The functional-integration method and diagram technique for spin systems

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Expressions are obtained for the partition functions of spin systems of spin $s = \frac{1}{2}$ and $s = 1$ in terms of the partition functions of systems in which the spin matrices are replaced by bilinear combinations of Fermi operators. This leads to a representation of the partition functions (and Green functions) of the spin ($s = \frac{1}{2}$ and $s = 1$) systems by functional integrals and to a simple temperature diagram technique with Matsubara frequencies $\omega_n = 2\pi T(n + \frac{1}{4})$ for $s = \frac{1}{2}$ and $\omega = 2\pi T(n + \frac{1}{3})$ for $s = 1$. The approach developed is illustrated in application to the Dicke model and the Ising-Heisenberg model. For the Dicke model rigorous results are obtained for the $N \to \infty$ asymptotic forms of the partition function and collective-excitation spectrum. The analogous formulas for the Heisenberg model are asymptotically exact in the limit of low temperatures.

1. INTRODUCTION

The Hamiltonians of many systems in statistical physics contain spin matrices. To this class belong ferromagnetic systems and quantum-optics models describing the interaction of an electromagnetic field with two-level atoms. The diagrammatic perturbation theory for spin systems is substantially more complicated than the standard Matsubara-Abrikosov-Gor'kov-Dzyaloshinkiĭ diagram technique.[1] Many authors have proposed variants of the diagram technique that are based on different representations of the spin matrices by Bose or Fermi operators.[2-7] However, the fact that the dimensionality of the space in which these operators act is always greater than the dimensionality of the spin matrices leads to the problem of the elimination of the superfluous states and to a substantial complication of the correspondence rules between the diagrams and their analytical expressions.

In the present paper we construct a simple diagrammatic technique for spin-$\frac{1}{2}$ and spin-1 systems that differs from the known techniques in the form of the Green function. For spin-$\frac{1}{2}$ systems the Green function

$$G=(i\omega-\varepsilon)^{-1}, \quad \omega=2\pi\beta^{-1}(n+{}^1/_4) \tag{1.1}$$

has in our approach Matsubara frequencies proportional to $n + 1/4$, while for spin 1 we have

$$G=(i\omega-\varepsilon)^{-1}, \quad \omega=2\pi\beta^{-1}(n+{}^1/_3) \tag{1.2}$$

and the Matsubara frequencies are proportional to $n + 1/3$. In other respects the diagram technique is standard and does not contain the complicated combinatoric rules characteristic of most of the known variants of the diagram technique for spin systems. The approach developed is illustrated in applications to the single-mode Dicke model and to the Ising-Heisenberg model with Hamiltonian

$$\hat{H} = -\frac{1}{2}\mu H\sum_{\mathbf{i}}\sigma_{\mathbf{i}}^z+\sum_{\mathbf{i}\neq\mathbf{j}}I(\mathbf{i}-\mathbf{j})\left(\frac{1}{4}\sigma_{\mathbf{i}}^z\sigma_{\mathbf{j}}^z+\lambda^2\sigma_{\mathbf{i}}^+\sigma_{\mathbf{j}}^-\right). \tag{1.3}$$

Here $\sigma_{\mathbf{i}}^z$ and $\sigma_{\mathbf{i}}^{\pm}$ are the spin matrices of the **i**th site, H is the magnetic field, μ is the magnetic moment, $I(\mathbf{i}-\mathbf{j})$ is the interaction constant of the **i**th and **j**th sites, and λ is a parameter (with $\lambda = 0$, (1.3) is the Ising model; with $\lambda = 1$ it is the spherically symmetric Heisenberg model].

The Dicke Hamiltonian

$$\hat{H}_D=\omega_0\psi^+\psi+\frac{\Omega}{2}\sum_{i=1}^N\sigma_i^z+\frac{g}{N^{1/2}}\sum_{i=1}^N(\sigma_i^+\psi+\psi^+\sigma_i^-) \tag{1.4}$$

contains not only σ-matrices describing the two-level atoms but also the creation operator ψ^+ and annihilation operator ψ for the single-mode radiation field.

2. THE DIAGRAM TECHNIQUE AND FUNCTIONAL INTEGRAL FOR SPIN SYSTEMS

We shall denote by $\hat{H}_\sigma$ the Hamiltonian of a spin system of the type (1.3) or (1.4). The derivation of the diagram technique for spin-$\frac{1}{2}$ systems is based on representing the σ-matrices

$$\sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{2.1}$$

by bilinear combinations of Fermi operators:

$$\sigma_i^z\to a_i^+a_i-b_i^+b_i,\quad \sigma_i^+\to a_i^+b_i,\quad \sigma_i^-\to b_i^+a_i \tag{2.2}$$

and on the basic formula for the partition function Z_σ of the spin system:

$$Z_\sigma=\mathrm{Sp}\exp(-\beta\hat{H}_\sigma)=i^N\,\mathrm{Sp}\exp\{-\beta(\hat{H}_F+i\pi\hat{N}/2\beta)\}. \tag{2.3}$$

Here $\hat{H}_F$ is the operator obtained from $\hat{H}_\sigma$ by the replacement (2.2), and

$$\hat{N}=\sum_{i=1}^N(a_i^+a_i+b_i^+b_i) \tag{2.4}$$

(N is the number of sites in the system).

The fundamental problem in using the fermion substitution (2.2) is that the σ-matrices (2.1) have dimensionality 2 while the fermion space corresponding to the ith site is four-dimensional, being generated by the vectors

$$a_i^+\Phi_0=|1,0\rangle_i,\quad b_i^+\Phi_0=|0,1\rangle_i,$$
$$\Phi_0=|0,0\rangle_i,\quad a_i^+b_i^+\Phi_0=|1,1\rangle_i. \tag{2.5}$$

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Here Φ_0 is the vacuum vector. We shall call the first two vectors (2.5) physical, and the other two nonphysical. The physical vectors generate a two-dimensional physical subspace characterized by the condition $\hat{N}_i\Phi = \Phi$, where $\hat{N}_i = a_i^+a_i + b_i^+b_i$ is the operator of the number of fermions at the ith site. The direct product of the physical subspaces of all the sites forms the sector in which the Hamiltonians $\hat{H}_\sigma$ and $\hat{H}_F$ coincide.

To prove the basic formula (2.3) we write

$$\hat{H}_F=\hat{H}_{Fi}+\hat{H}_{Fi}', \quad \hat{N}=\hat{N}_i+\hat{N}_i',$$

where $\hat{H}_{Fi}$ is that part of $\hat{H}_F$ which contains the spin operators of the ith site and $\hat{H}'_{Fi}$ is the entire remaining part. For Hamiltonians of the Ising–Heisenberg type (1.3) or Dicke type (1.4) we have

$$\hat{H}_{Fi}|unphys\rangle_i=0, \tag{2.6}$$

where $|\text{unphys}\rangle_i$ are the nonphysical vectors of the ith site. Therefore, the trace over the nonphysical states of the ith site vanishes:

$$\text{Sp}_{i\ unphys}\exp\{-\beta(\hat{H}_F+i\pi\hat{N}/2\beta)\}$$
$$=\exp\{-\beta(\hat{H}_{Fi}'+i\pi\hat{N}_i'/2\beta)\}\ \text{Sp}_{i\ unphys}(-i)^{\hat{N}_i}=0,$$

since $\text{Sp}_{i\ \text{unphys}}(-i)^{\hat{N}_i} = (-i)^0 + (-i)^2 = 1 - 1 = 0$ (on the nonphysical vectors $|0,0\rangle_i$ and $|1,1\rangle_i$ the operator $\hat{N}_i$ has eigenvalues 0 and 2, respectively). As a result, in the calculation of the trace all the nonphysical states are eliminated, while on the physical states $\hat{H}_F = \hat{H}_\sigma$ and $\hat{N}\Phi = N\Phi$. Therefore,

$$\text{Sp}\exp\{-\beta(\hat{H}_F+i\pi\hat{N}/2\beta)\}=(-i)^N\ \text{Sp}_{phys}\exp(-\beta\hat{H}_F)$$
$$=(-i)^N\ \text{Sp}\exp(-\beta\hat{H}_\sigma),$$

which proves (2.3). According to (2.3), when investigating spin systems with $s=\frac{1}{2}$ we must use the Hamiltonian $\hat{H}_F$ with the extra term $i\pi\hat{N}/2\beta$, i.e., with a purely imaginary chemical potential $\mu = -i\pi/2\beta$. On the basis of (2.3) we can construct the standard (for Fermi systems) diagram technique with the Green function

$$G=(i\omega_F-\varepsilon+\mu)^{-1}=(i\omega_F-\varepsilon-i\pi/2\beta)^{-1}.$$

Here $\omega_F = 2\pi\beta^{-1}(n+\frac{1}{2})$ is the fermion Matsubara frequency:

$$\omega_F-\frac{\pi}{2\beta}=\frac{2\pi}{\beta}\left(n+\frac{1}{2}-\frac{1}{4}\right)=\frac{2\pi}{\beta}\left(n+\frac{1}{4}\right).$$

Thus, we arrive at a Green function with Matsubara frequencies proportional to $n+\frac{1}{4}$.

We now consider a system with Hamiltonian $\hat{H}_s$ containing spin-1 σ-matrices:

$$s^z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad s^+=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad s^-=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}. \tag{2.7}$$

Here the basic formula has the form

$$Z_s=\text{Sp}\exp(-\beta\hat{H}_s)=\left(\frac{i}{3^{1/2}}\right)^N\text{Sp}\exp\left\{-\beta\left(\hat{H}_F+\frac{i\pi}{3\beta}\hat{N}\right)\right\}, \tag{2.8}$$

where $\hat{H}_F$ is obtained from $\hat{H}_s$ by the replacement

$$s_i^z\to a_i^+a_i-b_i^+b_i,\quad s_i^+\to a_i^+c_i+c_i^+b_i,\quad s_i^-\to b_i^+c_i+c_i^+a_i,$$
$$\hat{N}=\sum_{i=1}^N\hat{n}_i,\quad \hat{n}_i=a_i^+a_i+b_i^+b_i+c_i^+c_i. \tag{2.9}$$

The proof of the formula (2.8) is based on the fact that the Hamiltonians $\hat{H}_s$ and $\hat{H}_F$ coincide in the sectors with $\hat{n}_i\Phi = \Phi$, and also in the sectors with $\hat{n}_i\Phi = 2\Phi$, while the contributions to the right-hand side of (2.8) from the nonphysical states with $\hat{n}_i\Phi = 0$ and $\hat{n}_i\Phi = 3\Phi$ cancel each other in the same way as occurred in the case of spin $s=\frac{1}{2}$. Therefore,

$$\text{Sp}\exp\{-\beta(\hat{H}_F+i\pi\hat{N}/3\beta)\}=[\text{Sp}_{(\hat{n}_i=1)}\exp(-\beta\hat{H}_F)]$$
$$\times[\exp(-i\pi/3)+\exp(-2\pi i/3)]^N=[-2i\cos(\pi/6)]^N Z_s,$$

whence follows (2.8).

The Green function for a system of $s=1$ has the form

$$G=(i\omega_F-\varepsilon-i\pi/3\beta)^{-1}=(i\omega-\varepsilon)^{-1},$$

where $\omega = \omega_F - \pi/3\beta = 2\pi\beta^{-1}(n+\frac{1}{3})$, i.e., for a system with $s=1$ the Matsubara frequencies are proportional to $n+\frac{1}{3}$.

The basic formulas (2.3) and (2.8) make it possible to go over from spin matrices to Fermi operators and then to write the partition function and Green functions in the form of functional integrals.[8] For example, the ratio $Z_\sigma/Z_{0\sigma}$ of the partition function of a spin-$\frac{1}{2}$ system to the partition function of the corresponding free system is a formal quotient of functional integrals:

$$Z_\sigma/Z_{0\sigma}=\int D\mu\exp\left[S-\frac{i\pi}{2\beta}\int_0^\beta n(\tau)\,d\tau\right]\Big/$$
$$\int D\mu\exp\left[S_0-\frac{i\pi}{2\beta}\int_0^\beta n(\tau)\,d\tau\right], \tag{2.10}$$

where

$$S=\int_0^\beta d\tau\left\{\psi^*(\tau)\partial_\tau\psi(\tau)+\sum_{i=1}^N[a_i^*(\tau)\partial_\tau a_i(\tau)+b_i^*(\tau)\partial_\tau b_i(\tau)]\right\}$$
$$-\int_0^\beta H(\tau)\,d\tau,$$
$$n(\tau)=\sum_{i=1}^N[a_i^*(\tau)a_i(\tau)+b_i^*(\tau)b_i(\tau)]. \tag{2.11}$$

The quantity S is the Euclidean action on the interval $0\leqslant\tau\leqslant\beta$; $a_i^*(\tau), b_i(\tau)$ and $b_i^*(\tau)$ are anticommuting Grassmann variables corresponding to the Fermi operators a_i, a_i^+, b_i and b_i^+; $\psi(\tau)$ and $\psi^*(\tau)$ are variables corresponding to the remaining fields in $\hat{H}$ (e.g., the radiation field in the Dicke model); $H(\tau)$ is obtained from $\hat{H}$ by replacing the operators by the fields corresponding to them at the time τ.

The integration variables in (2.10) satisfy the boundary conditions

$$\psi(\beta)=\psi(0) \quad (\text{Bose}),$$
$$a_i(\beta)=-a_i(0), \quad b_i(\beta)=-b_i(0) \quad (\text{Fermi}). \tag{2.12}$$

By making the replacements

$$a_i(\tau)\to a_i(\tau)\exp\left(\frac{i\pi}{2\beta}\tau\right), \quad a_i^*(\tau)\to a_i^*(\tau)\exp\left(-\frac{i\pi}{2\beta}\tau\right),$$
$$b_i(\tau)\to b_i(\tau)\exp\left(\frac{i\pi}{2\beta}\tau\right), \quad b_i^*(\tau)\to b_i^*(\tau)\exp\left(-\frac{i\pi}{2\beta}\tau\right), \tag{2.13}$$

which cancel the terms with $n(\tau)$ in (2.10), we obtain

$$Z_\sigma/Z_{0\sigma}=\int e^S D\mu\Big/\int e^{S_0}D\mu, \tag{2.14}$$

where S is the action (2.11), and the integration is performed over the Grassmann fields with the boundary conditions

$$a_i(\beta)=ia_i(0), \quad a_i^*(\beta)=-ia_i^*(0),$$
$$b_i(\beta)=ib_i(0), \quad b_i^*(\beta)=-ib_i^*(0). \tag{2.15}$$

Analogously, for a system with spin $s=1$ we obtain the representation (2.14), in which the Grassmann Fermi fields satisfy the conditions

$$a_i(\beta)=e^{i\pi/3}a_i(0), \quad b_i(\beta)=e^{i\pi/3}b_i(0), \quad c_i(\beta)=e^{i\pi/3}c_i(0),$$
$$a_i^*(\beta)=e^{-i\pi/3}a_i^*(0), \quad b_i^*(\beta)=e^{-i\pi/3}b_i^*(0),$$
$$c_i^*(\beta)=e^{-i\pi/3}c_i^*(0). \tag{2.16}$$

3. THE PARTITION FUNCTION AND COLLECTIVE-EXCITATION SPECTRUM OF THE DICKE MODEL

In Ref. 9, the present authors obtained and rigorously proved asymptotic (for $N\to\infty$, where N is the number of atoms) formulas for the partition function of the "fermion" Dicke model. The Hamiltonian of this model

$$\hat{H}=\omega_0\psi^+\psi+\frac{\Omega}{2}\sum_{i=1}^{N}(a_i^+a_i-b_i^+b_i)+\frac{g}{N^{1/2}}\sum_{i=1}^{N}(\psi^+b_i^+a_i+a_i^+b_i\psi) \tag{3.1}$$

is obtained from the Hamiltonian of the Dicke model (1.4) itself after the fermion substitution (2.2). Now, having the representation (2.14) for $Z_\sigma/Z_{0\sigma}$, we can also obtain analogous results for the Dicke model itself. In the right-hand side of (2.14) S is an action of the form (2.11), constructed using the Hamiltonian (3.1), and the integration variables satisfy the boundary conditions (2.15).

By expanding the integration variables of (2.14) in Fourier series

$$\psi(\tau)=\beta^{-1/2}\sum_{\omega}\psi(\omega)e^{i\omega\tau}, \quad a_i(\tau)=\beta^{-1/2}\sum_{p}a_i(p)e^{ip\tau},$$
$$b_i(\tau)=\beta^{-1/2}\sum_{p}b_i(p)e^{ip\tau} \tag{3.2}$$

($\omega=2\pi n\beta\Pi^1$, $p=2\pi(n+\frac{1}{4})\beta^{-1}$), we write the action S and the measure $D\mu$ in the functional integrals (2.14) in the form

$$S=\sum_{p,i}\left(ip-\frac{\Omega}{2}\right)a_i^*(p)a_i(p)+\sum_{p,i}\left(ip+\frac{\Omega}{2}\right)b_i^*(p)b_i(p)$$
$$+\sum_{\omega}(i\omega-\omega_0)\psi^*(\omega)\psi(\omega)-\frac{g}{(\beta N)^{1/2}}\sum_{q-p=\omega}[\psi^*(\omega)b_i^*(p)a_i(q)$$
$$+a_i^*(q)b_i(p)\psi(\omega)], \tag{3.3}$$

$$D\mu=\prod_{\omega}d\psi^*(\omega)d\psi(\omega)\prod_{p,i}da_i^*(p)da_i(p)db_i^*(p)db_i(p). \tag{3.4}$$

From (3.3), by the standard rules, we obtain the following rules for the diagram technique: solid lines correspond to $G_{a,b}=(ip\mp\Omega/2)^{-1}$, with $p=2\pi\beta^{-1}(n+1/4)$, dashed lines corresponds to $D=(i\omega-\omega_0)^{-1}$, with $\omega=2\pi n/\beta$, and a vertex at which a dashed line terminates on a solid line corresponds to $g(\beta N)^{-1/2}$.

The ratio of functional integrals in (2.14) can be understood as the ratio of the finite-dimensional integrals obtained with cutoffs in the sums in (3.2) ($|\omega|<\omega_B, |p|<\omega_\sigma$) in the limit when these cutoffs are removed ($\omega_B,\omega_\sigma\to\infty$).

The variables $a_i(p), a_i(p), b_i(p)$, and $b_i^*(p)$ appear quadratically in the action, and we can integrate over them:

$$Z_\sigma/Z_{0\sigma}=\int D\mu(\psi)e^{S_0[\psi]}\det{}^N M(\psi,\psi^*)\Big/$$
$$\int D\mu(\psi)e^{S_0[\psi]}\det{}^N M(0,0). \tag{3.5}$$

Here

$$S_0[\psi]=\sum_{\omega}(i\omega-\omega_0)\psi^*(\omega)\psi(\omega), \tag{3.6}$$

where M is an operator with elements

$$M_{pq}=\begin{pmatrix}(ip+\Omega/2)\delta_{pq}, & g(\beta N)^{-1/2}\psi^*(p-q)\\ g(\beta N)^{-1/2}\psi(q-p), & (ip-\Omega/2)\delta_{pq}\end{pmatrix}. \tag{3.7}$$

Carrying over the factor $\det^N M(0,0)$ from the denominator to the numerator of (3.5) and making the replacement (for a justification, see Ref. 9)

$$\psi(\omega)\to\left(\frac{\pi}{\omega_0-i\omega}\right)^{1/2}\psi(\omega), \quad \psi^*(\omega)\to\left(\frac{\pi}{\omega_0-i\omega}\right)^{1/2}\psi^*(\omega), \tag{3.8}$$

we bring (3.5) to the form

$$Z_\sigma/Z_{0\sigma}=\int D\mu(\psi)\det{}^N(I+A)\exp\left[-\pi\sum_{\omega}\psi^*(\omega)\psi(\omega)\right]. \tag{3.9}$$

Here

$$\det(I+A)=\det M^{-1/2}(0,0)M(\psi,\psi^*)M^{-1/2}(0,0), \tag{3.10}$$

so that A is determined by the formulas

$$A_{pq}=\begin{pmatrix} 0 & B_{pq} \\ -C_{pq} & 0 \end{pmatrix},$$

$$B_{pq}=g(\pi/\beta N)^{1/2}\psi^*(q-p)$$

$$\times\{(\Omega/2+ip)[\omega_0-i(q-p)](\Omega/2-iq)\}^{-1/2},$$

$$C_{pq}=g(\pi/\beta N)\psi(p-q)\{(\Omega/2-ip)[\omega_0-i(p-q)](\Omega/2+iq)\}^{-1/2}. \tag{3.11}$$

The expression

$$S_{eff}=-\pi\sum_\omega\psi^*(\omega)\psi(\omega)+N\ln\det(I+A) \tag{3.12}$$

has the meaning of the effective action of the radiation field.

In (3.9) we have no longer a formal quotient of functional integrals, but the integral having a finite limit as ω_B, $\omega_\sigma\to\infty$. The representation (3.9) is convenient for the derivation of the asymptotic form of $Z_\sigma/Z_{0\sigma}$ for large N by the stationary-phase method. For $T>T_c$ there exists one point of stationary phase (the coordinate origin), while for $T<T_c$ we have a circle of stationary phase($|\psi(0)|^2=\rho_0,\psi(\omega)=\psi^*(\omega)=0$ for $\omega\neq 0$).

For the proof of the asymptotic formulas for $Z_\sigma/Z_{0\sigma}$ the space of the integration over the Bose fields $\psi(\omega)$ and $\psi^*(\omega)$ is divided into two regions: the neighborhood of the point (or circle) of stationary phase, and its complement. The integral over the first region can be calculated approximately, and the error that arises can be estimated rigorously. The integral over the second region can also be estimated rigorously. In our case of the Dicke model itself the technique used for the estimates repeats verbatim the technique developed in Ref. 9 for the fermion Dicke model. The only difference is that in the fermion model the frequencies in the operators M and A are proportional to $n+\frac{1}{2}$, while for the Dicke model itself they are proportional to $n+1/4$. Without repeating the calculations of Ref. 9, we give the asymptotic formulas for the integral (3.9):

$$Z_\sigma/Z_{0\sigma}=\prod_\omega[1-a(\omega)]^{-1}+O(N^{-1}),\quad T>T_c,$$

$$Z_\sigma/Z_{0\sigma}=AN^{1/2}e^{aN}\prod_\omega L^{-1}(\omega)[1+O(N^{-1/2})],\quad T<T_c. \tag{3.13}$$

Here,

$$a(\omega)=g^2(\omega_0-i\omega)^{-1}(\Omega-i\omega)^{-1}\,\mathrm{th}(\beta\Omega/2),$$

$$a=\ln[\mathrm{ch}(\beta\Omega_\Delta/2)/\mathrm{ch}(\beta\Omega/2)]-\omega_0\beta\Delta^2,$$

$$A=\left[\frac{\pi\beta\omega_0\Omega_\Delta^2}{g^2(1-\beta\Omega_\Delta/\mathrm{sh}\,\beta\Omega_\Delta)}\right]^{1/2},$$

$$L(\omega)=1+\frac{\omega_0\Omega\omega^2-\omega_0^2\Omega^2}{(\omega^2+\omega_0^2)(\omega^2+\Omega_\Delta^2)}, \tag{3.14}$$

where $\Omega_\Delta^2=\Omega^2+4g^2\Delta^2$, in which Δ (for $T<T_c$) is determined by the equation

$$\frac{g^2}{\omega_0\Omega_\Delta}\mathrm{th}\frac{\beta\Omega_\Delta}{2}=1. \tag{3.15}$$

The temperature $T_c=\beta_c^{-1}$ of the transition to the superradiative state is determined from this equation with $\Delta=0$ and $\Omega_\Delta=\Omega$. We note that for the fermion model $\tanh(\beta\Omega_\Delta/2)$ in (3.15) is replaced by $\tanh(\beta\Omega_\Delta/4)$ and the transition temperature is found to be lower by a factor of 2 than in the Dicke model itself.

The structure of the infinite products in the asymptotic formulas (3.13) for $Z_\sigma/Z_{0\sigma}$ carries information about the spectrum of the collective excitations of the system.

The Bose spectrum is obtained by equating to infinity the general factor in the infinite product for the partition function (3.13), after replacing $i\omega\to E$ in this factor. The equations for the spectrum

$$1-a(\omega)=0\quad(T>T_c);\qquad L(\omega)=0\quad(T<T_c) \tag{3.16}$$

(in which we must replace $i\omega\to E$) have solutions

$$E_{1,2}={}^1/_2\{\Omega+\omega_0\mp[(\Omega-\omega_0)^2+4g^2\,\mathrm{th}(\beta\Omega/2)]^{1/2}\}\quad(T>T_c),$$

$$E_1=0,\quad E_2=[(\Omega+\omega_0)^2+4g^2\Delta^2]^{1/2}\quad(T<T_c). \tag{3.17}$$

Equations (3.13) are the most exact of the known ones (and, furthermore the most rigorously proved results) for the Dicke model itself. Multimode variants of models of the Dicke type, and also Dicke models with an interaction that takes nonresonance terms into account, can be investigated analogously.

4. DIAGRAM TECHNIQUE FOR THE ISING-HEISENBERG MODEL

The representation (2.14) for $Z_\sigma/Z_{0\sigma}$ is also true for the Ising-Heisenberg model (1.3). Going over to the momentum representation, with allowance for the boundary conditions (2.15) by means of the formulas

$$a_i(\tau)=(\beta N)^{-1/2}\sum_{\mathbf{k},p}\exp[i(p\tau+i\mathbf{k})]a(\mathbf{k},p),$$

$$b_i(\tau)=(\beta N)^{-1/2}\sum_{\mathbf{k},p}\exp[i(p\tau+i\mathbf{k})]b(\mathbf{k},p), \tag{4.1}$$

we obtain

$$Z_\sigma/Z_{0\sigma}=\int e^{S[a,b]}D\mu[a,b]\Big/\int e^{S_0[a,b]}D\mu[a,b], \tag{4.2}$$

where $S=S_0+S_1$,

$$S_0=\sum_{p,\mathbf{k}}\left(ip+\frac{1}{2}\mu H\right)a^*(\mathbf{k},p)a(\mathbf{k},p)$$

$$+\sum_{p,\mathbf{k}}\left(ip-\frac{1}{2}\mu H\right)b^*(\mathbf{k},p)b(\mathbf{k},p), \tag{4.3}$$

$$S_1=\sum_{\omega,\mathbf{k}}J(\mathbf{k})\left[\frac{1}{4}\sigma^z(\mathbf{k},\omega)\sigma^z(-\mathbf{k},-\omega)+\lambda^2\sigma^+(\mathbf{k},\omega)\sigma^-(\mathbf{k},\omega)\right], \tag{4.4}$$

in which

$$\sigma^z(\mathbf{k},\omega)$$
$$=(\beta N)^{-1/2}\sum_{\substack{\mathbf{k}_1-\mathbf{k}_2=\mathbf{k}\\ p_1-p_2=\omega}}[a^*(\mathbf{k}_1,p_1)a(\mathbf{k}_2,p_2)-b^*(\mathbf{k}_1,p_1)b(\mathbf{k}_2,p_2)],$$

$$\sigma^+(\mathbf{k},\omega)=(\beta N)^{-1/2}\sum_{\substack{\mathbf{k}_1-\mathbf{k}_2=\mathbf{k}\\ p_1-p_2=\omega}}a^*(\mathbf{k}_1,p_1)b(\mathbf{k}_2,p_2),$$

$$\sigma^-(\mathbf{k},\omega)=(\beta N)^{-1/2}\sum_{\substack{\mathbf{k}_1-\mathbf{k}_2=\mathbf{k}\\ p_1-p_2=\omega}}b^*(\mathbf{k}_1,p_1)a(\mathbf{k}_2,p_2). \qquad (4.5)$$

As a consequence of the boundary conditions (2.15) the frequencies in the expansion (4.1) have the form $p=2\pi\beta^{-1}(n+\frac{1}{4})$.

The averages of products of the quantities σ^z and $\sigma^\pm$ are of immediate interest. Therefore, it is convenient to rearrange the representation (4.2) by introducing into the numerator and denominator Gaussian integrals over auxiliary Bose-type fields $\varphi_i(\mathbf{k},\omega)$ $(i=1,2,3)$:

$$\int e^{S_0[\varphi]}D\mu[\varphi], \qquad (4.6)$$

$$S_0[\varphi]=-\frac{1}{4}\sum_{i,\mathbf{k},\omega}J^{-1}(\mathbf{k})\varphi_i(\mathbf{k},\omega)\varphi_i(-\mathbf{k},-\omega). \qquad (4.7)$$

Making next the shift

$$\varphi_1(\mathbf{k},\omega)\to\varphi_1(\mathbf{k},\omega)+\lambda J(\mathbf{k})(\sigma^+(-\mathbf{k},-\omega)+\sigma^-(\mathbf{k},\omega)),$$
$$\varphi_2(\mathbf{k},\omega)\to\varphi_2(\mathbf{k},\omega)+i\lambda J(\mathbf{k})(\sigma^-(\mathbf{k},\omega)-\sigma^+(-\mathbf{k},-\omega)),$$
$$\varphi_3(\mathbf{k},\omega)\to\varphi_3(\mathbf{k},\omega)+J(\mathbf{k})\sigma^z(\mathbf{k},\omega), \qquad (4.8)$$

which cancels the four-fermion term in S, we obtain

$$Z_\sigma/Z_{0\sigma}=\int e^{\tilde S}D\mu[a,b,\varphi]\Big/\int e^{\tilde S_0}D\mu[a,b,\varphi], \qquad (4.9)$$

$$\tilde S[a,b,\varphi]=S_0[\varphi]+S_0[a,b]-\sum_{\mathbf{k},\omega}\Big[\frac{1}{2}\varphi_3(\mathbf{k},\omega)\sigma^z(-\mathbf{k},-\omega)$$
$$+\lambda\psi^*(\mathbf{k},\omega)\sigma^-(\mathbf{k},\omega)+\lambda\psi(\mathbf{k},\omega)\sigma^+(\mathbf{k},\omega)\Big], \qquad (4.10)$$

where

$$\psi(\mathbf{k},\omega)={}^1/_2[\varphi_1(\mathbf{k},\omega)+i\varphi_2(\mathbf{k},\omega)],$$
$$\psi^*(\mathbf{k},\omega)={}^1/_2[\varphi_1(-\mathbf{k},-\omega)-i\varphi_2(-\mathbf{k},-\omega)]. \qquad (4.11)$$

The averages of σ^z and $\sigma^\pm$ are related to averages of the new variables φ_3, ψ, and ψ^* by simple formulas:

$$\langle\sigma^z(\mathbf{k},\omega)\rangle=-J^{-1}(\mathbf{k})\langle\varphi_3(\mathbf{k},\omega)\rangle,$$
$$\langle\sigma^z(\mathbf{k},\omega)\sigma^z(-\mathbf{k},-\omega)\rangle$$
$$=-2J^{-1}(\mathbf{k})+J^{-2}(\mathbf{k})\langle\varphi_3(\mathbf{k},\omega)\varphi_3(-\mathbf{k},-\omega)\rangle,$$
$$\langle\sigma^+(\mathbf{k},\omega)\sigma^-(-\mathbf{k},-\omega)\rangle=-J^{-1}(\mathbf{k})+J^{-2}(\mathbf{k})\langle\psi^*(\mathbf{k},\omega)\psi(\mathbf{k},\omega)\rangle, \qquad (4.12)$$

where $\langle\ldots\rangle$ denotes functional averaging with weight $\exp\tilde S$.

We shall consider the diagram technique that arises in the calculation of the integrals in (4.9). In deriving the technique we must take into account that for $H\neq 0$ the average $\langle\varphi_3(0,0)\rangle$ is nonzero, while at $T=T_c$ the system undergoes a phase transition associated with the onset, as $H\to 0$ and $N\to\infty$, of a nonzero magnetization (this corresponds to Bose condensation of the field φ_3) We shall take account of the Bose condensate of the field φ_3 by making the shift

$$\varphi_3(\mathbf{k},\omega)=\tilde\varphi_3(\mathbf{k},\omega)+\rho(\beta N)^{1/2}\delta_{\mathbf{k}0}\delta_{\omega 0}, \qquad (4.13)$$

after which the action takes the form

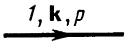
$$\tilde S=S_0[\tilde\varphi]+S_\rho[a,b]+S_{int}[a,b,\tilde\varphi]-{}^1/_4\beta N\rho^2J^{-1}(0), \qquad (4.14)$$

where

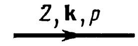
$$S_\rho[a,b]=\sum_{p,\mathbf{k}}\Big[ip+\frac{1}{2}(\rho+\mu H)\Big]a^*(\mathbf{k},p)a(\mathbf{k},p)$$
$$+\sum_{p,\mathbf{k}}\Big[ip-\frac{1}{2}(\rho+\mu H)\Big]b^*(\mathbf{k},p)b(\mathbf{k},p),$$

$$S_{int}=-\frac{1}{2}\sum_{\mathbf{k},\omega}\tilde\varphi_3(\mathbf{k},\omega)\sigma^z(-\mathbf{k},-\omega)-\lambda\sum_{\mathbf{k},\omega}\Big[\psi^*(\mathbf{k},\omega)\sigma^-(\mathbf{k},\omega)$$
$$+\sigma^+(\mathbf{k},\omega)\psi(\mathbf{k},\omega)\Big]-\frac{1}{2}(\beta N)^{1/2}J^{-1}(0)\rho\tilde\varphi(0,0). \qquad (4.15)$$

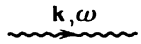
Taking as the free action the quantity $S_0[\tilde\varphi]+S_\rho[a,b]$, we obtain a diagram technique with the following elements (lines and vertices):



$-\langle a(\mathbf{k},p)a^*(\mathbf{k},p)\rangle=[ip+{}^1/_2(\rho+\mu H)]^{-1}$,



$-\langle b(\mathbf{k},p)b^*(\mathbf{k},p)\rangle=[ip-{}^1/_2(\rho+\mu H)]^{-1}$,

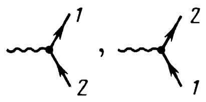


$-\langle\psi(\mathbf{k},\omega)\psi^*(\mathbf{k},\omega)\rangle=J(\mathbf{k})$,

$-\langle\varphi_3(\mathbf{k},\omega)\varphi_3(-\mathbf{k},-\omega)\rangle=2J(\mathbf{k})$,

$-{}^1/_2(\beta N)^{1/2}\rho J^{-1}(0)$,

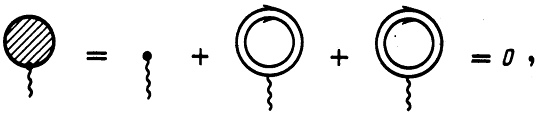
$-1/2(\beta N)^{1/2}$,



$-\lambda/(\beta N)^{1/2}$, (4.16)

in which the momentum and frequency are conserved at each vertex.

The parameter ρ defining the shift of the field φ_3 is found by equating the average $\langle\varphi_3(0,0)\rangle$ to zero, corresponding to a zero contribution from all diagrams with one external line. Graphically, the condition $\langle\tilde\varphi_3\rangle=0$ has the form



(4.17)

where the double lines correspond to the exact Green functions $G_{1,2}(\mathbf{k},p)$ of the fields a and b. Analytically, Eq. (4.17) has the form

$$\rho=2J(0)(\beta N)^{-1}\sum_{\mathbf{k},p}[G_1(\mathbf{k},p)-G_2(\mathbf{k},p)]. \qquad (4.18)$$

We note that the parameter ρ equals, apart from a factor $[2J(0)]^{-1}$, with the magnetization of the system. A nonzero solution of Eq. (4.16) for $N\to\infty$ and $H\to 0$ corresponds to the ferromagnetic state.

Only the Green functions of the fields $\tilde{\varphi}_3$, ψ, and ψ^* depend on the momentum, and, therefore, the expansion of the averages (4.11) in loops containing these Green functions corresponds to an expansion in the inverse range of the interaction.

The diagram technique constructed differs from the technique for Fermi systems only in the form of the frequencies appearing in the Green functions of the fields $a(\mathbf{k},p)$ and $b(\mathbf{k},p)$. We note that the Dyson equations for the exact Green functions of the fields $\tilde{\varphi}_3$, ψ, ψ^*, a, and b are equivalent to the Larkin equations.[2,4] However, the diagrammatic series for the self-energy parts are constructed considerably more simply in the present approach than in Refs. 2 and 4.

The variables a, a^*, b, and b^* appear in S quadratically, and the integral over them is equal to the determinant of the matrix M of this quadratic form:

$$M_{\mathbf{k},p;\mathbf{k}'q}=\begin{vmatrix}\left(ip+\frac{\mu H}{2}\right)\delta_{pq}\delta_{\mathbf{kk}'}+\frac{\varphi(\mathbf{k}-\mathbf{k}',p-q)}{2(\beta N)^{1/2}}, & \frac{\lambda\psi(\mathbf{k}-\mathbf{k}',p-q)}{(\beta N)^{1/2}}\\ \frac{\lambda\psi^*(\mathbf{k}'-\mathbf{k},q-p)}{(\beta N)^{1/2}}, & \left(ip-\frac{\mu H}{2}\right)\delta_{pq}\delta_{\mathbf{kk}'}-\frac{\varphi(\mathbf{k}-\mathbf{k}',p-q)}{2(\beta N)^{1/2}}\end{vmatrix}. \qquad (4.19)$$

This gives one further represenation of $Z_\sigma/Z_{0\sigma}$:

$$Z_\sigma/Z_{0\sigma}=\int D\mu[\varphi]\exp(S_{eff})\Big/\int D\mu[\varphi]\exp\{S_0[\varphi]\}. \qquad (4.20)$$

Here

$$S_{eff}=S_0[\varphi]+\ln\det\{M[\varphi_3,\psi]M^{-1}[0,0]\} \qquad (4.21)$$

is the effective action, depending on variables φ_3, ψ, and ψ^* that are directly related to the spin operators σ^z and $\sigma^\pm$. The effective action is nonpolynomial in the fields φ_3, ψ, and ψ^*, and it is essentially this which leads to difficulties in the formulation of the perturbation theory directly in terms of the spin operators.

A representation of the partition function of the Heisenberg model with arbitrary spin in the form of a functional integral with a nonpolynomial action containing three Bose fields, with the nonpolynomial part of the action having the same structure as $\ln\det(MM_0^{-1})$ in (4.21), has been obtained by Kolokolov.[10,11]

For the Ising model $\det M$ can be calculated in closed form. We have

$$\det(MM_0^{-1})=\prod_{i=1}^{N}\prod_{p=2\pi\beta^{-1}(n+1/4)}\left[1+\frac{(\mu H+x_i)^2}{4p^2}\right]\left(1+\frac{\mu^2H^2}{4p^2}\right)^{-1}=\prod_{i=1}^{N}\frac{\operatorname{ch}[\beta(\mu H+x_i)/2]}{\operatorname{ch}(\beta\mu H/2)}, \qquad (4.22)$$

where

$$x_i=N^{-1/2}\sum_{\mathbf{k}}\varphi_3(\mathbf{k},0)e^{i\mathbf{k}\mathbf{i}},$$

while for the partition function of the Ising model we obtain the well-known representation

$$Z_{I\sigma}=\left\{\int\exp\left[-\frac{\beta}{4}\sum_{\mathbf{i},\mathbf{j}}x_iR(\mathbf{i}-\mathbf{j})x_j\right]\prod_{\mathbf{i}}2\operatorname{ch}\frac{\beta}{2}(\mu H+x_i)\,dx_i\right\}\times\left\{\int\exp\left[-\frac{\beta}{4}\sum_{\mathbf{i},\mathbf{j}}x_iR(\mathbf{i}-\mathbf{j})x_j\right]\prod_{\mathbf{i}}dx_i\right\}^{-1}, \qquad (4.23)$$

where

$$R(\mathbf{i}-\mathbf{j})=N^{-1}\sum_{\mathbf{k}}J^{-1}(\mathbf{k})\exp\{i(\mathbf{k},\mathbf{i}-\mathbf{j})\}. \qquad (4.24)$$

In the general case of the Heisenberg model we can calculate the $N\to\infty$ asymptotic form of the partition function approximately by the method of stationary phase. For example, for the spherically symmetric model ($\lambda=1$) S_{eff} has a nonzero point of stationarity $[\varphi_3(\mathbf{i},\tau)=\rho,\ \psi=\psi^*=0]$, and Eq. (4.17) for the parameter ρ reduces in the one-loop approximation to the molecular-field equation

$$\rho=J(0)\operatorname{th}[\beta((\mu H+\rho)/2]. \qquad (4.25)$$

Calculation of the second variation of S_{eff} gives

$$\delta^2S_{eff}=-\frac{1}{4}\sum_{\mathbf{k}}\varphi_3(\mathbf{k},0)\varphi_3(-\mathbf{k},0)[J^{-1}(\mathbf{k})-b'(\rho)]-\frac{1}{4}\sum_{\mathbf{k},\omega\neq0}\varphi_3(\mathbf{k},\omega)\varphi_3(-\mathbf{k},-\omega)J^{-1}(\mathbf{k})-\sum_{\mathbf{k},\omega}\psi^*(\mathbf{k},\omega)\psi(\mathbf{k},\omega)\left[J^{-1}(\mathbf{k})-\frac{b(\rho)}{\Omega(\rho)-i\omega}\right], \qquad (4.26)$$

where

$$b(\rho)=\operatorname{th}[\beta(\mu H+\rho)/2],\quad b'(\rho)=db(\rho)/d\rho,\qquad \Omega(\rho)=\mu H+\rho. \qquad (4.27)$$

For the asymptotic form of the partition function of the spherical Heisenberg model in the one-loop approximation we obtain

$$Z_0 \approx \exp\{S_{eff}(0)\} \prod_{\mathbf{k}} [1 - J(\mathbf{k}) b'(\rho)]^{-1/2} \times \prod_{\mathbf{k},\omega} \left[1 - \frac{J(\mathbf{k}) b(\rho)}{\Omega(\rho) - i\omega} \right]^{-1}, \tag{4.28}$$

where

$$S_{eff}(\rho, 0) = -N \left[\frac{1}{4} \beta \rho^2 J^{-1}(0) - \ln 2\mathrm{ch} \frac{\beta}{2} \Omega(\rho) \right]. \tag{4.29}$$

From (4.28), by equating $1 - J(\mathbf{k}) b(\rho) [\Omega(\rho) - i\omega]^{-1}$ ($i\omega \to E$) to zero, we obtain the well-known expression for the spectrum in the molecular-field approximation:

$$E(\mathbf{k}) = \Omega(\rho) - J(\mathbf{k}) b(\rho). \tag{4.30}$$

It is also not difficult to obtain corrections to the molecular-field approximation.

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