to excitation of previously unknown interchange-Jeans modes. In the cylinder, the oscillatory Jeans instability is associated with excitation of a rotational branch, this also being responsible for the beam gravitational instability. This is the reason why this instability and the anisotropic gravitational instability have so much in common.

1. INTRODUCTION

Stellar systems of the most diverse scales—globular clusters, elliptical galaxies, various subsystems of spiral galaxies, etc.—are, being collisionless,[1] characterized by an anisotropic velocity distribution function of the stars.[2] From the evolutionary point of view, the occurrence of anisotropy is entirely natural if it is assumed that the stellar systems[1)] were formed in a collapse process. When a system of noncolliding gravitational masses collapses, the released potential energy is expended in the first place on an increase in the kinetic energy of the motion in the direction of contraction (the longitudinal direction). It is therefore natural to expect the formation of strongly anisotropic systems in which the velocity dispersion in the longitudinal direction is appreciably greater than the dispersion of the transverse velocities.

However, since real systems have only a moderate degree of anisotropy, the question arises of the mechanisms of isotropization that act in collisionless gravitating systems.

One of the known mechanisms is Lynden-Bell mixing (violent relaxation).[3] Despite the large number of studies devoted to this mechanism, there are various arguments which raise doubts about the possibility of Lynden-Bell relaxation in real stellar systems. We recall that Lynden-Bell relaxation was investigated[3] for the example of a strongly nonequilibrium system. Two stages of relaxation were found.[3] After a first "violent" period, which takes place over the dynamic time τ, the system slowly relaxes to the unique Fermi distribution f_F. Since as a result of the first stage the relaxation function f differs little from the equilibrium (Fermi f_F) function, universality of the Lynden-Bell mechanism would have the consequence that among stellar systems with age $t \gg \tau$ there should be none that differ appreciably from equilibrium systems. Nevertheless, such systems are observed (see Refs. 4 and 5 and the literature quoted there).

A second argument against Lynden-Bell relaxation in stellar systems is the presence of strong instabilities in systems in which, according to Ref. 3, Lynden-Bell relaxation should occur. If there is to be no Jeans instability leading to growth of transverse perturbations[6,7] in the system considered by Lynden-Bell,[3] the system must have a sufficiently large velocity dispersion in the directions at right angles to the radius: $2\langle v_r^2\rangle/\langle v_\perp^2\rangle \lesssim 1.5$–$2.5$ (Refs. 2 and 8–10). However, this contradicts modern cosmological ideas.[11,12]

In this paper we shall not dwell further on the group of problems relating to the Lynden-Bell relaxation mechanism but will concentrate on the search for and elucidation of possible collective instabilities.

It is well known that in a collisionless plasma isotropization is brought about by instabilities: the firehose instability when $T_\parallel > T_\perp$ (the indices $\parallel$ and $\perp$ denote the directions along and at right angles to the magnetic field $\mathbf{B}_0$) and the anisotropic instability when $T_\parallel < T_\perp$. In gravitating systems, the mechanism of the firehose instability proceeds in essentially the same way as in plasma systems. The condition for the firehose instability is satisfied if the centrifugal force which arises when the system is subject to bending (transverse) perturbations exceeds the stabilizing force of the magnetic pressure (in a plasma) or the gravitational attraction (for a gravitating system). The gravitational force which stabilizes the firehose instability leads, in its turn, to the Jeans instability. Therefore, one sometimes says that the firehose and Jeans instabilities complement each other, by which is meant that they develop in adjacent parameter regions (see Refs. 1 and 2).

A dispersion relation for an anisotropic gravitating system was derived for the first time in Refs. 13 and 14 for a rotating cylinder of infinite radius. Attention was drawn in these studies to the analogy between this relation and the dispersion relation of Harris[15] for an anisotropic plasma in a magnetic field. Wu[13] found[2)] that the eigenfrequencies are, in general, complex, and this was the first example of an oscillatory Jeans instability. While giving deserved recognition to this nontrivial fact (which Wu in his paper passes by without comment, to which we will refer below), we nevertheless feel we must point out an error in one of Wu's main results. In Ref. 13, Wu determines a critical value of the anisotropy, $\beta^2_{cr} = c_\parallel^2/c_\perp^2 \approx 0.293$ ($c_\parallel$ and $c_\perp$ are the velocity dispersions of the particles along the rotation axis and in the perpendicular direction), which, in his opinion,[13] separates the unstable solutions (having $\beta^2 < \beta^2_{cr}$) from the stable ones (with $\beta^2 > \beta^2_{cr}$). The value of β^2_{cr} is given in the abstract of Ref. 13 as a main result. This result is incorrect. There is no upper limit for the anisotropy value at all—an anisotropic rotating cylinder is unstable both when $\beta^2 < 1/2$ and when $\beta^2 > 5.3$ (Refs. 1 and 2). In the latter case, perturbations of the boundary that have a wavelength long com-

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pared with the radius R of the cylinder develop; this is the firehose instability (see Refs. 1 and 2). Wu's model,[13] which is radially unbounded, is in principle incapable of permitting study of the possibility of such instability.[3)]

The aim of the present paper is to investigate the physics of the anisotropic gravitational instability using as an example two models of anisotropic gravitating systems possessing very different properties—a layer of finite thickness at rest and a rotating cylinder of finite radius. In the theory of plasma instabilities, the analogy between the two mechanisms of beam and anisotropic instability of a plasma in a strong magnetic field is well known (see, for example, Ref. 18). As is shown in Ref. 5 of the present paper, such a situation is also characteristic of a gravitating medium. The mechanism of beam instability in a bounded gravitating system, first investigated in Ref. 16, is also based on the oscillatory Jeans instability. Thus, the detailed analysis of the physics of this instability made in the present paper will help to reveal the common basis of the mechanisms of the an isotropic and beam instabilities in gravitating systems.

2. QUALITATIVE ANALYSIS OF JEANS INSTABILITIES

2.1. Why the maximal growth rate of gravitational instabilities cannot exceed the Jeans growth rate

It is well known that the basic "working force" of electrostatic instabilities of a plasma is the perturbed electric field. Therefore, the maximal growth rates of such instabilities cannot exceed the electron plasma frequency ω_{pe}. Similarly, the maximal growth rates of the instabilities in gravitating systems due to the perturbed gravitational field cannot exceed the Jeans frequency ω_0. This is a natural result; for the minimal time of restructuring of a system as a result of gravitational interaction is the time of free fall and is of the order of ω_0^{-1}

2.2. Aperiodic nature of the Jeans instability in a homogeneous isotropic medium

During the more than 70-year history of the Jeans instability, its aperiodic nature appeared to be established once and for all. Indeed if an arbitrary volume in a gravitating medium is slightly compressed, it will either re-expand and an acoustic wave will arise, or it will continue to contract. In the latter case, one says that the system is gravitationally unstable to perturbations with characteristic size corresponding to the compressed volume. The physics of the instability is extremely simple—the gravitational force exceeds the pressure force. For this, the mass of the body and its size at constant density must not be too small; more precisely, they must exceed certain critical values. This last fact was first pointed out by Jeans, and this is why we speak of the "critical Jeans length" λ_J. Thus, if the scale of the perturbation exceeds λ_J, $\lambda > \lambda_J$, then an irreversible contraction—the Jeans instability—occurs in the gravitating medium.

2.3. Possibility of acoustic oscillations in a heterogeneous isotropic gravitating medium when $\lambda > \lambda_J$

Grishchuk and Zel'dovich[19] (see also our Ref. 20) pointed out a qualitative change in the nature of perturbations with $\lambda > \lambda_J$ in a heterogeneous medium, i.e., a medium consisting of several subsystems. Suppose that a perturbation with $\lambda > \lambda_J$ arises in the simplest two-component system. Then besides the classical Jeans instability there will also arise in the system characteristic oscillations which we called in Ref. 20 asynphase (analogous, for example, to Langmuir oscillations in a plasma or optical vibrations of a crystal lattice). Oscillating in antiphase, the density maxima of one subsystem are compensated by the minima of the other in the same spatial regions, so that the total density of the system is hardly changed. It is this that explains the possible existence of such oscillations with a wavelength appreciably exceeding the Jeans value.

In the quoted Refs. 19 and 20 it was shown that in an n-component gravitating system there exist $n-1$ asynphase oscillations and one synphase oscillation, this leading to Jeans instability when $\lambda > \lambda_J$. We shall see that each type of characteristic perturbation leads either to collapse or to oscillations. A mixed process—the participation of one and the same characteristic perturbation simultaneously in the oscillatory and collapsing regimes—does not, on the basis of the classical results of Jeans[21] and the recent studies of Refs. 19 and 20, appear to be possible.

2.4. Oscillatory Jeans instability in anisotropic systems

The greater complexity of the considered systems, introduced to make them more realistic models, must lead to a greater complexity of the spectrum of natural oscillations. In this paper, we shall show that a minimal extension of the framework of the investigated models—the consideration of anisotropic systems—already leads to a new physical phenomenon: the oscillatory Jeans instability. A spectrum of complex eigenfrequencies was first obtained, but without a discussion of the physics of the process, by Wu[13] in an investigation of anisotropic gravitational instability and by Mikhaĭlovskiĭ and Fridman[16] in an investigation of the gravitational beam instability.

With a view to bringing out the diverse physical characteristics of the phenomenon, we investigated in the present paper in two opposite limiting cases: a flat gravitating layer of finite thickness at rest and an infinitely long rotating cylinder of radius R. In both cases, we find exact solutions of the problem. Before we turn to the description of the finding of these solutions in Secs. 3 and 4, we shall attempt to present the basic results at the qualitative level. We begin with the flat anisotropic gravitating layer.

2.4.1. Gravitating anisotropic layer

We take the layer to lie in the xy plane. Its thickness (along the z axis) is determined by its "transverse" (at right angles to the plane of the layer) "temperature" $T_\perp$:

$$T_\perp = T_z \sim \int f_0 v_z^2 \, d^3 v,$$

where f_0 is the distribution function of the particles in the six-dimensional phase space. We consider the case when the wave vector of the perturbations lies mainly in the plane of the layer, but its transverse component is not small; specifically, we shall assume that in the direction of the z axis the eigenfunctions have n nodes. Then if $T_\perp \gg T_\parallel$, i.e., the system along the plane is so cold that the longitudinal wavelength satisfies $\lambda_\parallel > \lambda_J$, Jeans instability will develop. Two fundamentally new properties distinguish this instability from the classical Jeans instability (see Sec. 3): 1) Each characteristic perturbation, oscillating along the z axis, simultaneously collapses in the transverse direction along the plane of the layer. 2) The oscillation mode with n nodes along the z axis has an instability growth rate smaller than the Jeans growth rate by $n^{1/2}$ times. The first property follows directly from what we have said above, but the second is also obvious. Indeed, assuming the density along z to be homogeneous, we can obtain eigenfunctions with n nodes along the z axis by, for example, compressing an elementary layer of thickness h/n (h is the total thickness of the layer) of the coldest particles next to the plane $z=0$. From the condition of equilibrium of the particles with respect to $z \sim v_z/\omega_0$, where $\omega_0 \equiv (4\pi GNm)^{1/2}$ is the Jeans frequency (N is the total number of particles in unit volume, and m is the mass of each particle) it follows that such particles have a velocity dispersion n times less than $v_T \sim h\omega_0$, and, therefore, their volume density $N' \sim N/n$ is

$$N' \approx \frac{N}{2v_T} \int_{-v_T/n}^{v_T/n} dv_z e^{-v_z^2/v_T^2} \approx N/n.$$

The same estimate is also obtained for the distribution function (2) used below in Sec. 3. Therefore, the Jeans frequency ω_0' of such particles is $n^{1/2}$ times less than the Jeans frequency of the total layer, and this is what we wanted to show.

2.4.2. Rotating cylinder

We now consider a cylindrical system of collisionless particles rotating with angular velocity Ω. Suppose that along the generator z the dispersion velocity v_z is zero, while in the plane of rotation it is $\sim v_\perp$. For simplicity, we consider in this paper perturbations with short wavelength along z, $k_z R \gg 1$ (k is the wave vector). The time of acceleration of the velocity component v_z from 0 to $v_\perp$ is the characteristic time τ of the anisotropic instability, for it is in this time that the velocity dispersions of the particles in the different directions are equalized. We obtain this time τ from the zth equation of motion. The instability growth rate $\gamma \sim \tau^{-1}$ is estimated as

$$\gamma \sim \frac{1}{\tau} \sim \frac{1}{v_z}\frac{dv_z}{dt} \sim \frac{1}{v_z}\frac{\partial \Phi}{\partial z} \sim \frac{k_z}{v_z}\Phi. \tag{1}$$

Here, we have used the zth equation of motion and a dependence of the perturbated gravitational potential Φ on z in the form $\sim \exp(ik_z z)$. Setting $k_z \sim 1/R$ and $\Phi \sim v_\perp^2$, and also assuming that during the time of the instability v_z reaches a value $v_\perp$, we obtain $\sim v_\perp/R$ for the extreme right-hand expression in (1). It is just such a growth rate, $\gamma \sim v_\perp/R$, that is obtained in the rigorous derivation in Sec. 4.

3. FLAT ANISOTROPIC LAYER

For simplicity, we consider short-wave ($k_\perp h \gg 1$) perturbations in a strongly anisotropic ($\alpha \equiv v_{T_x}/v_{T_z} \ll 1$) flat layer with a distribution function

$$F_0 = \frac{\rho_0}{\pi\omega_0 h} A^{-1/2}\theta(A), \qquad A \equiv 1 - \frac{z^2}{h^2} - \frac{v_z^2}{\omega_0^2 h^2}, \tag{2}$$

($\theta(x)$ is the Heaviside unit step function). For perturbations with $k_z/k_x \ll \alpha^{-1}$ we can ignore on the right-hand side of the perturbed kinetic equation the term $(\partial\Phi_1/\partial z)(\partial f_0/\partial v_z)$ compared with $(\partial\Phi_1/\partial x)(\partial f_0/\partial v_x)$, while the dependence of the perturbed potential on z can be chosen, as in the case of perpendicular oscillations (see Refs. 1 and 2), in the form of polynomials in powers of z (moreover, to obtain the characteristic equation for the eigenfrequencies it is sufficient to retain in the calculations only the term with the leading power of z).

In the given case, however, it is easy not only to derive the characteristic equation but also to establish the explicit form of the eigenfunctions. Namely, we shall show that they have the form

$$\Phi_1^{(N)} = P_N(z) \quad (N=0,\ 1,\ 2,\ \ldots), \tag{3}$$

where $\Phi_1^{(N)}$ is the perturbation of the gravitational potential, and P_N is a Legendre polynomial. Putting $\Phi_1 \propto \exp(ikx - i\omega t)$ and solving the equation for the displacements ξ_x of the particles along x,

$$\frac{d^2\xi_x}{dt^2} = -\frac{\partial\Phi_1}{\partial x} = -ik\Phi_1,$$

we obtain

$$\xi_x = ik \int_{-\infty}^{0} dt e^{-i\omega t} P_N(z\cos t + v_z \sin t).$$

Here and in what follows, we assume that z and v_z are normalized, respectively, by h and $\omega_0 h$. In the given case, the density perturbation

$$\rho_1 = -\rho_0(\partial\bar{\xi}_x/\partial x + \partial\bar{\xi}_z/\partial z)$$

($\bar{\xi}$ is the displacement vector averaged in accordance with (2)) is

$$\rho_1 \approx -\rho_0 \partial\bar{\xi}_x/\partial x = -ik\rho_0\bar{\xi}_x.$$

Similarly, neglecting $\partial^2\Phi_1/\partial z^2$ in Poisson's equation compared with $\partial^2\Phi_1/\partial x^2 = -k^2\Phi_1$, we find

$$P_N(z) = -\int_{-\infty}^{0} te^{-i\omega t}\, dt\, \frac{1}{\pi}\int_{-\varkappa}^{\varkappa} P_N(z\cos t + v_z \sin t)\frac{dv_z}{(\varkappa^2 - v_z^2)^{1/2}}, \tag{4}$$

where $\varkappa \equiv (1-z^2)^{1/2}$. It only remains to make the substitution $v_z = \varkappa \sin\varphi$ and use the addition theorem

$$P_N[z\cos t + (1-z^2)^{1/2}\sin t \sin\varphi] = P_N(z)P_N(\cos t) + 2\sum_{k=1}^{\infty}\frac{(N-k)!}{(N+k)!}P_N^k(z)P_N^k(\cos t)\cos\left(\frac{k\pi}{2} - k\varphi\right). \tag{5}$$

At the same time, only the first term in (5) contributes to the inner integral in (4), which after the substitution reduces to

$$\int_{-\pi/2}^{\pi/2} d\varphi P_N[z\cos t + (1-z^2)^{1/2}\sin t \sin\varphi],$$

and the integral is simply transformed into $\pi P_N(z)P_N(\cos t)$. This completes the proof of (3), and we obtain a characteristic equation for the perturbations of the considered type ($k_x/k_z \gg \alpha$) in the form

$$1 + \int_{-\infty}^{0} dt t e^{-i\omega t} P_N(\cos t) = 0. \tag{6}$$

The characteristic equation for the perpendicular oscillations of the layer can be expressed by the similar formula[2]

$$1+\int_{-\infty}^{0} dt \sin t e^{-i\omega t} P_N(\cos t)=0. \quad (6')$$

One can show that Eq. (6) is equivalent to

$$1+\sum_l{}' \frac{\alpha_l}{(\omega-l)^2}=0, \quad (7)$$

where

$$\alpha_l \equiv \frac{(N+l-1)!!(N-l-1)!!}{(N+l)!!(N-l)!!}, \quad (8)$$

where the summation in (7) is from $-N$ to N, and the prime on the summation sign means that l has the same parity as N.

If the velocities of the particles along x are distributed in accordance with the function $f_0(v_x)$, then instead of (7) we shall have the equation

$$1-\sum_l{}' \frac{\alpha_l}{k} \int \frac{(\partial f_0/\partial v_x)\,dv_x}{\omega-kv_x-l}=0. \quad (9)$$

Characteristic equations in the form (7) and (9) are "expansions with respect to resonances" like the ones widely used, for example, in the theory of plasma instabilities in a magnetic field. These equations can be obtained directly in such a form if one uses the method proposed in Ref. 22. We denote $x_1=v_z$, $x_2=z$ and introduce the three-dimensional space (x_1, x_2, x_3) with the x_3 axis perpendicular to the plane (x_1, x_2). Further, in the space (x_1, x_2, x_3) we take a sphere of unit radius and consider the motion of the layer particles, not on the phase plane $(z, v_z)=(x_1, x_2)$, but in the projection onto the surface of this sphere (the projection is made parallel to the x_3 axis).

After specifying the potential in the form (3), it is necessary to find the perturbation of the distribution function by solving the linearized kinetic equation. For the considered perturbations, this equation has on its right-hand side

$$\frac{\partial f_0}{\partial v_x} ik\Phi_1 = ik\frac{\partial f_0}{\partial v_x} P_N(z) = ik\frac{\partial f_0}{\partial v_x} Y_{N0}(\theta', \varphi'),$$

where Y_{Nm} is a spherical harmonic, and θ' and φ' are spherical angles for a polar axis along the axis $x_2=z$. Since the left-hand side of the kinetic equation is $(-i\bar{\omega}+\partial/\partial\varphi)f$, where $\bar{\omega}\equiv\omega-kv_x$, and θ and φ are spherical angles for the position of the polar axis along the x_3 axis, it is natural to make the transformation

$$Y_{N0}(\theta', \varphi')=\sum_m a_m Y_{Nm}(\theta, \varphi), \quad (10)$$

i.e., expand the eigenfunctions of the one representation of the group SO(3) with respect to the eigenfunctions of the other representation of this group. The expansion (10) in the given case is a representation of the potential

$$P_N[z=(2E)^{1/2}\cos w]$$

($I=E$ and w are action-angle variables, $E=(z^2+v_z^2)/2$ being the energy of a particle) in the form of a superposition of angular harmonics:

$$P_N[(2E)^{1/2}\cos w]=P_N[(1-2E)^{1/2}]P_N(0) + \sum_l^N{}' \frac{(N-l)!}{(N+l)!} P_N{}^l[(1-2E)^{1/2}](e^{ilw}+e^{-ilw}). \quad (10')$$

The matrix elements a_m of this transformation can be found in terms of the Eulerian angles of the direction of the polar axis of the primed system (θ', φ') in the unprimed system (θ, φ); in the given case, they can be expressed, as is readily seen, in terms of the values of the associated Legendre functions $P_N^m(0)$ (see, for example, Ref. 23). Then the solution of the kinetic equation is found in the form

$$f=\sum_m a_m \frac{ik(\partial f_0/\partial v_x)F}{-i(\omega-m)} Y_{Nm}(\theta, \varphi).$$

The next step must be to calculate the density perturbation, i.e., the integral

$$\rho_1=-\sum_m \int dv_x \frac{k}{\bar{\omega}-m}\frac{\partial f_0}{\partial v_x} \int_{-(1-z^2)^{1/2}}^{(1-z^2)^{1/2}} dv_z \frac{Y_{Nm}(\theta, \varphi)}{(1-z^2-v_z^2)^{1/2}}.$$

But in the given case

$$\int dv_z (1-z^2-v_z^2)^{-1/2} \sim \int_0^{2\pi} d\varphi',$$

so that it is convenient to make the inverse transformation

$$Y_{Nm}(\theta, \varphi)=\sum_{m'} \alpha_{m'}^{(m)} Y_{Nm'}(\theta', \varphi'),$$

and in the result there remain only the $\alpha_0^{(m)}$, which must also be expressed in terms of $P_N^m(0)$. Using Poisson's equation with

$$\partial^2\Phi/\partial z^2 \ll \partial^2\Phi/\partial x^2,$$

we obtain the characteristic equation in the form (9).

We first investigate Eqs. (7) and (9) for perturbations having short wavelength with respect to z; $N\gg 1$. We find the asymptotic behavior of α_1, introducing first the notation $N+l=2p$, $N-1=2m$. For small l. we have $p\gg 1$, $m\gg 1$. We can therefore use Stirling's formula, and we then obtain $\alpha_1 \approx 2/\pi N$. In the case of large l, we have $2p\to 2N$, $2m\to 0$. Then in accordance with (8) we have $\alpha_1 \approx (\pi N)^{-1/2}$. Therefore, if N is large, $\alpha_1 \ll 1$, and α_1 does not depend on l. Therefore, only the resonance frequencies $\omega \approx l$ will make the main contribution to Eq. (7) (as in a plasma; see, for example, Ref. 18). Thus, for each of the resonances we have the equation

TABLE I.

N	2	4	6	8	10
z	−0.2156	−0.1273	−0.0908	−0.0706	−0.0578
z_0	−0.2105	−0.1264	−0.0905	−0.0705	−0.0577

TABLE II.

N	2	4		6			8			
l	2	2	4	2	4	6	2	4	6	8
$\mathrm{Re}\,z_0$	4	4	16	4	16	36	4	16	36	64
$\mathrm{Im}\,z_0$	2.449	1.581	4.183	1.281	2.806	5.700	1.109	2.327	3.884	7.010
$\mathrm{Re}\,z$	3.608	3.865	15.698	3.906	15.897	35.742	3.930	15.923	35.915	63.769
$\mathrm{Im}\,z$	2.354	1.502	4.078	1.237	2.691	5.587	1.077	2.255	3.745	6.971

$$1+\alpha_l/(\omega-l)^2\approx 0, \tag{11}$$

whence for the unstable root

$$\omega=l+i\gamma, \tag{12}$$

where $\gamma\approx\alpha_l^{1/2}$ is the instability growth rate. It can be seen that for matter cold in the (x,y) plane all short-wave perturbations are unstable. It also follows from (12) that the maximum of the growth rate of the anisotropic instability (at the limit of applicability of the approximation, i.e., as $\alpha\to 1$) is equal to the Jeans value ω_0.

In the case of even N, Eq. (7) reduces to an algebraic equation of degree $N+1$ in the variable $z\equiv\omega^2$. One of the roots (z) is found to be real (and negative); it corresponds to the Jeans mode of the considered scale. In Table I we give the exact (z) and approximate (z_0, calculated in accordance with (11) for $l=0$) values of this root. It can be seen that the approximate formula gives a good representation of the root even at small N.

The remaining roots of Eq. (7) are $N/2$ complex-conjugate pairs, so that the corresponding instabilities are oscillatory. Table II gives the real and imaginary parts of z as determined numerically from (7) and also as found in accordance with the approximate formula (11). Here too it is found that the approximate estimate is in fact sufficiently good already for $N=2$.

We now consider the dispersion relation with allowance for the thermal motion of the particles. For $v_{T_x}\neq 0$, the dispersion relation for each of the resonances has the form

$$1=\frac{\alpha_l}{\rho_0 k}\int\frac{\partial f_0/\partial v_x\,dv_x}{\delta\omega-kv_x}, \tag{13}$$

where $\delta\omega\equiv\omega-k$, $|\delta\omega|\ll 1$, $\delta\omega=\delta\omega_1+i\delta\omega_2$. For a Maxwell ion distribution in the plane (x,y), we obtain from (13)

$$1-\frac{2\alpha_l}{k^2v_{Tx}^2}\left[1+i\pi^{1/2}\frac{\delta\omega}{kv_{Tx}}W\left(\frac{\delta\omega}{kv_{Tx}}\right)\right]\approx 0. \tag{14}$$

In particular, for the critical wave number k_c separating the stable and unstable perturbations, we obtain from (14)

$$k_c=(2\alpha_l/v_{Tx}^2)^{1/2}=2^{1/2}\gamma/v_{Tx},$$

the perturbations with $k<k_c$ being unstable.

4. ANISOTROPIC CYLINDER OF FINITE RADIUS

We consider the very simple example of a cylinder of homogeneous density with a particle distribution function in the (x,y) plane[1,2,24]

$$f_0=\frac{\rho_0}{\pi}\delta[(1-\tilde\gamma^2)(1-r^2)-v_x^2-v_y^2]\qquad(|\tilde\gamma|\leqslant 1), \tag{15}$$

where the cylinder radius R and the angular velocity Ω_0 of the particles in the circular orbits are set equal to unity. Restricting ourselves, as in the case of a homogeneous layer, to short-wave ($k_zR\gg 1$) perturbations of a strongly anisotropic ($\alpha\equiv v_{T_z}/v_{T_\perp}\ll 1$) cylinder, we can set

$$\Phi_1\propto\bar\Phi_1\exp(-i\omega t+ik_zz)r^{2n}(r\exp(i\varphi))^m. \tag{16}$$

We write down the characteristic equation obtained here for the case of radial oscillations ($m=0$)[4)]

$$1+\frac{2i}{2^nk_z}\int dv_z\frac{\partial f_0}{\partial v_z}\int_{-\infty}^{0}dt\exp\{-i(\omega-k_zv_z)\}\times\frac{1}{2\pi}\int_0^{2\pi}d\varphi A_{\tilde\gamma}(t,\varphi)=0, \tag{17}$$

where

$$A_{\tilde\gamma}(t,\varphi)=\{e^{2it}[(1-\tilde\gamma^2)+(1-\tilde\gamma^2)^{1/2}(\cos\varphi-i\tilde\gamma\sin\varphi)]+2\tilde\gamma^2+2i\tilde\gamma(1-\tilde\gamma^2)^{1/2}\sin\varphi+e^{-2it}[(1-\tilde\gamma^2)-(1-\tilde\gamma^2)^{1/2}(\cos\varphi+i\tilde\gamma\sin\varphi)]\}^n. \tag{17'}$$

In particular, from (17) and (17′) we obtain for a cylinder cold in the z direction, when $f_0(v_z)=\delta(v_z)$, and for the value $n=1$

$$1+(1-\tilde\gamma^2)\left[\frac{1}{(\omega-2)^2}+\frac{1}{(\omega+2)^2}\right]+\frac{2\tilde\gamma^2}{\omega^2}=0, \tag{18}$$

while for $n=2$ we have

$$1+\frac{3}{4}(1-\tilde\gamma^2)\left[\frac{1}{(\omega-4)^2}+\frac{1}{(\omega+4)^2}\right]+3\tilde\gamma^2(1-\tilde\gamma^2)\left[\frac{1}{(\omega-2)^2}+\frac{1}{(\omega+2)^2}\right]+\frac{(1-3\tilde\gamma^2)^2}{2\omega^2}=0. \tag{19}$$

For any n, the characteristic equation has a form similar to that of (18) and (19) and can be reduced to an algebraic equation of degree $(2n+1)$ in $z\equiv\omega^2$. One of the roots, the Jeans one, is always real and negative, while the remainder form n complex-conjugate pairs. The dependence of the real and imaginary parts of z on the parameter $\tilde\gamma$ for Eqs. (18) and (19) is given in Fig. 1. For a system with circular orbits ($\tilde\gamma=1$), Eq. (18) gives: 1) $\omega^2=-2(=\omega_0^2)$, which corresponds to the Jeans instability, and 2) $\omega^2=4$, which gives roots that correspond to a rotational mode of oscillations[16] and are real in the considered limit. However, the case $\tilde\gamma=1$ is degenerate, and the rotational oscillations become unstable for $\tilde\gamma\neq 1$, i.e., in the presence of a thermal spread of the

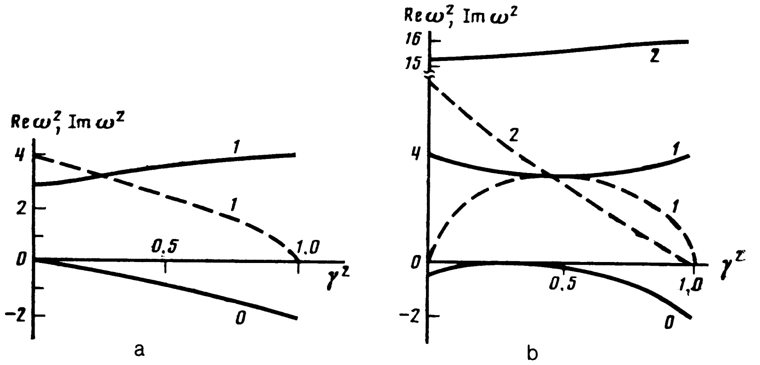

FIG. 1. Real parts Re ω^2 (continuous curves), and imaginary parts, Im ω^2 (broken curves), of the square ω^2 of the eigenfrequency for perturbations of the cylindrical model: *a*) the mode $n = 1$, $m = 0$, b) the mode $n = 2$, $m = 0$; curves 0 represent the anisotropic Jeans instability, and curves 1 and 2 the oscillatory instabilities.

particle velocities in the plane of rotation. Indeed, suppose $1 - \tilde{\gamma}^2 \equiv \varepsilon \neq 0$ although $\varepsilon \ll 1$.Then from Eq. (18) we obtain in addition to the root corresponding to the Jeans instability two other unstable roots:

$$\omega = \pm 2 + i\varepsilon^{1/2}. \tag{20}$$

The obtained solution describes an instability in the collisionless cylinder (15) analogous to the instability of a plane layer considered in Sec. 3. This last obviously corresponds most closely to the instability of a cylinder at rest ($\tilde{\gamma} = 0$), for which

$$\omega^2 = 3 \pm i\sqrt{15}. \tag{21}$$

Dispersion relations analogous to (18) and (19) are also obtained for the modes with $m \neq 0$. For example, for $m = 1$, $n = 0$ we have instead of (17)

$$1 = \frac{1}{k} \int dv_z \frac{\partial f_0}{\partial v_z} \left[\frac{1-\tilde{\gamma}}{\bar{\omega}-1-\tilde{\gamma}} + \frac{1+\tilde{\gamma}}{\bar{\omega}-1+\tilde{\gamma}} \right], \tag{22}$$

and instead of (18) in the case of a cold system

$$1 + \frac{1-\tilde{\gamma}}{(\omega-1-\tilde{\gamma})^2} + \frac{1+\tilde{\gamma}}{(\omega-1+\tilde{\gamma})^2} = 0. \tag{23}$$

The results of (23) are also analogous to those described above for the simplest radial mode. For $\tilde{\gamma} = 1 - \varepsilon$, $\varepsilon \ll 1$ the anisotropic instability has a growth rate Im $\omega \approx (2\varepsilon/3)^{1/2}$ (and a frequency Re $\omega \approx 1 + \tilde{\gamma}$); for $\tilde{\gamma} = 0$, $\omega \approx 1 \pm i2^{1/2}$. The described instability is stabilized by the velocity dispersion of the particles along z, and the critical values of the degree of anisotropy, $(v_{T_z}/v_{T_\perp})_{cr}$, are readily found in the same way as in the case of a flat layer.

5. CONCLUSIONS

Thus, we have described above numerous manifestations of the anisotropic instability of two very simple collisionless systems—the layer and the cylinder. To be specific, let us now consider the layer. For each given scale specified by the number N, the eigenfrequencies are complex and describe oscillatory instabilities. At the same time, the real parts of the eigenfrequencies are close to resonance values: Re $\omega \approx l\omega_0$. In the special case $l = 0$, we obtain an aperiodic instability, which in the limit of a layer cold in its plane corresponds to the unique negative $z \equiv \omega^2$ in Table I. We have obtained characteristic equations for perturbations with $k_x \gg k_z$ in the case of a layer. The opposite case ($k_x \ll k_z$) can also be investigated analytically, but is more cumbersome. The limiting case $k_x = 0$, which describes perpendicular oscillations of the layer, was investigated by Antonov[25] (see also Refs. 1 and 2). In this limit, there is a finite set of oscillation frequencies for each scale (N), and with decreasing scale of the perturbation (with increase in N) all frequencies approach the resonance values. Each of the frequencies in the spectrum of perpendicular oscillations found in Ref. 25 is a limit point toward which a corresponding individual branch is "attached" at $k_x \neq 0$. The behavior of these branches in the asymptotic limit $k_x \gg k_z$ is determined from the characteristic equation (9).

The aperiodic modes stand on their own. In the $k_x = 0$ limits they correspond to displacements that are not along z, as for all the remaining modes, but to indifferent-equilibrium displacements of different $E =$ const layers along the x direction (see Fig. 2). For $k_x = 0$, a frequency $\omega = 0$ obviously corresponds to all such perturbations. It is natural to call these modes interchange-Jeans modes (they have a small scale for $N \gg 1$). For $N = 0$, we have the simplest interchange Jeans mode, which does not have nodes along z and in the limit $k_x = 0$ is transformed into horizontal (along x) displacement of the layer as a whole. This structure of the characteristic mode is also characteristic of a gaseous or a collisionless isotropic layer. However, in this case it is unique—neither small-scale Jeans modes nor, *a fortiori*, oscillatory instabilities exist for such systems.

The frequencies of the perpendicular small-scale oscillations of the collisionless layer are close to the resonance

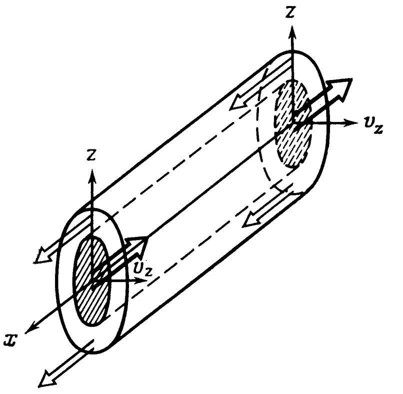

FIG. 2. Horizontal displacements of layers, $E_z =$ const, in the interchange-Jeans modes (in the figure, there is a relative displacement of the inner cylinder along the heavy arrows) and the outer shell (along the thin arrows).

values because of the weak influence of the self-gravitation on the corresponding interchange modes that exist without allowance for self-gravitation (their frequencies are exactly equal to integral multiples of ω_0). The self-gravitation of a wave with $k_x \neq 0$ leads in the case of a cold system to the occurrence of an instability, although the real part of the frequency for small-scale perturbations is still close to the resonance value. If the perturbation of the potential (2) is represented in the form of the superposition of harmonics (10′), then one can say that for $N \gg 1$ each individual harmonic evolves independently. In particular, in the case of a cold layer a corresponding characteristic equation (11) is obtained for each harmonic. It obviously describes the instability of Jeans type due to the part of the total perturbation (10′) that corresponds to the given harmonic ($\sim \exp(ilw)$). Accordingly, the instability growth rate is the $(\alpha_l)^{1/2}$ fraction of the ordinary Jeans growth rate $\omega_0 = (4\pi G\zeta_0)^{1/2}$ ($=1$, in the chosen units) and the effect is the same as would be produced by an effective decrease of the density ρ_0 by the factor $\alpha_1 : \rho_0 \to \alpha_1 \rho_0$.

We also note an analogy which suggests itself, namely, Eq. (7) is more similar to the dispersion relation of Ref. 20:

$$1+\sum_{l}^{N}\frac{\omega_{0l}{}^2}{(\omega-kV_l)^2-k^2c_l{}^2}=0, \tag{24}$$

which describes perturbations in a homogeneous gravitating medium with a corresponding number of cold ($c_1 = 0$) moving subsystems-streams (with velocities V_1, velocities of sound c_l, and densities ρ_{0l}, $\omega_{0l}^2 \equiv 4\pi G\rho_{0l}$); Eq. (24) goes over into (7) under the substitutions $\omega_{0l}^2 \to \alpha_l$, $kV_l \to l$. For sufficiently large velocity differences, Eq. (24) obviously describes Jeans contractions taking place in each of the subsystems in the comoving frames of reference. If the velocity differences of the streams are small, they cannot be treated independently, and the perturbations in them are coupled. The nature of the interaction of the subsystems is even more complicated in the case of hot subsystems, when the velocity of sound c becomes of order V. In Ref. 20, we described the instability that arises when $V > c$ as a beam instability; when the difference $V - c$ is increased, it obviously goes over into the Jeans instability in each of the streams, so that in general it would be more correct to describe this instability as a beam-Jeans instability.

What was said above applies to the hydrodynamic beam instability.

With this, we conclude the discussion of the instabilities of the anisotropic flat layer. The interpretation of the instabilities in an anisotropic cylinder (Sec. 4) is analogous, especially, of course, in the absence of rotation, i.e., in the case that is the most important from the point of view of applications. The case of rapid rotation ($1 - \gamma^2 \ll 1$) is of interest above all because of the existence of a close, at least at first sight, analogy with the traditional treatment of anisotropic instability of plasmas.[18] Indeed, the dispersion relation that describes long-wave ($k_\perp \rho_e \ll 1$, where ρ_e is the electron Larmor radius) perturbations of a strongly anisotropic ($\overline{v}_{\perp e} \gg \overline{v}_{\parallel e}$) plasma corresponds to a system of two streams:

$$1-\frac{\omega_{pe}{}^2}{\omega^2}-\frac{1}{2}k_\perp{}^2\rho_e\omega_{pe}{}^2\left[\frac{1}{(\omega-\omega_{Be})^2}+\frac{1}{(\omega+\omega_{Be})^2}\right]=0, \tag{25}$$

where ω_{pe} and ω_{Be} are, respectively, the plasma and Larmor frequencies. The equations obtained from (18) and (19) for perturbations of a strongly anisotropic gravitating cylinder when $(1-\tilde{\gamma}^2) \ll 1$ can also be represented in the two-stream form

$$1+\frac{2}{\omega^2}+a_n(1-\tilde{\gamma}^2)\left[\frac{1}{(\omega-2)^2}+\frac{1}{(\omega+2)^2}\right]=0, \tag{26}$$

where $a_n = 1$ for $n = 1$ and $a_n = 3$ for $n = 2$ (obviously, a_n plays the part of $k_\perp^2$ in (25)). The quantities that occur in (25) and (26) have completely analogous significances (with allowance for the way in which they have been made dimensionless). A difference is in the opposite signs of the terms in (25) and (26). However, this difference has a decisive importance, reflecting a fundamental difference between electric and gravitational forces. The former lead to repulsion of identical charges, the latter to attraction. Formally, this is manifested in the opposite signs of the right-hand sides of Poisson's equation for these two cases, and this leads in the present case to the noted difference between Eqs. (25) and (26).

We recall (see Ref. 18) that if a beam propagates along a magnetic field in a plasma a beam instability arises. If the magnetic pressure is much greater than the plasma pressure, the oscillations which arise as a result of the instability can be regarded as potential (electrostatic) perturbations. The oscillations grow because of the presence of two types of resonance in the plasma-beam system: Cherenkov and cyclotron. The first, the purely beam resonance, arises when the phase velocity of the wave is equal to the velocity of the beam. The second is due to the anisotropy of the distribution function in the magnetic field. Therefore, the dispersion relation for an anisotropic plasma in a magnetic field is analogous to the dispersion relation of a plasma-beam system in a magnetic field. Comparing Eqs. (7) and (24), we see that the same conclusions are valid for the two types of gravitating systems characterized by the presence of a stream and anisotropy, respectively.

[1)] The same applies to the elements of the cellular structure of the universe considered in modern cosmology ("pancakes" and "filaments," consisting of neutrinos or other collisionless particles).

[2)] No solution of the dispersion relation in quadratures was obtained in Ref. 14.

[3)] Allowance for finiteness of the radius of the rotating system leads to an important connection between k_z and $k_\perp$,[16] and also to a definite equilibrium condition. The absence of the factors mentioned above in unbounded systems gives rise to imaginary effects, which were criticized in Ref. 17.

[4)] For $m \neq 0$, the characteristic equation for perturbations of the considered type is obtained without any additional complications (compared with $m = 0$); however, we shall not require it.

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