Linear waves in two-fluid relativistic gasdynamics

M. B. Gavrikov and L. S. Solov'ev

M. V. Keldysh Applied Mathematics Institute, Academy of Sciences of the USSR
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This paper is devoted to the development of a theory of waves propagating in a two-component gaseous medium. In all cases considered we use only the method of two-fluid relativistic electromagnetic gasdynamics in the framework of the special relativity theory. We pay special attention to the problem of the interaction in a mixture of both neutral and charged gases when they move relative to one another. This interaction is for charged gases responsible for the appearance of ohmic resistance to an electrical current.

INTRODUCTION

The study of waves propagating in a plasma is a traditional part of plasma physics and a large number of papers have been devoted to it.[1,2] For the solution of various problems were invoked both two- and one-fluid approaches and also kinetic considerations without sufficiently analyzing the regions where each method is applicable and without justification of the corresponding limiting approaches. One must note also the incorrectness of using nonrelativistic equations of motion together with the relativistic equations of electrodynamics. In the present paper we apply consistently the mathematical apparatus of two-fluid relativistic electromagnetic gasdynamics (REMGD) including the complete set of Maxwell equations and the equations of the relativistic gasdynamics for the electron and ion gases. We take the finite plasma temperature, the effect of phenomena leading to wave damping, limiting cases of slow waves, and so on, into account in the framework of a single gasdynamic approach based upon using the equations of a two-fluid REMGD.

In this paper we consider linear waves in a homogeneous two-component medium. In the case of a plasma which is a mixture of electron and ion gases this medium may be anisotropic due to the pressure of a uniform external magnetic field. When we take dissipative processes which lead to the damping of propagating waves into account, we restrict ourselves to studying the interaction between the components of the mixture of two gases. For the plasma components such an interaction is equivalent to the appearance of an ohmic resistance due to the "friction" of the electron and ion gases when they move relative to one another.

The complete set of REMGD equations contains the Maxwell equations and the gasdynamic equations of motion obtained by setting the divergence of the total relativistic energy-momentum tensor and the set of dissipative tensors equal to zero. When describing the interaction to two gases we use an "interaction tensor"; for its construction we invoke the requirement of relativistic invariance and satisfaction of the correspondence principle.

All our considerations are made in the framework of Einstein's special relativity theory and allow us to take the limit to the non-relativistic case. We take into account the existence of uniform isotropic radiation in local thermodynamic equilibrium with matter of the two relativistic factors—the fact that the macroscopic velocity of the gas and the random velocities of the particles in it are comparable to the velocity of light—only the latter factor effects the propagation of linear waves in a medium at rest and it leads to an effective increase of weight of the particles of a hot gas.

1. RELATIVISTIC ELECTROMAGNETIC GASDYNAMIC EQUATIONS

In the framework of the special relativity theory the REMGD equations are derived from the conservation law for the divergence of the total energy-momentum 4-tensor

$$\hat{T}^k_{i;k}=\partial\hat{T}_i^k/\partial x^k=0,\quad \hat{T}_i^k=T_i^k+\mathcal{T}_i^k+\hat{\tau}_i^k \tag{1}$$

in four-dimensional space-time $x^i=(ct,x^\alpha), ds^2=g_{ik}dx^idx^k=c^2dt^2-d\mathbf{r}^2$, where we use the invariant velocity 4-vector $u^i=dx^i/ds=(1/\Gamma, v^\alpha/c\Gamma), \Gamma=(1-v^2/c^2)^{1/2}$.

The symmetric energy-momentum tensor $\hat{T}_{ik}$ contains the material tensor T_{ik}, the electromagnetic tensor $\mathcal{T}_{ik}$, and the sum of the dissipative tensors $\hat{\tau}_{ik}$. The tensors T_{ik} and $\mathcal{T}_{ik}$ defined by their mixed components are, respectively, equal to[3]

$$T_i^k=\rho c^2Wu_iu^k-p\delta_i^k,\quad \mathcal{T}_i^k=-F_{il}F^{kl}+{}^1/_4\delta_i^kF_{ln}F^{ln}, \tag{2}$$

where F_{ik} is the antisymmetric electromagnetic field tensor, p the pressure, and $\rho=mn$ the rest-mass density. The enthalpy c^2W and entropy S per unit rest mass are determined by the expressions

$$c^2W=c^2+\mathcal{E}+p/\rho,\quad TdS=c^2dW-dp/\rho, \tag{3}$$

where $\mathcal{E}$ is the internal energy and T the temperature.

Multiplying Eq. (1) by u^i (and after that summing over i) and using $u^iu_i=1, d/ds=u^i\partial/\partial x^i=\Gamma^{-1}d/cdt$, we get the mass-entropy conservation law:[4,5]

$$c^2W(\rho u^k)_{;k}+\rho T\,dS/ds+u^i\hat{\tau}^k_{i;k}=0, \tag{4}$$

where the electromagnetic field tensor $\mathcal{T}_i^k$ does not contribute to this equation. Equation (4) combines the rest-mass energy and entropy conservation laws. For a known energy production and with account taken of the change in rest-mass energy, Eq. (4) splits into two:[5,6]

$$c^3(\rho u^k)_{;k}=-\rho\varepsilon,\quad c\rho T dS/ds=\rho\varepsilon W-cu^i\hat{\tau}_i. \tag{5}$$

Here $\rho\varepsilon(n,T)$ is the energy production (in cm^3/c) in the rest system of an element of the fluid, $\hat{\tau}_i\equiv\hat{\tau}^k_{i;k}$. The right-hand side of Eq. (5) determines the heat release both in this process and in dissipative processes described by the tensor $\hat{\tau}_i^k$. When $\rho\varepsilon=0$ the first Eq. (5) becomes the rest-mass conservation equation for the gas.

The time component of Eq. (1) gives the conservation

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law of the total energy, which in the absence of an electromagnetic field has the form

$$\frac{\partial}{\partial t}\left(\frac{\rho W c^2}{\Gamma^2}-p\right)+\operatorname{div}\frac{\rho W c^2}{\Gamma^2}\mathbf{v}=-c\tau_0. \quad (6)$$

The partial derivative with respect to time pertains to the total energy $\rho\mathscr{E}^*$ which is equal to the sum of the relativistic rest-mass ($\rho\mathscr{E}_0$), thermal ($\rho\mathscr{E}_T$), and kinetic ($\rho\mathscr{E}_K$), energies:[1]

$$\rho\mathscr{E}^*=\rho W c^2/\Gamma^2-p=\rho\mathscr{E}_0+\rho\mathscr{E}_T+\rho\mathscr{E}_K,$$
$$\rho\mathscr{E}_0=\rho c^2/\Gamma,\quad \rho\mathscr{E}_T=(\rho\mathscr{E}+p)/\Gamma^2-p,\quad \rho\mathscr{E}_K=\rho\mathscr{E}_0(1/\Gamma-1). \quad (7)$$

With allowance for (5), we can write the energy-momentum conservation law in the form

$$\rho c^2\frac{d}{ds}Wu_i=\frac{\partial p}{\partial x^i}+\rho\varepsilon W u_i-\hat{\tau}_i. \quad (8)$$

The complete set of relativistic gasdynamics equations for a neutral gas can thus be written as the set of vector equations

$$\frac{\partial}{\partial t}\frac{\rho}{\Gamma}+\operatorname{div}\frac{\rho}{\Gamma}\mathbf{v}=-\frac{\rho\varepsilon}{c^2},\quad \frac{\rho T}{\Gamma}\frac{dS}{dt}=\rho\varepsilon W-cu^i\hat{\tau}_i,$$
$$\frac{\rho}{\Gamma}\frac{d}{dt}\frac{W}{\Gamma}\mathbf{v}=-\nabla p+\frac{\rho\varepsilon}{\Gamma c^2}W\mathbf{v}-\hat{\boldsymbol{\tau}}, \quad (9)$$

and the energy conservation law (6) will be their consequence.

Together with the viscosity and the thermal conductivity, the interaction of gases moving with different velocities is an important dissipative process. In particular, this process is responsible for the occurrence of resistance to an electrical current in a plasma.

The invariant 4-vector of the interaction of two gases, which satisfies the momentum conservation law and which is a linear combination of the 4-vectors u_i^+ and u_i^-, can be written in the form[2)]

$$\tau_i=\pm(\alpha u_i^+-\beta u_i^-). \quad (10)$$

The rate at which heat is released in each gas in its own system of coordinates, $Q=\rho T dS/ds=-cu^i\tau_i$, will then be equal to

$$Q_+=-c(\alpha-\beta/\Gamma'),\quad Q_-=-c(\beta-\alpha/\Gamma'), \quad (11)$$

and the total heat release is

$$Q_\Sigma=Q_++Q_-=c(\alpha+\beta)(1/\Gamma'-1)=2c^2\nu(1/\Gamma'-1).$$

The invariant quantity $1/\Gamma'=u_i^+u^i_-=(1-\mathbf{v}_+\mathbf{v}_-/c^2)/\Gamma_+\Gamma_-$ which occurs here can be expressed in term of the invariant relativistic relative velocity

$$\Gamma'=\left(1-\frac{v_{\rm rel}^2}{c^2}\right)^{1/2},\quad v_{\rm rel}=\frac{\{(\mathbf{v}_+-\mathbf{v}_-)^2-[\mathbf{v}_+\mathbf{v}_-]^2/c^2\}^{1/2}}{1-\mathbf{v}_+\mathbf{v}_-/c^2},$$

which is less than the velocity of light, $v_{\rm rel}<c$. The interaction parameter ν has the dimension of g/s·cm^3.

If we determine α and β from the conditions $Q_+/Q_-=m_-/m_+$, $\alpha+\beta=2c\nu$, we find

$$\tau_i=\pm\frac{2c\nu}{(m_++m_-)(1+1/\Gamma')}\left[\left(\frac{m_+}{\Gamma'}+m_-\right)u_i^+-\left(\frac{m_-}{\Gamma'}+m_+\right)u_i^-\right]. \quad (12)$$

When $m_+=m_-$ and in the nonrelativistic limit for any m_+ and m_- this expression changes accordingly to

$$\tau_i=\pm c\nu(u_i^+-u_i^-).$$

The heat releases $Q_\pm$ and Q_Σ are given according to (11) by the formulae

$$Q_\pm=\frac{2\nu c^2 m_\mp}{m_++m_-}\left(\frac{1}{\Gamma'}-1\right),\quad Q_\Sigma=2\nu c^2\left(\frac{1}{\Gamma'}-1\right). \quad (13)$$

The expressions for $Q_\pm$ are positive and invariant, if $\nu<0$ is an invariant function. The entropy of each gas then increases. The rate of release of thermal energy in the "laboratory" system of coordinates are

$$q_\pm=Q_\pm\Gamma_\pm. \quad (14)$$

The rate of change of total energy of each of the gases in the laboratory frame of reference, $U=-c\tau_0=\mp c(\alpha/\Gamma_+-\beta/\Gamma_-)$, can be written in the form

$$U=\mp\frac{2\nu(\mathbf{v}_+-\mathbf{v}_-)}{\Gamma_+\Gamma_-(1+1/\Gamma')(m_++m_-)}\left(\frac{m_+\mathbf{v}_+}{\Gamma_+}+\frac{m_-\mathbf{v}_-}{\Gamma_-}\right). \quad (15)$$

They vanish in the relativistic mass center system.

In the case of neutral gases, when the densities are conserved, the change in energy is the sum of the changes of their thermal and kinetic energies: $U=q+U_K$. According to Eqs. (13)–(15) it follows that the rates of change of the kinetic energies of each of the interacting gases are

$$U_K=\frac{2\nu}{\Gamma_+\Gamma_-(1+1/\Gamma')(m_++m_-)}\left\{\mp\left(\frac{m_+}{\Gamma_+}+\frac{m_-}{\Gamma_-}\right)(\mathbf{v}_+-\mathbf{v}_-)\mathbf{v}_\pm+\frac{m_\mp}{c^2\Gamma_\mp}[\mathbf{v}_+\mathbf{v}_-]^2\right\}. \quad (16)$$

The total set of two-fluid REMGD equations contains the Maxwell equations for the electromagnetic field:

$$\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\operatorname{rot}\mathbf{E},\quad \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\operatorname{rot}\mathbf{B}-\frac{4\pi}{c}\mathbf{j}, \quad (17)$$

and the gasdynamics equations for each of the charged gases:

$$\frac{\partial}{\partial t}\frac{\rho}{\Gamma}+\operatorname{div}\frac{\rho}{\Gamma}\mathbf{v}=-\frac{\rho\varepsilon}{c^2},\quad \frac{T}{\Gamma}\frac{dS}{dt}=\rho\varepsilon W-cu^i\hat{\tau}_i,$$
$$\frac{\rho}{\Gamma}\frac{d}{dt}\frac{W}{\Gamma}\mathbf{v}=-\nabla p+\frac{en}{\Gamma}\left(\mathbf{E}+\frac{1}{c}[\mathbf{vB}]\right)-\hat{\boldsymbol{\tau}}. \quad (18)$$

The remaining pair of Maxwell equations,

$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{div}\mathbf{E}=4\pi\rho_e, \quad (19)$$

are satisfied automatically if they were satisfied at the initial instant of time. Equations (18) are a set of equations for each gas. The charge and current densities are given by the formulae

$$\rho_e=\sum en/\Gamma,\quad \mathbf{j}=\sum en\mathbf{v}/\Gamma, \quad (20)$$

which connect these electrodynamic quantities with the gasdynamic velocity $\mathbf{v}$ and the density $n=\rho m$, where ρ is the rest-mass density. Under the condition $\hat{\tau}_i^+=-\hat{\tau}_i^-$ we get from (17) and (18) the total energy conservation law:

$$\frac{\partial}{\partial t}\left\{\sum\left(\rho\frac{Wc^2}{\Gamma^2}-p\right)+\frac{E^2+B^2}{8\pi}\right\}+\operatorname{div}\left(\sum\rho\frac{Wc^2}{\Gamma^2}\mathbf{v}+\frac{c}{4\pi}[\mathbf{EB}]\right)=0. \quad (21)$$

containing the electromagnetic field energy density $(E^2+B^2)/8\pi$ and the Poynting vector $c|\mathbf{E}\times\mathbf{B}|/4\pi$.

The interaction 4-vector of two oppositely charges gases, which determines the ohmic resistance, can be obtained from (12) through the substitution $\nu\to|e_+e_-|n_+n_-/\sigma$ (σ is the electrical conductivity):

$$\tau_i=\pm\frac{2|e_+e_-|n_+n_-c}{\sigma(1+1/\Gamma')}\left[u_i^+-u_i^-+\left(\frac{1}{\Gamma'}-1\right)\frac{m_+u_i^+-m_-u_i^-}{m_++m_-}\right]. \quad (22)$$

It follows from (18) and (22) that the invariant rates of release of Joule heat in ion and electron gases in their own coordinate systems are given by the expressions

$$Q=\frac{\rho}{\Gamma}\frac{dS}{dt}=-cu^i\tau_i=\frac{2|e_+e_-|n_+n_-m_\mp c^2}{\sigma(m_++m_-)}\left(\frac{1}{\Gamma'}-1\right). \quad (23)$$

The ratio $Q_+/Q_-=m_-/m_+$ and the total heat release

$$Q_\Sigma=2\sigma^{-1}|e_+e_-|n_+n_-c^2(1/\Gamma'-1)$$

are independent of the ion and electron masses and in the nonrelativistic limit for a neutral gas ($|e_+n_+|=|e_-n_-|$) change to the expression $Q_\Sigma=j^2/\sigma$ which is known from electrodynamics.

The thermodynamic function occurring in the REMGD equations for a perfect gas in equilibrium with isotropic radiation (photon gas) are given by the equations

$$p=knT+{}^1/_3aT^4,\quad \rho\mathscr{E}=knT/(\gamma-1)+aT^4, \quad (24)$$

which depend only on the density n and temperature T. Here γ is the adiabatic index, $a=\pi^2k^4/15\hbar^3c^3=7.56\cdot10^{-15}$ erg/cm^{3}K^4 is the Stefan-Boltzmann constant, $k=1.38\cdot10^{-16}$ erg/K is the Boltzmann constant. For an approximate treatment of the dynamics of a mixture of neutral gases one normally uses the effective particle mass $m=\rho/n=\Sigma m_\alpha n_\alpha/\Sigma n_\alpha$. The equations of state for degenerate gases which have a constant entropy are given in Refs. 7 and 8.

Moreover, for a study of waves in a uniform two-component plasma one considers the case $\rho\varepsilon=0$ and takes into account only the single dissipative force (22). The equation of state (24) with account taken of equilibrium radiation is used solely for a single-temperature gas.

2. SOUND WAVES

The equations of relativistic gasdynamics for a neutral gas without dissipation and when the rest mass is conserved, $\rho\varepsilon=0$ are, according to (9) and (3) the set

$$\frac{\partial}{\partial t}\frac{\rho}{\Gamma}+\operatorname{div}\frac{\rho}{\Gamma}\mathbf{v}=0,\quad T\frac{dS}{dt}\equiv c^2\frac{dW}{dt}-\frac{1}{\rho}\frac{dp}{dt}=0,$$
$$\frac{\rho}{\Gamma}\frac{d}{dt}\frac{W}{\Gamma}\mathbf{v}=-\nabla p. \quad (25)$$

In the linear approximation when waves are propagating in a uniform medium at rest $\Gamma=(1-v^2/c^2)^{1/2}\approx1$ and Eqs. (25) differ from the nonrelativistic ones only because of the heavier particle mass, $m\to m^*=mW$. Plane longitudinal sound waves can propagate in a neutral gas; in the linear approximation they are proportional to $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$ and the perturbation of the velocity $\mathbf{v}$ in them is directed along the wave vector $\mathbf{k}$.

It follows from (25) that the phase velocity of the sound waves $\omega/k=c_s$ is given by the formula

$$c_s=\left[\frac{1}{W}\left(\frac{dp}{d\rho}\right)_s\right]^{1/2}. \quad (26)$$

If we assume that W and p are function of n and T, the expression for c_s^2 can be written in the form

$$c_s^2=\frac{1}{mW}\frac{W_Tp_n-W_np_T}{W_T-p_T/\rho c^2}. \quad (27)$$

The subscripts n and T indicate partial derivatives with respect to n and T. In the case when $W_T=0$ we have $c_s^2=\rho c^2W_\rho/W$.

If we neglect radiation we have for a perfect gas from (27)

$$c_s^2=\frac{\gamma p/\rho}{1+\gamma p/(\gamma-1)\rho c^2}. \quad (28)$$

From this it is clear that the sound phase velocity increases when the dimensionless parameter $\gamma p/\rho c^2$ increases, and in the limit $\gamma p/\rho c^2\gg1$ it tends to $(\gamma-1)^{1/2}c$. As the sound phase velocity is the same as the group velocity $V_{gr}=d\omega/dk$ which cannot exceed the light velocity c, it follows from (28) that there is a restriction on the magnitude of the adiabatic index $(1<\gamma<2)$ which determines the sound velocity.

Evaluating the sound velocity in a perfect gas, taking equilibrium radiation into account, leads, according to Eqs. (24) and (27), to

$$c_s^2=\frac{c^2}{3\delta+1/(\gamma-1)}\,\frac{\delta^2+5\delta+\gamma/(\gamma-1)}{\mu+\delta+\gamma/(\gamma-1)}, \quad (29)$$

where $\mu=mc^2/kT$ and $\delta=4aT^3/3kn$ are dimensionless parameters. The requirement $c_s^2<c^2$ leads as in the earlier case also to the restriction $1<\gamma<2$ and c_s is a maximum as $\mu\to0$, $\delta\to0$, when $c_s\to(\gamma-1)^{1/2}c$.

The limiting value of the sound velocity $c_s=c$ is reached as $\rho\to\infty$ in the case of a so-called extremely rigid equation of state[9] $p=c^2b\rho^2, \rho\mathscr{E}=p, W=1+2b\rho$ corresponding to $\gamma=2$, when

$$c_s^2=2b\rho c^2/(1+2b\rho). \quad (30)$$

Here $c_s\to c$ as $b\rho\to\infty$, also in agreement with the formula $c_s=(\gamma-1)^{1/2}c$ for the limit of the sound velocity.

3. SOUND WAVES IN A MIXTURE OF NEUTRAL GASES

The sound velocity in a mixture of neutral gases is usually unique and determined by the properties of the mixture as a whole. Such a result can be the consequence only of a rather strong interaction between the separate components of the mixture. Below we consider the mixture of two neutral gases which interact with one another through a "friction force" which in the non-relativistic limit is proportional to the difference of the velocities of the gases.

When we take the interaction of the gases, which is described by the 4-vector (12), into account we can write the linearized equations of motion (25) for the mixture in the form

$$\partial\rho/\partial t=-\rho\operatorname{div}\mathbf{v},\quad dp=\rho c^2dW,$$
$$\rho W\partial\mathbf{v}/\partial t=-\nabla p\mp\nu(\mathbf{v}_+-\mathbf{v}_-) \quad (31)$$

(as the change in the entropy $\propto v^2$).

When there is no interaction ($\nu = 0$) the sound velocities in each of the components are different:

$$c_s^2 = \frac{1}{W}\frac{dp}{d\rho} = \frac{1}{W}\frac{\rho c^2 \partial W/\partial \rho}{1-\rho c^2 \partial W/\partial \rho} \quad (32)$$

and are determined by their "own" functions $W_\pm(\rho, p)$. For a longitudinal wave in a mixture of gases (indicated by the indexes $\pm$) we find, assuming the perturbations $\tilde{\rho}$, $\tilde{p}$, v_x to be $\propto \exp[i(kx-\omega t)]$, the set of equations

$$\left(\frac{\omega^2}{k^2} - c_s^2\right) v = \mp \frac{i\nu\omega}{\rho^* k^2}(v_+ - v_-), \quad (33)$$

where $\rho^* = \rho W$. If we temporarily write $\nu\omega/\rho^* k^2 = \beta$ we can write the dispersion relation that follows from (33) in the form of a biquadratic equation in $\omega/k = u$:

$$u^4 - [c_+^2 + c_-^2 - i(\beta_+ + \beta_-)]u^2 + c_+^2 c_-^2 - i(\beta_+ c_-^2 + \beta_- c_+^2) = 0. \quad (34)$$

Here $c_\pm = c_s^\pm$ are the sound velocities in the separate interacting gases, determined by Eqs. (32).

When $c_+ = c_- = c_s$ Eq. (34) splits into two equations:

$$u^2 - c_s^2 = 0, \quad u^2 + ik(\beta_0^+ + \beta_0^-)u - c_s^2 = 0 \quad (\beta_0 = \nu/\rho^* k^2). \quad (35)$$

Hence it follows that in that case there are solutions corresponding to undamped and damped sound waves. It is clear from (33) that in the undamped wave $v_+ = v_-$, i.e., both gases oscillate with the same velocities.

The formal solution of Eq. (34) will be

$$u^2 = \tfrac{1}{2}\{c_+^2 + c_-^2 - i(\beta_+ + \beta_-) \pm [(c_+^2 - c_-^2)^2 - 2i(\beta_+ - \beta_-)(c_+^2 + c_-^2) - (\beta_+ + \beta_-)^2]^{1/2}\}. \quad (36)$$

In the case of a weak interaction, $\beta/c_s^2 \ll 1$ it follows from (36) that

$$\frac{\omega^2}{k^2} \approx \begin{cases} c_+^2 - i\beta_+ \\ c_-^2 - i\beta_- \end{cases}, \quad \frac{\omega}{k} \approx \begin{cases} c_+ - i\nu/2k\rho_+^* \\ c_- - i\nu/2k\rho_-^* \end{cases}, \quad (37)$$

i.e., there are two weakly damped waves with phase velocities which are the same as the sound velocities in the non-interacting gases.

In the opposite limiting case of strongly interacting gases, $\beta/c_s^2 \gg 1$, one of the roots of Eq. (36) describes a strongly damped wave, while we get for the second the expression

$$\frac{\omega^2}{k^2} = \frac{\rho_+^* c_+^2 + \rho_-^* c_-^2}{\rho_+^* + \rho_-^*} - \frac{2i\beta_+\beta_-}{(\beta_+ + \beta_-)^3}(c_+^2 - c_-^2)^2, \quad (38)$$

from which it follows that

$$\frac{\omega}{k} = \left(\frac{\rho_+^* c_+^2 + \rho_-^* c_-^2}{\rho_+^* + \rho_-^*}\right)^{1/2} - \frac{ik}{\nu}\left(\frac{\rho_+^* \rho_-^*}{\rho_+^* + \rho_-^*}\right)^2 \frac{(c_+^2 - c_-^2)^2}{\rho_+^* c_+^2 + \rho_-^* c_-^2}. \quad (39)$$

There is thus in that case a weakly damped sound wave with a phase velocity and a damping rate which are invariant under an exchange of the indexes $\pm$ and which are determined by the general properties of the gas mixture. When $c_+ = c_-$ the damping rate vanishes. One can show that the set of propagating waves consists only of the cases (37) and (39), as all other waves are strongly damped.

For a mixture of non-relativistic gases, $\rho^* = \rho$, $c_s^2 = \gamma p/\rho$, the phase velocity is, according to (39) equal to

$$V_{\text{ph}} = [(\gamma_+ p_+ + \gamma_- p_-)/(\rho_+ + \rho_-)]^{1/2}. \quad (40)$$

Putting $V_{\text{ph}}^2 = \gamma(p_+ + p_-)/(\rho_+ + \rho_-)$, we find an expression for the effective adiabatic index for the gas mixture: $\gamma = (\gamma_+ p_+ + \gamma_- p_-)/(p_+ + p_-)$.

The coefficient of the velocity difference in (31) is of the order of $\nu/\rho \sim 1/\tau$ where τ is the relaxation time. One verifies easily that the dimensionless small expansion parameter used in deriving (39) is of the order of $\omega\tau = 2\pi\tau/T_\lambda$, i.e., the relaxation time τ must be small compared to the wave period T_λ.

4. ELECTROMAGNETIC WAVES IN TWO-FLUID REMGD

The linearized equations describing the propagation of waves in a uniform two-component plasma at rest when we take into account the finite conductivity and when there is a uniform magnetic field present can, according to (17), (18), and (22), be written in the form

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta \mathbf{E} = -4\pi\left(\nabla \rho_e + \frac{1}{c^2}\frac{\partial \mathbf{j}}{\partial t}\right),$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \Delta \mathbf{B} = \frac{4\pi}{c}\operatorname{rot}\mathbf{j},$$

$$\frac{\partial n}{\partial t} = -n \operatorname{div}\mathbf{v}, \quad (41)$$

$$m^* n\frac{\partial \mathbf{v}}{\partial t} = -m^* c_s^2 \nabla n \pm en\left(\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right) \mp \frac{e^2 n^2}{\sigma}(\mathbf{v}_+ - \mathbf{v}_-).$$

We have put here

$$e_+ = -e_- = e, \quad m^* = mW, \quad \rho_e = e(n_+ - n_-), \quad \mathbf{j} = en(\mathbf{v}_+ - \mathbf{v}_-),$$

where in the unperturbed state $\mathbf{v} = 0$, $\mathbf{E} = 0$, $n_+ = n_- = n$. Substituting into (41) the required solution $\propto \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ for the perturbations, we get

$$\left(\frac{\omega^2}{c^2} - k^2\right)\tilde{\mathbf{E}} = 4\pi i\left(\mathbf{k}\rho_e - \frac{\omega}{c^2}\mathbf{j}\right), \quad \left(\frac{\omega^2}{c^2} - k^2\right)\tilde{\mathbf{B}} = \frac{4\pi i}{c}[\mathbf{jk}],$$

$$\nabla \tilde{n} = \frac{in}{\omega}(\mathbf{kv})\mathbf{k},$$

$$\omega^2 \mathbf{v} - c_s^2(\mathbf{kv})\mathbf{k} \mp \frac{4\pi e^2 n}{m^*}\frac{(\mathbf{kV})\mathbf{k} - \mathbf{V}\omega^2/c^2}{\omega^2/c^2 - k^2}$$

$$= \pm\frac{ie\omega}{m^* c}\left([\mathbf{vB}] - \frac{enc}{\sigma}\mathbf{V}\right), \quad (42)$$

where $\mathbf{V} = \mathbf{v}_+ - \mathbf{v}_-$ and the tilde indicates perturbations of the corresponding quantities.

Choosing a Cartesian system of coordinates such that the x axis is directed along $\mathbf{k}$ and the xy plane contains the vector $\mathbf{B}$, we find a set of six equations for the components of the velocities $\mathbf{v}_+$ and $\mathbf{v}_-$:

$$(u^2 - c_s^2)v_x \mp \alpha^2 c^2 V_x = \mp\frac{iu}{k}(v_z \Omega_y - \beta V_x),$$

$$v_y \pm \alpha^2 \frac{V_y}{1 - u^2/c^2} = \pm\frac{i}{ku}(v_z \Omega_x - \beta V_y), \quad (43)$$

$$v_z \pm \alpha^2 \frac{V_z}{1 - u^2/c^2} = \pm\frac{i}{ku}(v_x \Omega_y - v_y \Omega_x - \beta V_z).$$

Here

$$u = \omega/k, \quad \mathbf{\Omega} = e\mathbf{B}/m^* c, \quad \beta = \varkappa^2 c^2/4\pi\sigma,$$
$$\alpha^2 \equiv \varkappa^2/k^2 = 4\pi e^2 n/m^* c^2 k^2,$$

and α^2 is a dimensionless parameter proportional to the density n.

A. Waves in an ideal plasma without a magnetic field

It is clear from (43) that when there is no magnetic field ($\mathbf{\Omega} = 0$) there are two kinds of waves, longitudinal and transverse, with the velocity vector $\mathbf{v}$ in them directed, respectively, along and at right angles to the wave vector $\mathbf{k}$. Neglecting the interaction between the electron and ion gases ($\beta = 0$), the phase and group velocities of the transverse waves are equal to

$$V_{\rm ph} = \omega/k = c[1+(\varkappa_+^2+\varkappa_-^2)/k^2]^{1/2},$$
$$V_{\rm gr} = d\omega/dk = c[1+(\varkappa_+^2+\varkappa_-^2)/k^2]^{-1/2}. \qquad (44)$$

From this it is clear that $V_{\rm ph} \geqslant c, V_{\rm gr} \leqslant 1$ and in free space as $\varkappa_+^2 + \varkappa_-^2 \to 0$ the phase and group velocities tend to the light velocity c.

The dispersion equation for longitudinal waves which follow from the first Eq. (43) has, when $\Omega_y = 0$ and $\beta = 0$, the form

$$(u^2-c_+^2-\alpha_+^2c^2)(u^2-c_-^2-\alpha_-^2c^2) = \alpha_+^2\alpha_-^2c^4,$$

where $c_\pm = c_s^\pm$ are the sound velocities in the positively and negatively charged gases. Solving the biquadratic equation we find

$$u^2 = \frac{1}{2}\left\{c_+^2+c_-^2+(\varkappa_+^2+\varkappa_-^2)\frac{c^2}{k^2} \pm\left(\left[c_+^2-c_-^2+(\varkappa_+^2-\varkappa_-^2)\frac{c^2}{k^2}\right]^2 + \varkappa_+^2\varkappa_-^2\frac{c^4}{k^4}\right)^{1/2}\right\}. \qquad (45)$$

The relation obtained determines two velocities for the fast and the slow longitudinal waves. In the case $\alpha^2 \ll c_s^2/c^2$ it follows from (45) that

$$\omega/k = c_+, \qquad \omega/k = c_-, \qquad (46)$$

i.e., there are two phase velocities which are equal to the sound velocities in two non-interacting neutral gases. For the opposite sign of the inequality $\alpha^2 \gg c_s^2/c^2$ we get

$$\frac{\omega}{k} = (\varkappa_+^2+\varkappa_-^2)^{1/2}\frac{c}{k}, \qquad \frac{\omega}{k} = \left(\frac{\varkappa_+^2c_-^2+\varkappa_-^2c_+^2}{\varkappa_+^2+\varkappa_-^2}\right)^{1/2} = \left(\frac{m_+^{*}c_+^2+m_-^{*}c_-^2}{m_+^{*}+m_-^{*}}\right)^{1/2}. \qquad (47)$$

Here both phase velocities are collectivized, as they depend on the general characteristics of the plasma. In a plasma the fast wave is called the Langmuir wave and the slow one the sound wave.

The simplest dispersion relation is obtained for gases consisting of particles with equal mass, $m_+ = m_- = m$, where $c_+ = c_- = c_s$ and $\varkappa_+ = \varkappa_- = \varkappa$. In that case the phase and group velocities equal, respectively,

$$\frac{\omega}{k} = \begin{cases}(c_s^2+2\varkappa^2c^2/k^2)^{1/2}\\ c_s\end{cases}, \qquad \frac{d\omega}{dk} = \begin{cases}c_s^2(c_s^2+2\varkappa^2c^2/k^2)^{-1/2}\\ c_s\end{cases}. \qquad (48)$$

For the slow wave the phase and group velocities are exactly the same as the relativistic sound velocity c_s whereas the phase velocity of the fast wave is larger than c_s and the group velocity less than c_s. It is clear from (48) that the group velocity of the Langmuir waves vanishes only when $c_s = 0$, i.e., at zero temperature.

B. Waves propagating along the magnetic field

It is clear from (43) that when there is a magnetic field present the simplest case to analyze is a wave moving along the magnetic field when $\Omega = \Omega_x$. The magnetic field in that case does not affect the longitudinal wave and, if there is no dissipation, it follows from (43) for the transverse wave that

$$v_y \pm \alpha^2\frac{v_y^+-v_y^-}{1-u^2/c^2} = \pm\frac{i\Omega}{ku}v_z, \qquad v_z \pm \alpha^2\frac{v_z^+-v_z^-}{1-u^2/c^2} = \mp\frac{i\Omega}{ku}v_y. \qquad (49)$$

Introducing the complex velocity vector $v = v_y + iv_z$ we get a set of two equations for $v_\pm$:

$$v \pm \alpha^2\frac{v_+-v_-}{1-u^2/c^2} = \pm\frac{\Omega}{ku}v. \qquad (50)$$

The dispersion equal for $u = \omega/k$ which follows from this has the form

$$(1-u^2/c^2)(u-\Omega_+/k)(u+\Omega_-/k)+(\alpha_+^2+\alpha_-^2)u^2 = 0. \qquad (51)$$

For waves with the opposite circular polarization $u \to -u$ and $v = v_y - iv_z$.

For particles with equal masses this equation reduces to a biquadratic one, and this enables us to find its exact solution:

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left\{c^2+\frac{2\varkappa^2c^2}{k^2}+\frac{\Omega^2}{k^2} \pm\left[\left(c^2+\frac{2\varkappa^2c^2}{k^2}+\frac{\Omega^2}{k^2}\right)^2-\frac{4\Omega^2c^2}{k^2}\right]^{1/2}\right\}. \qquad (52)$$

For low-frequency oscillations, $u^2/c^2 \ll 1$, it follows from (51) that

$$\omega = {}^1\!/_2\{\Omega_+-\Omega_-\pm[(\Omega_++\Omega_-)^2+4\Omega_+\Omega_-(\alpha_+^2+\alpha_-^2)]^{1/2}\} (1+\alpha_+^2+\alpha_-^2)^{-1}. \qquad (52a)$$

Hence we get for a rather dense plasma, $\alpha_+^2 + \alpha_-^2 \gg 1$,

$$\frac{\omega}{k} = \frac{1}{k}\left(\frac{\Omega_+\Omega_-}{\alpha_+^2+\alpha_-^2}\right)^{1/2} = \frac{B}{[4\pi n(m_+^{*}+m_-^{*})]^{1/2}}. \qquad (52b)$$

In the non-relativistic limit transverse waves propagate in this case with the Alfvén velocity in agreement with classical MHD.

C. Waves propagating at an angle to the magnetic field

Solving the first two Eqs. (43) with respect to v_x and v_y and substituting them into the third equation we get a dispersion equation for waves propagating at an arbitrary angle to the magnetic field:

$$\left[b_+ - \frac{1-u^2/c^2}{k^2u^2}\left(\frac{u^2\Omega_{+,y}^2}{c^2D_1}a_- + \frac{1-u^2/c^2}{D_2}\Omega_{+,x}^2b_-\right)\right] \times\left[b_- - \frac{1-u^2/c^2}{k^2u^2}\left(\frac{u^2\Omega_{-,y}^2}{c^2D_1}a_+ + \frac{1-u^2/c^2}{D_2}\Omega_{-,x}^2b_+\right)\right]$$
$$= \alpha_+^2\alpha_-^2\left[1+\frac{1-u^2/c^2}{k^2u^2}\left(\frac{u^2\Omega_{+,y}\Omega_{-,y}}{c^2D_1} - \frac{1-u^2/c^2}{D_2}\Omega_{+,x}\Omega_{-,x}\right)\right]^2. \qquad (53)$$

We have used here the notation

$$a = (u^2-c_s^2)/c^2-\alpha^2, \qquad D_1 = a_+a_- - \alpha_+^2\alpha_-^2,$$
$$b = 1-u^2/c^2+\alpha^2, \qquad D_2 = b_+b_- - \alpha_+^2\alpha_-^2. \qquad (54)$$

In the general case Eq. (53) is rather complicated as it contains all earlier special cases. We therefore restrict ourselves to remarking that with B_x^2 and B_y^2 as the coordinates Eq. (53) will describe a set of second-order curves.

In the case of particles with equal masses, $m_+ = m_-$, Eq. (53) simplifies considerably and transforms into a pair of equations:

$$b - \frac{1-u^2/c^2}{k^2u^2}\left(\frac{u^2\Omega_y^2}{c^2D_1}a + \frac{1-u^2/c^2}{D_2}\Omega_x^2 b\right)$$
$$= \pm\alpha^2\left[1 + \frac{1-u^2/c^2}{k^2u^2}\left(\frac{u^2\Omega_y^2}{c^2D_1} - \frac{1-u^2/c^2}{D_2}\Omega_x^2\right)\right]. \quad (55)$$

Substituting here the expressions for a, b, D_1, and D_2 we get two equations which are cubic in u^2:

$$\left[\frac{u^2-c_s^2}{c^2} - \alpha^2 \mp \alpha^2\right]\left[u^2\left(1-\frac{u^2}{c^2}+2\alpha^2\right) - \left(1-\frac{u^2}{c^2}\right)\frac{\Omega_x^2}{k^2}\right]$$
$$- u^2\left(1-\frac{u^2}{c^2}+\alpha^2\pm\alpha^2\right)\frac{\Omega_y^2}{c^2k^2} = 0 \quad (56)$$

in accordance with the $\pm$ -signs in Eqs. (55).

In the cases $\Omega_y = 0$ and $\Omega_x = 0$ when the waves propagate along and at right angles to the magnetic field, Eqs. (56) reduce to biquadratic ones. In the first case both Eqs. (56) lead to the single dispersion equation (52), and in the second case we have two solutions:

$$u^2 = c_s^2 + 2\alpha^2c^2 + \Omega^2/k^2,$$
$$u^2 = {}^1/_2\{c^2+2\alpha^2c^2+c_s^2+\Omega^2/k^2$$
$$\pm[(c^2+2\alpha^2c^2-c_s^2-\Omega^2/k^2)^2+8\alpha^2c^2\Omega^2/k^2]^{1/2}\}, \quad (57)$$

corresponding to longitudinal and transverse polarisations $[\mathbf{k}\times\mathbf{v}] = 0$ and $\mathbf{k}\cdot\mathbf{v} = 0$. One verifies easily that all roots are real and that the group velocity does not exceed the velocity of light as long as $c_s < c$.

When $\alpha^2 \gg 1$ it follows in the nonrelativistic case from (56) that

$$u^2 = {}^1/_2\{c_s^2+V_A^2 \pm [(c_s^2+V_A^2)^2 - 4c_s^2V_{Ax}^2]^{1/2}\},$$

where $V_A = B(4\pi\rho)^{1/2}$ and $\rho = 2mn$, i.e., we get the known expressions for the fast and slow magnetosonic waves in MHD[10] achieving the limiting transition from two-fluid REMGD to one-fluid MHD without the usual assumption that $m_+ \gg m_-$.

D. Waves in a plasma with account of ohmic resistivity

When we take into account the effect of the finite conductivity of the plasma on the wave propagation we restrict ourselves to the case when there is no external magnetic field.

For longitudinal wave $[\mathbf{k}\times\mathbf{v}] = 0$ we get from (43) the dispersion equation

$$u^4 - [c_+^2+c_-^2+c^2(\alpha_+^2+\alpha_-^2)(1-iku/4\pi\sigma)]u^2$$
$$+c^2(\alpha_+^2c_-^2+\alpha_-^2c_+^2)(1-iku/4\pi\sigma)+c_+^2c_-^2=0. \quad (58)$$

When the conductivity is sufficiently small, $\omega/4\pi\sigma \gg 1$, it follows from this that

$$u^2 = \frac{\alpha_+^2c_-^2+\alpha_-^2c_+^2}{\alpha_+^2+\alpha_-^2}$$
$$-\frac{4\pi i\sigma}{kc^2}\frac{\alpha_+^2\alpha_-^2(c_+^2-c_-^2)^2}{(\alpha_+^2+\alpha_-^2)^{5/2}(\alpha_+^2c_-^2+\alpha_-^2c_+^2)^{1/2}}. \quad (59)$$

In the case of equal masses, $m_+ = m_-$, Eq. (58) has an exact solution:

$$\omega = \pm kc, \quad \omega = \pm[2\varkappa^2c^2+k^2c_s^2-(\varkappa^2c^2/4\pi\sigma)^2]^{1/2} - i\varkappa^2c^2/4\pi\sigma. \quad (60)$$

There are therefore weakly damped waves both in the case of high and in the case of low electrical conductivity, and in the case of equal masses there exists an undamped sound wave.

For transverse wave $\mathbf{k}\cdot\mathbf{v} = 0$, it follows from (43) that

$$u^3-(1+\alpha_+^2+\alpha_-^2)c^2u-i(\alpha_+^2+\alpha_-^2)(c^2-u^2)kc^2/4\pi\sigma=0. \quad (61)$$

For high conductivity it follows from this that

$$\omega = \pm c(k^2+\varkappa_+^2+\varkappa_-^2)^{1/2} - \frac{ic^2}{8\pi\sigma}\frac{(\varkappa_+^2+\varkappa_-^2)^2}{k^2+\varkappa_+^2+\varkappa_-^2}, \quad (62)$$

and in the opposite case of a low conductivity

$$\omega = \pm kc - 2\pi i\sigma.$$

Here also there exist in both limiting cases propagating waves with a damping which tends to zero, respectively, as $\sigma\to\infty$ and as $\sigma\to 0$.

BASIC RESULTS

1. To take into account the effects of the interaction between the two gases when they move relative to one another, we have introduced a relativistically invariant 4-vector of the exchange of momentum. In the non-relativistic limit the corresponding friction force is proportional to the difference between the velocities of the gases and for a plasma is equivalent to ohmic resistance.

2. We have obtained dispersion equations for electromagnetic waves propagating in a two-component relativistic plasma.

3. Starting from the requirement that the sound velocity is limited since it cannot exceed the light velocity, we obtained a restriction on the adiabatic index, $1<\gamma<2$.

4. We have shown that when we take into account the interaction in a mixture of neutral gases, weakly damped "collective" sound waves propagate with a phase velocity which is determined by the general characteristics of the mixture as a whole.

5. We have shown that sound waves in a plasma (in a mixture of charged particles) become collective at sufficiently large densities even without special account of the interaction between the plasma components.

6. We have considered plasma-wave damping caused by the finite electrical conductivity and we have noted the existence of undamped sound waves in the case of particles of equal masses.

7. We have obtained general solutions of the dispersion equation for waves in a plasma propagating along and at right angles to the magnetic field in the case of equal masses of the charged particles.

8. We have shown that the limiting transition to results described by one-fluid MHD is accomplished in the case of high densities $4\pi e^2n/mc^2k^2 \gg 1$ and low wave velocities $\omega^2/k^2c^2 \ll 1$ and low particle velocities $v_T^2 \ll c^2$.

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APPENDIX

Ohm's law

We can obtain the relativistic expression for Ohm's law in two-fluid REMGD by considering stationary plasma flow in a longitudinal electrical field E. We then look for the interaction 4-vector in the form

$$\tau_i^{\pm}=\alpha_{\pm}u_i^{\pm}+\beta_{\pm}u_i^{\mp}.$$

From the conditions of conservation fo thermal energy in both ($\pm$) plasma components $u^i_{\pm}\,\tau_i^{\pm}=0$ it follows that

$$\alpha_{\pm}+\beta_{\pm}/\Gamma'=0,\qquad \tau_i=-\beta_{\pm}(u_i^{\pm}/\Gamma'-u_i^{\mp}).$$

The conditions for the cancellation of the momentum changes by the electrical field E give $\beta_+=\beta_-=-ce^2n^2/\sigma$, and hence, when $n_+=n_-$

$$\tau_i=\frac{e^2n^2c}{\sigma}\left(\frac{u_i^{\pm}}{\Gamma'}-u_i^{\mp}\right).$$

The expression for the electrical field following from the two equations of momentum conservation takes correspondingly the form[6]

$$E=\frac{en}{\sigma}\,\frac{v_+-v_-}{\Gamma_+\Gamma_-}=\frac{env_{\rm rel}}{\sigma\Gamma'}.$$

In the non-relativistic limit this formula becomes Ohm's law $E=j/\sigma$.

[1)] The corresponding splitting was carried out in Ref. 5 for a gravitating sphere in GRT.

[2)] The plus and minus are here the indexes of the components of the mixture of the two gases.

[1] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, Academic Press, New York, 1973.

[2] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics*, Pergamon, 1978.

[3] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, Pergamon, 1975.

[4] K. P. Stanyukovich, *Unsteady Motion of a Continuous Medium* Pergamon, 1960.

[5] L. S. Solov'ev, Preprint Inst. At. En.-3362/1, 1980.

[6] V. Ts. Gurovich and L. S. Sovol'ev, Zh. Eksp. Teor. Fiz. **91**, 1144 (1936) [Sov. Phys. JETP **64**, 677 (1986)].

[7] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon, 1969.

[8] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Chicago University Press, 1939.

[9] Ya. B. Zel'dovich and I. D. Novikov, *Theory of Gravitation and Stellar Evolution*, Chicago University Press, 1983.

[10] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, 1984.

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