Vortex dynamo in a convective medium with helical turbulence

S. S. Moiseev, P. B. Rutkevich, A. V. Tur, and V. V. Yanovskiĭ

Institute of Space Research, Academy of Sciences of the USSR
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A possible mechanism for the generation of large-scale vortices by helical turbulence is analyzed. At certain values of the helical-turbulence parameter, a new type of instability occurs in the liquid. It results in the generation of a large vortex with a nontrivial topology.

1. INTRODUCTION

Various structures in turbulence are presently the subject of active research (Refs. 1 and 2, for example). In this regard, the type of turbulence richest in structures is one in which symmetries of some sort have been broken. Well developed turbulence, however, is known to exhibit a tendency toward restoration of broken symmetries.[3] For example, a large-scale disruption of homogeneity or isotropy is generally reversed at a smaller scale. This is the basis of the local Kolmogorov-Obukhov theory.[4,5] The interaction of a large-scale perturbation with turbulence takes the form of damping of this perturbation because of the turbulent viscosity and transfer of the energy of the perturbation to small-scale turbulent fluctuations. In such a situation, the existence of long-lived structures with a scale $L \gg \lambda$ (λ is the scale of the turbulence) would seem improbable.

This situation is different if the symmetry broken by the turbulence is not restored. One example of a symmetry breaking of this type, and one which is compatible with the theory of the local structure of turbulence, is the breaking of reflection invariance (parity). Such turbulence is of course called "helical" and is characterized by a nonzero pseudoscalar $\langle \mathbf{v}^T \mathrm{curl}\, \mathbf{v}^T \rangle$ (the helicity). From the physical standpoint, this turbulence arises in a force field having pseudovector properties (a magnetic field, the Coriolis force, etc.). The anomalous properties of helical turbulence were first discovered in MHD.[6,7] It is found that a helical turbulence generates and sustains large-scale magnetic fields (the α effect[6]). Although the equation for curl $\mathbf{v}^T$ is analogous to the equation for a magnetic field, the α effect does not occur for the vorticity in a homogeneous and isotropic helical turbulence of an incompressible fluid.[8] This result is easy to understand if we expand the Reynolds stress tensor in a series in the average velocity and assume that the gradients of this velocity are small:

$$\langle v_i^T v_j^T \rangle = T_{ij}(x) = T_{ij}(0) + \alpha_{ijk}\langle v_k \rangle + \beta_{ijkl}\nabla_k \langle v_l \rangle + \dots . \quad (1)$$

In the initial stage of the evolution we can asume $\langle v \rangle \ll v^T \sim \langle (\mathbf{v}^T)^2 \rangle^{1/2}$, and we can ignore the effect of the average field on the turbulence. A necessary condition for the existence of a large-scale instability which generates vortex structures is that the tensor α_{ijk} be nonzero. In nonhelical turbulence, there is simply no possibility of constructing such a tensor. In helical, homogeneous, and isotropic turbulence we would have $\alpha_{ijk} \sim$ const (ε_{ijk} is the Levi-Cività density), but the symmetry of the tensor T_{ij} means that the pseudoscalar coefficient in α_{ijk} must vanish. As a result, we are left with only terms corresponding to turbulent viscosity in T_{ij}. The first example of the α effect in hydrodynamics was found in Refs. 9 and 10 for the case of a compressible fluid and homogeneous and isotropic helical turbulence. In this case the nonlinear term containing the Reynolds stress is not a symmetry tensor. It is clear from the discussion above that in an incompressible fluid the helicity alone would not suffice for the occurrence of an α effect. We would also need some additional symmetry-breaking factors which would make it possible to construct a nonvanishing tensor α_{ijk}. The first example of this type was proposed in Refs. 11–13; we will study it in detail in the present paper. In this example the additional factors which break the symmetry are the gravitational force and a temperature gradient.

Two other examples in which a vortex α effect is possible have also been identified. In the first of these examples, one considers a homogeneous and isotropic helical turbulence superposed on a given large-scale flow.[14,15] In the second, one finds an anisotropic α effect in a reflection-invariant flow.[16] Essentially all of the additional factors are of the nature of a release mechanism which makes it possible to pump some of the energy of the helical turbulence into large-scale vortex structures. This energy pumping is naturally associated with the suppression of the flow at small scales in the helical turbulence.[17–19] As a result, the helical turbulence must seek an additional channel for shedding deviations from equilibrium; this additional channel turns out to be the generation of large-scale structures. This process results in a transfer of some of the turbulence energy to larger scales. Such a process can naturally be interpreted as a vortex dynamo. In the present paper we examine a homogeneous and isotropic small-scale helical turbulence created by a helical external force. This system is in a gravitational force field $\mathbf{g}$ with a small vertical temperature gradient. In this situation, the rule that the tensor α_{ijk} in (1) cannot vanish is removed. As a result, the equations of motion averaged over the turbulence do indeed contain terms representing an anisotropic α effect. This effect differs substantially in structure from an ordinary turbulent viscosity. One might say that the resulting equations of motion describe the effect of a small-scale helical turbulence on the ordinary convection process. From the standpoint of convection, this effect reduces to the following: If the helicity is zero, the convective instability begins at Rayleigh numbers $\mathrm{Ra} > \mathrm{Ra}_{cr}$, as we know,[20] and it has a horizontal length scale $k_\perp^{-1} \sim h$, where h is the thickness of the liquid layer. As the helicity parameter increases, the parameter Ra_{cr} decreases, and the horizontal length scale of the instability increases. When the helicity reaches a certain critical value the horizontal length scale formally becomes infinite. This event means that the convection has undergone a complete change in structure: In place of the large number of convective cells the system finds it prefera-

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ble to form a single large cell (a vortex), whose horizontal dimension is now determined by the horizontal variations in the problem. The large vortex which has appeared has a toroidal field, coupled with a weaker poloidal field (horizontal and vertical circulations). It has a nontrivial streamline topology, as we will see below. This property is common to all large-scale vortices which are generated in a helical turbulence. Since the atmospheric turbulence is a helical turbulence,[21] this effect may also be pertinent to certain natural vortices, e.g., tropical cyclones.[22,23]

2. EQUATIONS OF MOTION OF A VORTEX DYNAMO

The type of convection simplest to treat theoretically is that which occurs in a plane-parallel layer of an incompressible liquid which is being heated from below. This problem is described by a Navier-Stokes equation, an entropy equation, and the condition that the liquid be incompressible. This system of equations is supplemented with an equation of state, which ignores the pressure dependence of the density: $\rho = \rho_0(1-\beta T)$, where $\beta = -\rho_0^{-1}(\partial\rho/\partial T)$ is the thermal expansion coefficient. We will analyze this situation for instabilities for perturbations of the velocity $\mathbf{v}$, the temperature Θ, and the pressure p_1 against the background of a ground state $T_0(z)$, $p_0(z)$. This ground state is a consequence of the heating and is specified by a constant temperature gradient $T_0(z) = -A\mathbf{e}$, where A is a positive constant, and $\mathbf{e}$ is a unit vector directed vertically upward. The system of equations for the perturbations can be written in the Boussinesq approximation as follows[20,24]:

$$\frac{\partial v_i}{\partial t} - \nu\Delta v_i + v_k\nabla_k v_i + \frac{\nabla_i p_1}{\rho} - \beta\Theta g e_i = 0, \qquad (2)$$

$$\frac{\partial\Theta}{\partial t} - \chi\Delta\Theta + v_k\nabla_k\Theta - Ae_jv_j = 0, \qquad (3)$$

$$\nabla_k v_k = 0. \qquad (4)$$

Here ν is the kinematic viscosity, and χ is the thermal diffusivity. The simplest boundary conditions on system (2)–(4) turn out to be the so-called free boundary conditions (Γ is the upper or lower boundary of the layer)

$$v_z|_\Gamma = 0, \quad \left.\frac{\partial v_x}{\partial z}\right|_\Gamma = \left.\frac{\partial v_y}{\partial z}\right|_\Gamma = 0, \quad \Theta|_\Gamma = 0. \qquad (5)$$

In the Navier-Stokes equation we introduce a random external force F_i, which creates a small-scale helical turbulence; we assume $\langle F_i\rangle = 0$. For simplicity we assume that this small-scale turbulence is homogeneous, isotropic, and steady-state. The correlation function for a random velocity field of this sort in the Fourier coordinate representation is known:

$$Q_{ij}^T(t_1-t_2, \mathbf{k}) = B(t_1-t_2, k)(\delta_{ij} - k_ik_j/k^2) + iG(t_1-t_2, k)\varepsilon_{ijl}k_l, \qquad (6)$$
$$\langle \mathbf{v}\operatorname{rot}\mathbf{v}\rangle \sim \int k^2 G(t_1-t_2, k)\,dk,$$

where $\int \langle \mathbf{v}\operatorname{rot}\mathbf{v}\rangle d\Omega = I_T$ is a topological invariant in the inviscid case.[25]

The most interesting part of the correlation function (6) is the term which contains the pseudotensor ε_{ijl} and the pseudoscalar G. It is the presence of this term which is responsible for the appearance of nontrivial physical effects.

Our problem is to derive a closed average equation from system (2)–(4). In this paper we consider the simplest case, of small Reynolds numbers, in which we can calculate the Reynolds stress tensor exclusively. In other words, we are actually examining the effect of a small-scale helical turbulent noise on a convection process. Even in this simplified formulation of the problem, we can see all the basic physical effects which arise.

Accordingly, we assume that the nonlinear terms in Eqs. (2) and (3) are small, i.e., that the Reynolds number satisfies $\mathrm{Re} = u\lambda/\nu \ll 1$, where u is the velocity scale of the turbulent fluctuations, and λ is the external scale of the turbulence. We can then solve the equation for the temperature perturbation iterative and find a single equation for the velocity:

$$L_{ij}v_j = -D_\chi P_{im}\nabla_k(v_kv_m) - \beta AgP_{im}e_me_j\nabla_k\left(v_k\frac{1}{D_\chi}v_j\right) + F_i. \qquad (7)$$

Here L_{ij} is a linear operator which figures in convection theory:

$$L_{ij} = D_\nu D_\chi\delta_{ij} - \beta AgP_{im}e_me_j, \qquad (8)$$
$$D_\nu = \partial/\partial t - \nu\Delta, \quad D_\chi = \partial/\partial t - \chi\Delta,$$

and $P_{im} = \delta_{im} - \nabla_i\nabla_m/\Delta$ is a projection operator which eliminates the potential part of the velocity field. We understand the differential operators in the denominator as integral operators with corresponding Green's functions.

To carry out the procedure of averaging Eq. (7), we write the velocity field v_i as the sum of an average part ($\langle v_i\rangle$) and a fluctuating part v_i' ($\langle v_i'\rangle = 0$):

$$v_i = v_i' + \langle v_i\rangle.$$

If $\langle v_i\rangle$ is zero, the random part of the velocity is caused by the external force F_i. This part of the velocity field is denoted as v_i^T. This is a uniform, isotropic, and helical random field. We are now interested in the evolution of the small average field $\langle v_i\rangle$, under the assumption

$$\langle v\rangle \ll v^T \sim \langle(\mathbf{v}^T)^2\rangle^{1/2}.$$

In this case the random part of the velocity acquires a small nonuniform increment $\tilde{v}_i$,

$$\tilde{v}_i \ll v^T,$$

and can be written in the form

$$v_i' = v_i^T + \tilde{v}_i.$$

As a result, the total velocity field can be written in the form

$$v_i = \langle v_i\rangle + v_i^T + \tilde{v}_i. \qquad (9)$$

Here $\tilde{v}_i$ is a functional of $\mathbf{v}^T$ and $\langle\mathbf{v}\rangle$: $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\{\mathbf{v}^T, \langle\mathbf{v}\rangle\}$.

Taking the average of Eq. (7), and subtracting the average equation from the complete equation (7), we find the following system of equations, in lowest order:

$$L_{ij}\langle v_j\rangle = -D_\chi P_{im}\nabla_k(\langle v_k^T\tilde{v}_m\rangle + \langle\tilde{v}_kv_m^T\rangle)$$
$$-\beta AgP_{im}\nabla_ke_me_j(\langle\tilde{v}_kD_\chi^{-1}v_j^T\rangle + \langle v_k^TD_\chi^{-1}\tilde{v}_j\rangle), \qquad (10)$$

$$L_{ij}\tilde{v}_j = -D_\chi P_{im}\nabla_k(\langle v_k\rangle v_m^T + v_k^T\langle v_m\rangle)$$
$$-\beta AgP_{in}e_ne_m\nabla_k(v_k^TD_\chi^{-1}\langle v_m\rangle + \langle v_k\rangle D_\chi^{-1}v_m^T). \qquad (11)$$

Equation (10) for the average velocity contains averages of quadratic combinations (Reynolds stresses). They can be expressed in terms of the mean field $\langle v_i\rangle$ and the correlation function of the turbulence by making use of the functional dependence of the field $\tilde{v}_i$ on the turbulent field v_i^T, given by the Furutsu-Novikov formula

$$\langle v_k^T(t,\mathbf{x})\tilde{v}_m(t,\mathbf{x})\rangle = \lim_{\substack{t_1\to t\\ \mathbf{x}_1\to\mathbf{x}}} \int ds \int d\mathbf{y} \langle v_k^T(t,\mathbf{x}) v_r^T(s,\mathbf{y})\rangle \left\langle \frac{\delta \tilde{v}_m(t_1,\mathbf{x}_1)}{\delta v_r^T(s,\mathbf{y})} \right\rangle. \quad (12)$$

In order to use (12) we need to assume that the turbulent noise is Gaussian. In problems involving the interaction of a large-scale field with a small-scale turbulence, this assumption is legitimate even at large Reynolds numbers. The reason is that the interaction of large-scale motions with turbulent fluctuations is dominated by the vortices which contain the most energy, whose length scale is λ. At the same time, it is well known that the Gaussian approximation is quite applicable for the energy scale of turbulence.[3]

The generation of a large-scale mean velocity field is related to the first derivatives with respect to the coordinates. The terms with second derivatives give rise to a turbulent viscosity, which is dominated by the nonhelical part of the turbulence. Calculations of this process were carried out by Krause and Rüdiger.[8] The additional turbulent viscosity due to the helical turbulence can be ignored. Since the right side of Eq. (10) for the mean field contains the first derivatives with respect to the coordinates, we need to consider in Eq. (11) only the terms which do not contain gradients of the mean fields.

For the variational derivative we then find

$$\begin{aligned}\langle \delta\tilde{v}_j(t,\mathbf{x})/\delta v_s^T(s,\mathbf{y})\rangle &= -L_{ji}^{-1}\{D_\chi P_{im}\langle v_k\rangle \nabla_k \langle \delta v_m^T(t,\mathbf{x})/\delta v_r^T(s,\mathbf{y})\rangle \\ &\quad + \beta A g P_{im}\langle v_k\rangle \nabla_k D_\chi^{-1}\langle \delta v_j^T(t,\mathbf{x})/\delta v_r^T(s,\mathbf{y})\rangle\}, \end{aligned} \quad (13)$$

where the inverse operator is

$$L_{ji}^{-1} = \frac{(D_\nu D_\chi - \beta A g P_{mn} e_m e_n)\delta_{ji} + \beta A g P_{jl} e_l e_i}{D_\nu D_\chi (D_\nu D_\chi - \beta A g P_{mn} e_m e_n)}. \quad (14)$$

We restrict (13) to the linear dependence on the temperature gradient, and we substitute the variational derivative into the Furutsu-Novikov formula (12), in which we take Fourier transforms in the variables $\mathbf{x}$ and $\mathbf{x}_1$ and integrate over $d\mathbf{y}$. We find the following expression for a quadratic combination:

$$\begin{aligned}\langle v_k^T \tilde{v}_m\rangle &= -\lim_{\substack{t_1\to t\\ \mathbf{x}_1\to\mathbf{x}}} \int ds \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} e^{-i\mathbf{k}\mathbf{x}} e^{-i\mathbf{k}_1\mathbf{x}_1} G(t-s,k)\,\varepsilon_{krl} i k_l \\ &\quad \cdot \left[\frac{\hat{P}_{mr}^{(1)}}{\hat{D}_\nu(1)} + \beta A g \left(\frac{\hat{P}_{mn}(1) e_n e_i \hat{P}_{ir}(1)}{\hat{D}_\nu^2(1)\hat{D}_\chi(1)} + \frac{\hat{P}_{mn}(1) e_n e_r}{\hat{D}_\nu(1)\hat{D}_\chi^2(1)}\right)\right] \\ &\quad \cdot i k_{1a}\langle v_a(\mathbf{k}+\mathbf{k}_1, t_1)\rangle \delta(t_1 - s)\end{aligned}$$

(the 1 in parentheses means that the corresponding operators depend on t_1, $\mathbf{k}_1$).

Since we are assuming that the length scale $\langle v_i\rangle$ is much greater than the length scale of the turbulence, λ, and since we are ignoring the gradient of $\langle v_i\rangle$, we can integrate over $d(\mathbf{k}+\mathbf{k})$:

$$\begin{aligned}\langle v_k^T \tilde{v}_m\rangle &= -\lim_{t_1\to t} \int ds \int_0^\infty \frac{k^2\,dk}{(2\pi)^3} \int d\Omega\, G(t-s,k)\,\varepsilon_{krl} k_l k_a \left[\frac{\hat{P}_{mr}(1)}{\hat{D}_\nu(1)}\right. \\ &\quad \left. + \beta A g \left(\frac{\hat{P}_{mn}(1) e_m e_i \hat{P}_{ir}(1)}{\hat{D}_\nu^2(1)\hat{D}_\chi(1)} + \frac{\hat{P}_{mn}(1) e_n e_r}{\hat{D}_\nu(1)\hat{D}_\chi^2(1)}\right)\right] \\ &\quad \cdot \langle v_a(\mathbf{x}, t_1)\rangle \delta(t_1 - s). \end{aligned} \quad (15)$$

Integrating over the angles $d\Omega$, we find the following expression for a symmetric combination which appears in the equation for the mean velocity:

$$\begin{aligned}\langle v_k^T \tilde{v}_m\rangle + \langle \tilde{v}_k v_m^T\rangle &= -\frac{16\pi}{15} \beta A g (e_m \varepsilon_{kra} + e_k \varepsilon_{mra}) e_r \lim_{t_1\to t} \int ds \\ &\quad \cdot \int_0^\infty \frac{k^4\,dk}{(2\pi)^3} \left(\frac{1}{\hat{D}_\nu^2(1)\hat{D}_\chi(1)} + \frac{1}{\hat{D}_\nu(1)\hat{D}_\chi^2(1)}\right) \\ &\quad \cdot G(t-s,k)\langle v_a(t_1,\mathbf{x})\rangle \delta(t_1 - s). \end{aligned} \quad (16)$$

In order to calculate the inverse operators in (16), we need to know the Green's function, i.e., the fundamental solution of the corresponding operator.

The fundamental solution $\varepsilon(t_1)$ for the operator $\hat{D}_\nu(1)\hat{D}_\chi^2(1) = (\partial/\partial t_1 + \nu k^2)(\partial/\partial t_1 + \chi k^2)^2$ is

$$\varepsilon(t_1) = \eta(t_1)\left[\frac{t_1 e^{-\chi k^2 t_1}}{(\nu-\chi)k^2} + \frac{e^{-\nu k^2 t_1} - e^{-\chi k^2 t_1}}{(\nu-\chi)^2 k^4}\right]; \quad (17)$$

$$\eta(t_1) = \begin{cases} 1, & t_1 > 0, \\ 0, & t_1 < 0. \end{cases}$$

Using the fundamental solution (17), we find the following expression for a symmetric combination:

$$\begin{aligned}\langle v_k^T \tilde{v}_m\rangle + \langle \tilde{v}_k v_m^T\rangle &= -\frac{16\pi}{15} \beta A g (e_m \varepsilon_{kra} + e_k \varepsilon_{mra}) e_r \int_0^\infty \frac{k^4\,dk}{(2\pi)^3} \\ &\quad \cdot \int_{-\infty}^{t} ds \frac{t-s}{(\chi-\nu)k^2} (e^{-\nu k^2(t-s)} - e^{-\chi k^2(t-s)}) G(t-s,k)\langle v_a(s,\mathbf{x})\rangle. \end{aligned}$$

The integration over dk and ds can be carried out explicitly if the specific correlation function $G(t-s,k)$ is given. For simplicity, we take this function to be

$$G(t-s,k) = G_0 \frac{u^2\lambda^4}{(1+\lambda^2 k^2)^2} e^{-|t-s|/\tau}. \quad (18)$$

Assuming the time dependence of the mean field to be slow, i.e., ignoring the time derivatives of the mean field, we find

$$\begin{aligned}&\langle v_k^T \tilde{v}_m\rangle + \langle \tilde{v}_k v_m^T\rangle \\ &= -\frac{16\pi}{15}(e_m \varepsilon_{kra} + e_k \varepsilon_{mra}) e_r \langle v_a\rangle \frac{G_0}{(2\pi)^3} \frac{\pi}{4} \frac{\beta A g u^2 \tau^3}{\lambda} \\ &\quad \cdot \frac{\lambda}{(\tau\nu)^{1/2} + (\tau\chi)^{1/2}} \frac{3 + 3((\tau\nu)^{1/2} + (\tau\chi)^{1/2})/\lambda + \tau(\nu + \chi + (\nu\tau)^{1/2})/\lambda^2}{(1 + (\tau\nu)^{1/2}\lambda)^3 (1 + (\tau\chi)^{1/2}/\lambda)^3}. \end{aligned} \quad (19)$$

To calculate the second quadratic combination in expression (10) for the mean field, we again use the Furutsu-Novikov formula:

$$\begin{aligned}M_{kj}(t,\mathbf{x}) &\equiv \langle \tilde{v}_k(t,\mathbf{x}) D_\chi^{-1} v_j^T(t,\mathbf{x})\rangle + \langle v_k^T(t,\mathbf{x}) D_\chi^{-1}\tilde{v}_j(t,\mathbf{x})\rangle \\ &= \lim_{\mathbf{x}_1\to\mathbf{x}} \int ds \int d\mathbf{y} [\langle v_k^T(t,\mathbf{x}) v_r^T(s,\mathbf{y})\rangle D_\chi^{-1}(1) \\ &\quad \cdot \langle \delta\tilde{v}_j(t_1,\mathbf{x}_1)/\delta v_r^T(s,\mathbf{y})\rangle \\ &\quad + D_\chi^{-1}\langle v_k^T(t,\mathbf{x}) v_r^T(s,\mathbf{y})\rangle \langle \delta\tilde{v}_j(t_1,\mathbf{x}_1)/\delta v_r^T(s,\mathbf{y})\rangle] \end{aligned}$$

[the operator $D_\chi(1)$ acts on variables with subscript 1]. Proceeding as in the calculation of the first quadratic combination, we find

$$M_{kj}(t,\mathbf{x}) = -\lim_{t_1\to t}\int ds\int \frac{d\mathbf{k}}{(2\pi)^3} G(t-s,k)\left[\frac{\varepsilon_{krq} ik_q \hat{P}_{jr}}{\hat{D}_\chi(1)\hat{D}_\nu(1)} + \frac{\varepsilon_{jrq} ik_q \hat{P}_{kr}}{\hat{D}_\chi \hat{D}_\nu(1)}\right](-ik_a)\langle v_a(t_1,\mathbf{x})\rangle \delta(t_1-s).$$

To pursue the integration we use the fundamental solutions $\varepsilon_\nu(t)$ and $\varepsilon_{\nu\chi}(t)$ of the operators $\hat{D}_\nu$ and $\hat{D}_\chi \hat{D}_\nu$:

$$\varepsilon_\nu(t) = \eta(t) e^{-\nu k^2 t}, \qquad \varepsilon_{\nu\chi}(t) = \eta(t)\frac{e^{-\nu k^2 t} - e^{-\chi k^2 t}}{(\chi-\nu)k^2}.$$

The expression for the tensor $M_{kj}(t,\mathbf{x})$ becomes

$$M_{kj}(t,\mathbf{x}) = -\frac{4\pi}{3}\varepsilon_{kja}\langle v_a\rangle \int_0^\infty \frac{k^4\,dk}{(2\pi)^3}\left\{\int_{-\infty}^{t} ds \frac{e^{-\nu k^2(t-s)} - e^{-\chi k^2(t-s)}}{(\chi-\nu)k^2} G(t-s,k) - \int_{-\infty}^{t} ds\, e^{-\nu k^2(t-s)} \int_{-\infty}^{t} dt'\, e^{-\chi k^2(t-t')} G(t'-s,k)\right\}$$

Using the explicit expression for the correlation function, (18), we finally find

$$M_{kj}(t,\mathbf{x}) = \frac{4\pi}{3}\varepsilon_{kja}\langle v_a\rangle \frac{G_0}{(2\pi)^3}\frac{\pi}{2}\frac{u^2\tau^2}{\lambda} \frac{2\tau(\chi\nu)^{1/2}/\lambda^2 + [(\tau\nu)^{1/2} + (\tau\chi)^{1/2}]/\lambda}{\tau(\nu+\chi)\lambda^{-3}[(\tau\nu)^{1/2}+(\tau\chi)^{1/2}](1+(\tau\nu)^{1/2}/\lambda)^2(1+(\tau\chi)^{1/2}/\lambda)^2}. \tag{20}$$

As we have already mentioned, the nonhelical part of the correlation function gives rise to a turbulent viscosity ν_T and a turbulent thermal diffusivity χ_T. In the approximations of this paper, these properties are much smaller than the corresponding laminar coefficients. We can write equations for the large-scale convection, (10), in dimensionless form, introducing as a length scale the thickness of the liquid layer, h; we divide times by $T = h^2/\nu$ and velocities by h/T. The average equation takes the form

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\Pr\frac{\partial}{\partial t} - \Delta\right)\langle v_i\rangle - \mathrm{Ra}\, P_{im} e_m e_j \langle v_j\rangle = \mathrm{Ra}\, s P_{im}\nabla_k\left[\mu_1\left(\Pr\frac{\partial}{\partial t} - \Delta\right)(e_m\varepsilon_{kra} + e_k\varepsilon_{mra}) - \mu_2 e_m \varepsilon_{kra}\right] e_r\langle v_a\rangle. \tag{21}$$

Here $\mathrm{Ra} = \beta A g h^4/\nu\chi$ is the Rayleigh number, $\Pr = \nu/\chi$ is the Prandtl number, $s = (G_0/30\pi)\mathrm{Re}(\lambda/h)$ is the parameter of the helical turbulence, and

$$\mu_1 = \frac{\lambda^2}{T\nu}\frac{\Pr^{-1}}{1+\Pr^{-1/2}}\frac{3\lambda/(\tau\nu)^{1/2} + 3(1+\Pr^{-1/2}) + \lambda^{-1}(\nu\tau)^{1/2}(1+\Pr^{-1/2}+\Pr^{-1})}{(1+\lambda(\tau\nu)^{-1/2})^3(\Pr^{-1/2}+\lambda(\tau\nu)^{-1/2})^3},$$

and

$$\mu_2 = \frac{5}{2}\frac{\lambda^2/\tau\nu}{(1+\Pr^{-1/2})(1+\Pr^{-1})}\frac{1+\Pr^{-1/2}+2\lambda(\tau\nu)^{-1/2}\Pr^{-1/2}}{(1+\lambda/(\tau\nu)^{1/2})^2(\Pr^{-1/2}+\lambda/(\tau\nu)^{1/2})^2}.$$

3. LARGE-SCALE INSTABILITY AND RESTRUCTURING OF THE CONVECTION

The equations found for the mean field differ from the equations of ordinary convection in that they contain terms with the tensor ε_{ikl}. These terms lead to a positive feedback between the toroidal and poloidal components of the velocity and thus to a large-scale instability. In order to study this effect, we write the velocity field in the form

$$\langle\mathbf{v}\rangle = \langle\mathbf{v}_T\rangle + \langle\mathbf{v}_P\rangle, \qquad \langle\mathbf{v}_T\rangle = \mathrm{rot}(\mathbf{e}\psi),\ \langle\mathbf{v}_P\rangle = \mathrm{rot}\,\mathrm{rot}(\mathbf{e}\varphi). \tag{22}$$

Here $\langle\mathbf{v}_T\rangle$ and $\langle\mathbf{v}_P\rangle$ are respectively the toroidal and poloidal components of the solenoidal field of the velocity $\langle\mathbf{v}\rangle$, and ψ and φ are respectively pseudoscalar and scalar functions. For ψ and φ we find the following system of equations from (21):

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi = -\mathrm{Ra}\, s\mu_1(\mathbf{e}\nabla)^2\varphi, \tag{23}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\Pr\frac{\partial}{\partial t} - \Delta\right)\Delta\varphi - \mathrm{Ra}\,\Delta_\perp\varphi = -\mathrm{Ra}\, s\left\{\mu_1\left(\Pr\frac{\partial}{\partial t} - \Delta\right)[\Delta_\perp - (\mathbf{e}\nabla)^2] - \mu_2\Delta_\perp\right\}\psi, \tag{24}$$

where $\Delta_\perp$ is the Laplacian in the horizontal coordinates. It can be seen from this system that the toroidal and poloidal fields are coupled exclusively through the helicity parameter s. Eliminating the field ψ (for example) from the system of equations, we find a single equation for φ:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\Pr\frac{\partial}{\partial t} - \Delta\right)\Delta\varphi - \mathrm{Ra}\,\Delta_\perp\varphi + \mathrm{Ra}^2 s^2\left\{\mu_1^2\frac{\Pr\partial/\partial t - \Delta}{\partial/\partial t - \Delta}[(\mathbf{e}\nabla)^2 - \Delta_\perp] + \mu_1\mu_2\frac{\Delta_\perp}{\partial/\partial t - \Delta}\right\}(\mathbf{e}\nabla)^2\varphi = 0. \tag{25}$$

Boundary conditions on the function φ are found from (5):

$$\varphi(0) = \varphi(1) = \varphi_{zz}''(0) = \varphi_{zz}''(1) = \varphi_{zzzz}^{\mathrm{IV}}(0) = \varphi_{zzzz}^{\mathrm{IV}}(1). \tag{26}$$

A solution of Eq. (25) under the conditions (26) can be sought in the following form, as in the case of ordinary convection:

$$\varphi(z,\mathbf{r}_\perp,t) \sim e^{\gamma t} e^{i\mathbf{k}_\perp\mathbf{r}_\perp}\sin\pi z, \tag{27}$$

where $\mathbf{r}_\perp$ is the horizontal radius vector.

Substituting (27) into (25) and setting $\gamma = 0$, we find the following expression for the neutral-stability curve:

$$\mathrm{Ra}_{\mathrm{cr}}(k_\perp^2, s) = \frac{2\pi^4(1+k_\perp^2/\pi^2)^3}{k_\perp^2/\pi^2 + \{k_\perp^2/\pi^2 + 4(1+k_\perp^2/\pi^2)^3(\pi^3 s)^2[\mu_1^2(1-k_\perp^2/\pi^2) + \mu_1\mu_2\pi^{-2}(1+k_\perp^2/\pi^2)^{-1}k_\perp^2/\pi^2]\}^{1/2}}. \tag{28}$$

It is easy to see that if the helicity parameter s tends toward zero, the neutral-stability curve becomes the known curve for ordinary convection[20,24]:

$$\mathrm{Ra}_{\mathrm{cr}}(k_\perp^2, 0) = \pi^4 \frac{(1+k_\perp^2/\pi^2)^3}{k_\perp^2/\pi^2}. \tag{29}$$

This curve has a minimum at $\mathrm{Ra}_{\mathrm{min}} = 27\pi^4/4$; it is reached at $k^2_{\mathrm{min}} = \pi^2/2$. As a result, we find the well-known convection cells with a horizontal dimension on the order of the vertical dimension.

Incorporating a nonzero helicity $s \neq 0$ changes the physical picture of the instability substantially. To avoid obscuring the essential features of the situation, we examine in detail the simplest case, $\mu_2 \ll \mu_1$. This case is reached at small Prandtl numbers, $\mathrm{Pr} \ll 1$.

As the parameter s is increased, $\mathrm{Ra}_{\mathrm{min}}(s)$ decreases, and the minimum shifts toward smaller wave numbers $k_\perp$. In other words, the horizontal dimension of the cells increases. When the helicity parameter reaches the value

$$s = s_0 = 1/4\pi^3\mu_1, \tag{30}$$

the minimum reaches the value $k_{\mathrm{min}} = 0$ (Fig. 1). Formally, this result means that the horizontal dimension of the instability is infinite. This result in turn means a complete restructuring of the convection, with the result that the formation of one large cell (or vortex) is preferable to a set of convection cells for the system. The size of this vortex is actually set by the horizontal variations in the problem. We wish to stress that a vortex which results from an instability necessarily couples the toroidal and poloidal velocity fields, as can be seen from system (23), (24). As a result, there are topologically nontrivial streamline configurations.

To explicitly find the field configurations and the instability growth rate in the case $s > s_0$, we consider Eq. (25) slightly above the critical value: $\delta R = \mathrm{Ra} - \mathrm{Ra}(0,s) \ll 1$. It is necessary to introduce a horizontal variation in the problem; a very simple example is

$$\delta R(\mathbf{r}_\perp) = \delta R_0(1 - \mathbf{r}_\perp^2/r_0^2). \tag{31}$$

If the extent to which the critical value is exceeded is small and if $r_0 \gg 1$, Eq. (25) can be simplified by ignoring the highest derivatives in it with respect to both the time and the transverse coordinates ($\partial/\partial t \to \gamma$):

$$\gamma\varphi(\mathbf{r}_\perp) = \mu_1 s\pi\delta R(\mathbf{r}_\perp)\varphi(\mathbf{r}_\perp) + 2(1 - s_0/s)\Delta_\perp\varphi(\mathbf{r}_\perp). \tag{32}$$

Equation (32) has the form of a steady-state Schrödinger equation with some effective potential well, which in turn controls the eigenfunctions and the growth rate of the instability.

With (31) as an explicit horizontal variation, we can reduce (32) to a problem with a centrally symmetric potential. Setting $\varphi(\mathbf{r}_\perp) = \varphi(r_\perp)e^{im\Phi}$, where Φ is the polar angle, we find an ordinary differential equation for $\varphi(r_\perp)$:

$$\frac{d^2\varphi(r_\perp)}{dr_\perp^2} + \frac{1}{r_\perp}\frac{d\varphi(r_\perp)}{dr_\perp} - \frac{m^2}{r_\perp^2}\varphi(r_\perp) + W\varphi(r_\perp) = 0,$$
$$W = \frac{\mu_1 s\pi\delta R_0 - \gamma - \mu_1 s\pi\delta R_0 r_\perp^2/r_0^2}{2(1 - s_0/s)}. \tag{33}$$

A solution of Eq. (33) is constructed with the help of Laguerre polynomials: it falls off exponentially at infinity. The argument of the cutoff exponential function determines the horizontal length scale of the structure which results from the instability (expressed in dimensional variables):

$$L = h\left[\frac{r_0^2}{h^2}\frac{2(1-s^0/s)}{\mu_1 s\pi\delta R_0}\right]^{1/4} \gg h. \tag{34}$$

The eigenvalues of problem (33) determine the growth rates of the various instability modes. The lowest mode is determined by the maximum growth rate:

$$\gamma = \mu_1 s\pi\delta R_0\left\{1 - \frac{2h}{r_0}\left[\frac{2(1-s_0/s)}{\mu_1 s\pi\delta R_0}\right]^{1/2}\right\}\frac{\nu}{h^2}. \tag{35}$$

From the convection standpoint, helical turbulence can thus cause a substantial change in the nature of the instability and a complete restructuring of the convection structure.

When helicity appears, the minimum heating level required to trigger the convection process decreases, and the horizontal size of the cells increases, indicating the appearance of a factor which promotes upwelling of the light and warm liquid volume. One such factor is the toroidal velocity field in a convection cell, which is generated by the small-scale helical turbulence from the poloidal velocity field which is a usual features of convection. The toroidal field which arises operates through the turbulence itself to amplify the poloidal convection field, thus closing the feedback loop. This convection transfers the heat from the warm lower boundary of the layer to the upper layer more efficiently, and fewer cells per unit area become preferable from the energy standpoint in the liquid. In other words, the horizontal dimension of the cells increases.

With increasing value of the helicity parameter, the transverse dimensions of the cells become progressively larg-

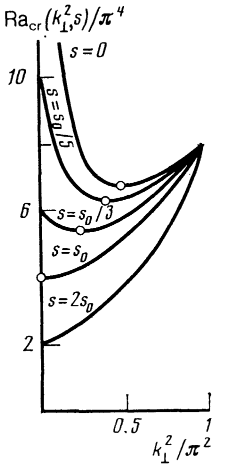

FIG. 1.

er, and at a certain finite value of the helicity parameter a convection with a single cell becomes preferable from the energy standpoint. The size of this single cell, $L \sim (r_0 h)^{1/2}$, is set by the boundaries of the heating region in the transvserse direction. The efficiency of the Archimedean forces and the suction effect of the toroidal field become comparable in competition with the viscous forces. Over a fairly wide heating region, the growth rate is given by

$$\gamma \sim \mu_1 s \pi \delta R_0 \nu / h^2.$$

From the standpoint of the helical turbulence, the convection removes the rule that the α effect cannot vanish, and it permits shedding of some of the turbulence energy into the large-scale structures which appear.

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