Gluino condensate in supersymmetric gluodynamics

A. Yu. Morozov, M. A. Ol'shanetskiĭ, and M. A. Shifman

Institute of Theoretical and Experimental Physics

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The method of evaluating the gluino condensate, proposed in a previous paper,[2] is extended to supersymmetric Yang-Mills theories (without matter) with gauge groups $Sp(N)$ and exceptional groups, but excluding E_8. In all cases, the condensate $\langle \mathrm{Tr}\lambda\lambda \rangle$ is nonzero and assumes $T(G)$ values, where $T(G)$ is one-half the Dynkin index for the adjoint representation of the group G.

1. INTRODUCTION

The question as to whether gluino condensation occurs in supersymmetric Yang–Mills theories without matter, i.e., whether

$$\langle \mathrm{Tr}\, \lambda\lambda \rangle \neq 0, \qquad (1)$$

has assumed particular importance since the suggestion[1] that the condensate (1) could be used in the shadow sector for supersymmetry (SUSY) breaking within the framework of the superstring approach. This question has attracted a relatively extensive literature, analyzed, for example, in Ref. 2. Summarizing briefly the overall situation, we may say that there is a variety of arguments in favor of gluino condensation but, unfortunately, they are not entirely rigorous and, more importantly, they do not assure a reliable evaluation of the condensate (1). The basic difficulty is that the condensate (1) is a dynamic parameter that arises in the theory with strong coupling.

In this paper, we continue to develop the construction proposed in Ref. 2, whereby the condensate (1) can be found by an indirect method. The main advantage of the construction is that all the calculations are performed in the weak-coupling regime and can be fully monitored theoretically. Then, using certain general properties of supersymmetric theories, we are able to extend the result to the strong-coupling regime.

The method used in Ref. 2 is inductive in character: we first evaluate $\langle \mathrm{Tr}\lambda\lambda \rangle$ for $SU(2)$ and then, assuming that $\langle \mathrm{Tr}\lambda\lambda \rangle \neq 0$ for $SU(N-1)$, we find $\langle \mathrm{Tr}\lambda\lambda \rangle$ for $SU(N)$. The analogous chain $O(6) \equiv SU(4) \to O(7) \to \ldots \to O(N)$ was constructed in Ref. 2 for orthogonal groups, as well. The logic of the method employed in Ref. 2, the basic elements of which are taken from the remarkable papers of Affleck *et al.*[3] and Amati *et al.*[4] is as follows.

(a) For a given group G, we introduce auxiliary superfields of matter in a specially selected representation of G (we shall refer to this model with auxiliary field as the *intermediate model*); (b) if the mass term in the matter fields is small, the group G is spontaneously broken down to a smaller group G' by the large vacuum expectation value of the scalar field, subject to the condition that the condensate $\langle \mathrm{Tr}\lambda\lambda \rangle$ is nonzero in the case of G'; (c) $\langle \mathrm{Tr}\lambda\lambda \rangle$ is evaluated in the intermediate model in terms of $\langle \mathrm{Tr}\lambda\lambda \rangle_{G'}$ and the result depends on $\langle \mathrm{TR}\lambda\lambda \rangle_{G'}$ and the mass parameter m; (d) we next let $m \to \infty$, so that the auxiliary matter vanishes from the spectrum, and then use the exact results for $\langle \mathrm{Tr}\lambda\lambda \rangle$ as a function of m in the intermediate model to determine $\langle \mathrm{Tr}\lambda\lambda \rangle_G$ in the limit as $m \to \infty$.

The aim of this paper is to generalize the method adopted in Ref. 2 to symplectic and exceptional groups. The condensate (1) can be evaluated in all cases with the exception of the group E_8, for which the program outlined above is unsuitable for technical reasons.

We note that the above approach is similar in spirit to that proposed earlier in Ref. 5, in which it was used to prove the spontaneous breaking of discrete chiral symmetry $Z_{2T}(T) \to Z_2$, which is the remainder of the anomalous $U(1)$ invariance of supersymmetric gluodynamics [$T(G)$ is half the Dynkin index for the adjoint representation of the gauge group]. The authors of Ref. 5 were able to introduce auxiliary fields in theories with the gauge group $O(N)$ in such a way as to completely break the gauge group and reduce the situation to a model with a weak coupling regime. The index $T(G)$ was then found for the degenerate vacuum states in this regime. According to Witten,[6] the number of vacuum states in supersymmetric theories is an invariant that does not change when matter is made more massive (in the course of its removal from the spectrum as $m \to \infty$).

2. GENERAL ANALYSIS

It will be convenient to introduce the following notation. The auxiliary matter superfields will be referred to as S and T. We shall not need more than two auxiliary fields; T may be identical with S in the case of real representations. We shall use T and T' to denote one-half of each Dynkin index for the adjoint representations of the group G and its subgroup G':

$$\mathrm{Tr}_{AdG} t^a t^b = T\delta^{ab}, \quad \mathrm{Tr}_{AdG'} t'^a t'^b = T'\delta^{ab}.$$

(We must watch the consistency of the normalizations of the generators when G' is embedded in G). The Dynkin index of the matter fields will be denoted by T_M, where $T_M = \Sigma_i T(R_i)$ and the sum is evaluated over all the supermultiplets of matter (S and T).

The following line of argument is used to evaluate the gluino condensate.[2]

(1) The theory without the mass term $mST|_F$ has a nonanomalous R-symmetry, for which both superfields S and T transform in the same way. The absence of anomaly means that there must be a statistical relationship between the chiral R-charges of the gluino, λ, and the matter fermions ψ_S and ψ_T: if $\lambda \to e^{i\alpha}\lambda$, then $\psi_{S,T} \to e^{-i\alpha T/T_M}\psi_{S,T}$. (For example, in the instanton field, there are $2T$ gluino zero modes and $2T_M$ zero modes of ψ_S and ψ_T.) Invariance of the Yukawa vertex $g\varphi_S^*(\psi_S\lambda)$ demands that, at the same time, $\varphi_{S,T} \to \exp[-i\alpha(T-T_M)/T_M]\varphi_{S,T}$.

(2) The dependence of the gluino condensate on the

mass parameter m can be determined exactly throughout the range of variation of m. Let the mass term be of the form $mST|_F + \overline{m}\,\overline{ST}|_F$. We note, first, that the condensate is an analytic function of m: $\partial\langle \mathrm{Tr}\lambda\lambda\rangle/\partial\overline{m} = 0$. In point of fact, this derivative is equal to the connected part of the vacuum correlator of $\lambda\lambda$ and the F-component of the superfield $\overline{ST}$, $D^2\overline{ST}$. However, the correlator of the two lower components of superfields of the same chirality must be zero:

$$\langle \mathrm{Tr}\,\lambda\lambda(0),\ \overline{D}^2\overline{ST}(x)\rangle_{connected}=0. \tag{2}$$

The dependence on the phase m can be found by using the R-transformations. For this, we note that the Lagrangian with the mass term $mST|_F + \overline{m}\,\overline{ST}|_F$ does not change when the R-transformations of the field are accompanied by the additional phase rotation

$$m\to m\exp(2i\alpha T/T_M),\qquad \overline{m}\to\overline{m}\exp(-2i\alpha T/T_M).$$

Since, under this transformation, $\langle \mathrm{Tr}\lambda\lambda\rangle\to e^{2i\alpha}\langle\mathrm{Tr}\lambda\lambda\rangle$, it is clear that

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle_G=Cm^{T_M/T}, \tag{3}$$

where the numerical constant C no longer depends on m. The next stage is necessary to prove that this constant is not zero, and to calculate it.

(3) The identity[7]

$$\frac{1}{8}\overline{D}^2\overline{S}e^V S=mST+\frac{1}{16\pi^2}\mathrm{Tr}\,W^2,$$

$$\left\langle mST+\frac{1}{16\pi^2}\mathrm{Tr}\,W^2\right\rangle=0$$

enables us to relate the gluino condensate to the expectation value $v=\langle\varphi_S\rangle=\langle\varphi_T\rangle$ throughout the range of variation of m:

$$\frac{1}{16\pi^2}\langle\mathrm{Tr}\,\lambda\lambda\rangle_G=mv^2.$$

The value of v^2 as $m\to 0$ is determined by the form of the superpotential for the matter fields S and T. The superpotential contains two contributions: the bare contribution mST and $A\Lambda^y(ST)^{-x}$, which is of nonperturbative origin. The functional form of the second term is uniquely determined by dimensional considerations ($-2x+y=3$) and by the R-invariance ($x=T_M/(T-T_M)$). The result is

$$y=\frac{3T-T_M}{T-T_M},\qquad v^2=(Axm^{-1}\Lambda^y)^{1/(x+1)},$$

and

$$\frac{1}{16\pi^2}\langle\mathrm{Tr}\,\lambda\lambda\rangle_G=m^{T_M/T}(Ax\Lambda^y)^{1/(x+1)}\sim m^{T_M/T}\Lambda^{3-T_M/T}. \tag{4}$$

The fact that the mass dependence is the same as (3) shows that there can be no other contributions to the superpotential. The required coefficient C in (3) is therefore expressed in terms of the coefficient A in front of the correction to the superpotential.

(4) The behavior of the theory in the limit as $m\to 0$ was suggested in Ref. 2 as a means of determining the mass-independent parameter A. In this limit, the initial gauge group G is broken by the vacuum expectation value v down to the subgroup G'. The result is a supersymmetric Yang–Mills theory with additional light chiral superfields: when S and T are suitably chosen, the additional light superfields are singlets in G'. This demands that all the components of S and T that are nonsinglets in G' should be expended in giving mass to the gauge bosons from G/G'. In that case, $T=T'+T_M$. The dynamics of the light fields is described by the superpotential $A\Lambda^y(ST)^{-x}$. The coefficient A can be expressed in terms of the gluino condensate in the theory with the group G'. Proceeding by induction, we can now reduce the question of the gluino condensate for all theories (except for E_8; see below) to the $SU(2)$ case examined in Ref. 2, and hence evaluate the condensates.

(5) To establish the relationship between A and $\langle\mathrm{Tr}\lambda\lambda\rangle_{G'}$, let us examine the evolution of the effective Lagrangian for a varying normalization point μ. When $\mu\gg v$, the gauge symmetry of G is not broken and the Lagrangian for the Yang–Mills superfields is $g_G^{-2}(\mu)\mathrm{Tr}_G W^2$. When $\mu\ll v$, only the gauge group G' survives. The coupling constant $g^2_{G'}$, in the corresponding Lagrangian $g_{G'}^{-2}(\mu)\mathrm{Tr}_{G'}W^2$ is different from g^2_G. These two constants are equal when $\mu=v$. Hence,

$$\begin{aligned}\frac{1}{g_{G'}^2(\mu)}&=\frac{1}{g_{G'}^2(v)}+\frac{3T'}{8\pi^2}\ln\frac{\mu}{v}\\&=\frac{1}{g_G^2(v)}+\frac{3T'}{8\pi^2}\ln\frac{\mu}{v}=\frac{1}{g_G^2(\mu)}\\&\quad-\frac{3T-T_M}{8\pi^2}\ln\frac{\mu}{v}+\frac{3T'}{8\pi^2}\ln\frac{\mu}{v}\\&=\frac{1}{g_G^2(\mu)}-\frac{3T-3T'-T_M}{8\pi^2}\ln\frac{\mu}{v}.\end{aligned} \tag{5}$$

Thus, when $\mu\ll v$, the Lagrangian acquires a contribution which we shall write in terms of the superfields S and T as follows:

$$\frac{1}{2}\frac{3T-3T'-T_M}{8\pi^2}\ln(ST)\mathrm{Tr}_{G'}W^2. \tag{6}$$

If the theory contains a nonzero condensate $\langle\mathrm{Tr}\lambda\lambda\rangle_{G'}$, then, for scales $\mu\ll\Lambda_{G'}$, this contribution may be looked upon as a superpotential for the light chiral superfields forming the condensate v.

All that remains is to express the condensate $(1/16\pi^2)\times\langle\mathrm{Tr}\lambda\lambda\rangle_{G'}=k_{G'}\Lambda^3_{G'}$ in terms of the original parameter $\Lambda=\Lambda_G$ and v. To do this, we use the renormalization-group formula (5), in which we substitute

$$\frac{1}{g_{G'}^2(\mu)}=\frac{3T'}{8\pi^2}\ln\frac{\mu}{\Lambda_{G'}},\qquad\frac{1}{g_G^2(\mu)}=\frac{3T-T_M}{8\pi^2}\ln\frac{\mu}{\Lambda}.$$

The result is

$$\Lambda_{G'}^3=v^3\left(\frac{\Lambda}{v}\right)^{(3T-T_M)/T'}$$

The superpotential for the light components of the fields S and T is

$$k_{G'}(3T-3T'-T_M)\Lambda^{(3T-T_M)/T'}(ST)^{-(3T-3T'-T_M)/T'} \tag{7}$$

The required coefficient A has thus been expressed in terms of the gluino condensate in the theory with the gauge group G'. Returning now to (4) with $m\to\infty$, we see that

$$\frac{1}{16\pi^2}\langle\mathrm{Tr}\,\lambda\lambda\rangle_G\sim m^{T_M/T}\left(k_{G'}\Lambda^{(3T-T_M)/(T-T_M)}\right)^{(T-T_M)/T}$$
$$=\Lambda_G^3 k_{G'}^{(T-T_M)/T},$$

i.e.,

$$(k_G)^T \sim (k_{G'})^{T'}. \quad (8)$$

This relationship clearly shows the T-fold degeneracy of the vacuum in the supersymmetric Yang–Mills theory with the gauge group G.

Just in case the long derivation obscures the simple essence of the construction, let us formulate the result in a somewhat different form. As noted in Ref. 2, the expression for $\langle \mathrm{Tr}\lambda\lambda \rangle$ in terms of the bare quantities m_0, g_0, and M_0, namely

$$\frac{1}{16\pi^2}\langle \mathrm{Tr}\,\lambda\lambda\rangle = \mathrm{const}\, m_0^{T_M/T} M_0^{3-T_M/T}\left(\frac{8\pi^2}{g_0^2}\right)\exp\left(-\frac{8\pi^2}{Tg_0^2}\right) \quad (9)$$

is generally exact for an arbitrary relationship between m_0 and M_0. Next, by varying m_0, we can continously pass from the G-gluodynamic limit ($m_0 = M_0$) to G'-gluodynamics $m_0 \ll M_0 \exp\{-8\pi^2/(3T - T_M)g_0^2\}$). For small m_0, only those terms survive in $\mathrm{Tr}\Lambda\Lambda$ that correspond to the unbroken group G', and the expression on the right-hand side of (9) reduces to $k_{G'}\Lambda^3_{G'}$. If we know the constant $k_{G'}$ in the expression

$$\frac{1}{16\pi^2}\langle \mathrm{Tr}\,\lambda\lambda\rangle_{G'} = k_{G'}\Lambda_{G'}^3$$

(and, by hypothesis, this constant is known), the relationship between the grouping

$$m_0^{T_M/T} M_0^{3-T_M/T}\left(\frac{8\pi^2}{g_0^2}\right)\exp\left(-\frac{8\pi^2}{Tg_0^2}\right)$$

and $\Lambda^3_{G'}$ for $m_0 \to 0$ determines the constant in (9). Passing to the limit as $m_0 \to M_0$, and using the relations

$$M_0^3\left(\frac{8\pi^2}{g_0^2}\right)\exp\left(-\frac{8\pi^2}{Tg_0^2}\right) = \Lambda_G^3,$$

we obtain from (9) the expression for $(1/16\pi^2)\langle \mathrm{Tr}\lambda\lambda \rangle_G$ in terms of Λ^3_G.

This method was used in Ref. 2 to reduce the question of gluino condensates to the determination, for each group G, of the chain $G \to G' \to \ldots \to SU(2)$ and of the matter superfields S and T necessary at each stage. The superfields S and T must have the property that their vacuum expectation values in the general case break G down to G' and, at the same time, their components that are nonsinglets in G' are expended in giving mass to the gauge bosons from G/G'. The reference to the general position is significant. If the vacuum expectation value of S and T in the general case were to break G down to some $G'' \neq G'$ and, only under certain particular expectation values, down to G', we would have to prove the stability of these vacuum expectation values with respect to perturbative and nonperturbative corrections. A further condition on the fields S and T is that there must be a mass term mST, i.e., the product of representations $S \times T$ must contain a singlet in G.

The general scheme for breaking G down to G' can be described as follows. The general position vector from the matter multiplet $M = \Sigma_i R_i$ is reduced by a transformation from the group G to some standard form. By definition, the group G' conserves this reduced vector. The set of G'-singlet vectors V forms a subspace H in M:

$$H = \{V \in M \mid gV = V \quad \text{for all} \quad g \in G'\}. \quad (10)$$

For example, if M is the adjoint representation of the group $SU(N)$, we have $G' = \{U(1)\}^N$ and H is the space of the diagonal matrices. There is an extension $\mathcal{N}(H)$ of the subgroup G' that consists of transformations that conserve the subspace H as a whole:

$$\mathcal{N}(H) = \{g \in G \mid gH = H\}.$$

In the above example, $\mathcal{N}(H)$ is the extension of the subgroup G' by transformations consisting of $N!$ permutations of the diagonal elements. The factor group $\mathcal{N}(H)/G'$ often contains continuous components and its Lie algebra is nontrivial. For example, in the case of the $\tilde{N}$-plet of the group $SU(n)$, the space H is homogeneous $H = (v,0,...,0)$, $G' = SU(N-1)$, $\mathcal{N}(H)/G' = U(1)$. Suppose the vectors of the representation M can be used to construct a number of independent polylinear invariants $I_1 \ldots I_k$. We are interested only in invariants that are singlets of G and consist of chiral superfields (without the participation of antichiral superfields) or, conversely, consist of antichiral superfields alone. After the breaking of the gauge group, the corresponding fields become light. It is clear that the invariants do not actually depend on all the coordinates in the multiplet M, but only on the coordinates on the subspace H (10). However, this requires a mathematical refinement, referred to as the Luna-Richardson theorem in mathematical literature.[8] The significance of this is that, after contraction on H, the polynomials $I_1 \ldots I_k$ become identical with the polynomials that are invariant under transformations from the factor group $\mathcal{N}(H)/G'$. It is precisely these combinations of coordinates in the space H that are the true variables in the broken theory. In the first of the above examples, the invariants constructed from the matrix A in the adjoint representation have the form $I_1 = \mathrm{Tr}A^2;...;I_{N-1} = \mathrm{Tr}A^N$ and depend on the eigenvalues of the matrix A. It is clear that symmetric polynomials of the eigenvalues of A generate all the invariants of the adjoint representation of $SU(N)$. A less trivial situation arises when we examine theories with gauge groups $Sp(2N)$, E_6 and E_7, which we shall examine later.

The G-singlet mass term of matter is constructed with the aid of the invariant I_1 that is quadratic in the vectors of the multiplet M. In the intermediate model, valleys arise in the limit as $m \to 0$, i.e., flat directions along which the D-terms of the Lagrangians are found to vanish. It follows from general theory[9] that, when the set of matter fields is not too extensive, the number of independent "valley parameters" is equal to the number of independent chiral invariants $I_1 \ldots I_k$. By selecting a certain vector from the multiplet M, we fix the gauge condition that the matter fields must satisfy (the analog of the unitary gauge). "Angle"-type variables have been removed. "Modulus"-type variables remain and correspond to invariant combinations of coordinates in the subspace H.

We emphasize that a precise relationship between the condensates $\langle \mathrm{Tr}\lambda\lambda \rangle_G$ and $\langle \mathrm{Tr}\lambda\lambda \rangle_{G'}$ (with all the coefficients) can be found only when the G-singlet chiral superfield is the only one, i.e., there is only one invariant I_1. If there are several invariants, each appears in (7) with its own coefficient. It is natural to suppose that all these coefficients are of the same order of magnitude, but the precise relationship between them can be established only by specific calculation.

The chains leading to $SU(2)$ from all the groups $SU(N)$ and $SO(N)$ were constructed in Ref. 2. Below, we shall describe the analogous chains for all the remaining simple groups with the exception of E_8: $Sp(N) \to Sp(N-1) \to Sp(1) \approx SU(2)$; $G_2 \to SU_3 \to \ldots$; $F_4 \to SO_8 \to \ldots$; $E_7 \to SO_8 \to \ldots$.) Several invariants occur in the case of F_4, E_6, and E_7, and the precise value of the condensate cannot be determined. For the group E_8 (the most interesting from the point of view of applications), the adjoint representation is the smallest, so that we cannot find G', S, and T, for which the condition $T = T' + T_M$ would be satisfied, and, in that case, the method of Ref. 2 is invalid.

In fact, to construct all the chains, it is sufficient to use the tables listing the possible ways of breaking the groups G by vacuum expectation values of matter fields in different representations, and then chose all the variants satisfying the above requirements. Such tables have been compiled by Elashvili.[10] Excerpts from them are reproduced in the Appendix. The $G \to G'$ transitions are indicated more explicitly below.

3. THE GROUP *Sp(N)*

The group $Sp(N)$ is the most direct generalization of $SU(N)$ in which complex numbers are replaced by quaternions. Let $q = q^0 + q^a e_a$ be a quaternion. We recall that the imaginary quaternion units e_a, $a = 1,2,3$ can be represented by the Pauli matrices: $e_a = i\sigma_a$. The $Sp(N)$ Lie algebra consists of anti-Hermitian quaternion $N \times N$ matrices Q:

$$Sp(N) = \{Q \mid \bar{Q}^T = -Q, \text{ i.e. } Q = \{q_{ij}\},\ q_{ij} = -\bar{q}_{ji};\ i, j = 1 \ldots N\}, \tag{11}$$

where the bar indicates the conjugate quaternion $\bar{q} = q^0 - q^a e_a$. By rewriting the quaternions in terms of the Pauli matrices, we can express matrices Q of the order N in terms of complex matrices of order $2N$.

The matrices Q operate on the column vector of N quaternions $x_k = x_k^0 e_0 + i\mathbf{x}_h \boldsymbol{\sigma} (k = 1 \ldots N)$ by multiplication from the left. This application of the group $Sp(N)$ conserves the vector length $(\Sigma_k^N (x_k^0)^2 + (\mathbf{x}_h)^2)^{1/2}$. When we pass to complex numbers, we have $2N$-dimensional representations R_1 and R_2, constructed as follows. The complex $2N$-vectors $S \in R_1$ and $T \in R_2$ are conveniently given in the form of a set of N pairs $S = \{S1 \ldots S_N\}$, $T = \{T_1 \ldots T_N\}$:

$$S_j = \begin{pmatrix} \xi_j^1 \\ \xi_j^2 \end{pmatrix} = \begin{pmatrix} x_j^0 + ix_j^3 \\ ix_j^1 - x_j^2 \end{pmatrix}, \quad T_j = \begin{pmatrix} \eta_j^1 \\ \eta_j^2 \end{pmatrix} = \begin{pmatrix} ix_j^1 + x_j^2 \\ x_j^0 - ix_j^3 \end{pmatrix}.$$

It is readily seen that an invariant can be constructed from the two multiplets, i.e., the length of the vector x_k. In terms of the components $(\xi_j^\alpha, \eta_j^\alpha)$, $\alpha = 1,2$, this invariant can be represented with the aid of an antisymmetric tensor of rank two:

$$ST = \sum_{j=1}^{N} \varepsilon_{\alpha\beta} \xi_j^\alpha \eta_j^\beta,$$

which indicates that the representation $2N$ is pseudoreal. Hence, a mass term can be formed from the two $2N$-plets. The theory contains no other invariants. If we apply the transformations from the group $G = Sp(N)$ to arbitrary vectors, we can reduce them to the standard form

$$S_1 = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad S_j = 0 \quad (j = 2 \ldots N), \quad T_1 = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix},$$
$$T_j = 0 \quad (j = 2 \ldots N). \tag{12}$$

Vectors such as (12) form the subspace H in the space of the representation $M = R_1 \oplus R_2$. Their common stationary subgroup is $G' = Sp(N-1)$. For this subgroup, the generators of the group G and two of its fundamental $2N$-plet representations are found to split as follows:

$$[Sp(N)] = [Sp(N-1)] + 2\cdot[2(N-1)] + 3[1], \tag{13}$$

$$[2N] = [2(N-1)] + 2\cdot[1], \quad [2N] = [2(N-1)] + 2[1]. \tag{14}$$

In (13), $[Sp(N)]$ represents the adjoint representation of $Sp(N)$, whose dimension is $N(2N+1)$. Pairing gives mass to the fields that are $[2(N-1)]$-plets with respect to the subgroup $G' = Sp(N-1)$. Of the four singlet matter fields (14) that generate the space H, three are paired with singlets in (13), i.e., the corresponding gauge fields become massive. The subgroup $G' = Sp(N-1)$ remains unbroken, and the light singlet corresponding to the mass-invariant $ST = \varepsilon_{\alpha\beta} \xi^\alpha \eta^\beta$. survives. We note that transformations from G can reduce S and T to the more specialized form

$$S = \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

All that needs to be done is to apply the transformations from the factor group $\mathcal{N}(H)/G' \approx SU(2)$ to the vectors (12). The variable v^2 that corresponds to the mass term is then the only invariant of the factor group $\mathcal{N}(H)/G'$

4. THE GROUP G_2

The fundamental 7-plet of the group V^a is a real representation, which means that we can write down the mass term in terms of this representation alone. By fixing an arbitrary vector out of the 7-plet, we break the group G_2 down to the subgroup $SU(3)$. Before explaining this, let us write out the expansions for the adjoint and fundamental representations of the G_2 algebra with respect to $SU(3)$:

$$[G_2] = [SU(3)] + 3 + \bar{3},$$
$$7 = 1 + 3 + \bar{3}. \tag{15}$$

We use the same notation here as in (13).

The singlet in the expansion for the 7-plet in (15) corresponds to a standard vector. In other words, the standard representation of the 7-plet is constructed so that the components that transform as 3 and $\bar{3}$ under the $SU(3)$-subalgebra are equal to zero. The only nonzero component is the $SU(3)$ singlet. Hence, it is clear that $SU(3)$ is a stationary subgroup. The mass term has the usual form: $mV^aV^a|_F$ + h.c. .... We then find that, apart from V^aV^a, no chiral invariants can be formed from the single 7-plet of V^a.

Operating on the standard vector with representations from G_2, we obtain the orbit of G_2. We know that this is the sphere $S^6 = G_2/SU(3)$. Hence, it follows that the vector is in the general position, and the group G_2 is naturally broken down to $SU(3)$. This gives mass to the triplets 3 and $\bar{3}$. The subgroup $SU(3)$ remains unbroken. Apart from the supersymmetric $SU(3)$ gluodynamics, the intermediate model is also found to contain the singlet corresponding to the invariant $|V|^2$. Thus, starting with $\langle \mathrm{Tr}\lambda\lambda \rangle_{SU(3)}$, we can calculate $\langle \mathrm{Tr}\lambda\lambda \rangle_{G_2}$.

5. THE GROUP F_4

The F_4 algebra has the dimension 52, and that of its fundamental representation is 26. It is real, so that the mass term can be constructed with the aid of a single 26-plet. The vectors of this multiplet can be written in the form of traceless Hermitian octonion matrices of order three[11]:

$$J=\left\{\begin{pmatrix}\alpha_1 & \zeta_3 & \bar\zeta_2\\ \bar\zeta_3 & \alpha_2 & \zeta_1\\ \zeta_2 & \bar\zeta_1 & \alpha_3\end{pmatrix},\quad \sum_{j=1}^{3}\alpha_j=0\right\}. \tag{16}$$

The F_4 algebra acts in this representation as follows. The 24 generators that do not belong to the SO_8 subalgebra take the form of anti-Hermitian octonion matrices with zero diagonal:

$$P=\begin{pmatrix}0 & \eta_3 & -\bar\eta_2\\ -\bar\eta_3 & 0 & \eta_1\\ \eta_2 & -\bar\eta_1 & 0\end{pmatrix}, \tag{17}$$

and operate on the matrices J like an ordinary commutator: $\delta J = PJ - JP$. The remaining 28 generators of the F_4 algebra form the SO_8 subalgebra. The latter annihilates the diagonal elements of the matrix J, and each matrix element ζ_j in (16) transforms in accordance with one of its three eight-dimensional representations (8_V-vector, 8_S^L-left spinor, 8_S^R-right spinor). Thus,

$$\delta^{F_4}J=[P,J]+\delta^{SO_8}J, \tag{18}$$

which corresponds to the following expansion of the 26-plet:

$$26=1+1+8_V+8_S{}^L+8_S{}^R=2\cdot[1]+3\cdot[8]. \tag{19}$$

For the adjoint representation

$$[F_4]=[SO_8]+8_V+8_S{}^L+8_S{}^R=28+3\cdot[8], \tag{20}$$

where the last three terms are identical with the analogous terms in (19). Thus, for the adjoint representation, we have

$$52=28+3\cdot[8]. \tag{21}$$

We now note that, apart from the quadratic invariant (mass term)

$$\langle J,J\rangle=\mathrm{Tr}\{J,J\}/2 \tag{22}$$

there is also a cubic F_4 singlet that is the determinant of the matrix J. It can be written in symmetric form with the aid of a new operation, referred to as the Freundenthal product[11]:

$$J_1\times J_2\equiv\frac{1}{2}\left[\{J_1,J_2\}-J_1\,\mathrm{Tr}\,J_2-J_2\,\mathrm{Tr}\,J_1 +I\left(\mathrm{Tr}\,J_1\,\mathrm{Tr}\,J_2-\frac{1}{2}\mathrm{Tr}\{J_1,J_2\}\right)\right], \tag{23}$$

where J_1,J_2 represents the anticommutator $J_1J_2 + J_2J_1$. The cubic invariant then has the form

$$[J_1,J_2,J_3]=\mathrm{Tr}\{(J_1\times J_2),J_3\}/2. \tag{24}$$

It can be verified that this is symmetric under permutation of the indices 1,2,3.

An arbitrary matrix from the 26-plet (16) can be reduced by transformations from F_4 to a diagonal form (Freundenthal's theorem)[11]:

$$H=\left\{\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3);\ \sum\alpha_j=0\right\}. \tag{25}$$

The presence of the two independent parameters is due to the existence of two independent invariants (22) and (24). In general, if α_1, α_2, and α_3 in (25) are different, it follows from the description of the effect of F_4 that this vector breaks F_4 down to SO_8. This breaking is therefore general in character. The SO_8 algebra annihilates any vector from the space H. On the other hand, F_4 contains a subgroup that conserves H as a whole. This is an extension of SO_8 by the group of permutations of α_j. The invariants (22) and (24) can be expressed in terms of the α_j in the form of polynomials that are invariant under permutations $\Sigma\alpha_j^2$ and $\alpha_1\alpha_2\alpha_3$, which is obvious in this situation. In the special case where two diagonal elements α_j coincide, the group F_4 breaks down to SO_9. However, it can be verified that this regime is unstable.

The requirement that the F-terms of the Lagrangian vanish at the vacuum point leads to the condition $\alpha_1 \sim \alpha_2$. The existence of $\langle \mathrm{Tr}\lambda\lambda \rangle$ in F_4-gluodynamics follows unambiguously from the fact that $\langle \mathrm{Tr}\lambda\lambda \rangle_{SO(8)} \neq 0$. The determination of the precise value of the constant in the relation

$$\langle \mathrm{Tr}\,\lambda\lambda\rangle_{F_4}=\mathrm{const}\cdot\Lambda_{F_4}{}^3 \tag{26}$$

becomes a much more complicated problem because of the presence of the two independent parameters α_1, α_2. The relationship between $\Lambda_{SO(8)}$ and the bare quantities depends on the behavior of the theory in the transition region and, in particular, on the masses of the "heavy" vector bosons from F_4/SO_8. The masses of all the heavy bosons were previously equal, but now depend on the additional dimensionless ratio α_1/α_2. Of course, when $\alpha_1 \sim \alpha_2$, the order of magnitude of the constant in (26) is immediately established. However, when an exact value of the constant is required, we have to consider the relationship between α_1 and α_2. This question is not, as yet, finally settled.

6. THE GROUP E_6

E_6 has 78 generators. In addition to the 52 generators of its F_4 subalgebra, there are a further 26 generators that can be written in the form of traceless Hermitian octonion matrices, such as (16)[11]:

$$E_6=F_4+\{T\},\qquad 78=52+26. \tag{27}$$

The fundamental representation of E_6 has the dimension 27. It is described by the same matrices (16), but the matrices need not now be traceless. The E_6 algebra acts on the 27-plet as follows:

$$\delta^{E_6}J=\delta^{F_4}J+\frac{i}{2}\{T,J\}=\delta^{SO_8}J+[P,J]+\frac{i}{2}\{T,J\}. \tag{28}$$

It is readily seen from these formulas that the E_6 algebra is broken down to the F_4 subalgebra on the scalar matrices cI, so that

$$27=1+26. \tag{29}$$

Representation 27 is complex. This means that it is insufficient to enable us to form the mass term, and we must therefore examine the adjoint representation 27 as well. To describe this representation, we must suppose that the matrices are formed from octonions with complex coefficients. If we use a bar to represent complex conjugates of these coefficients, we have

$$\delta^{E_6}\bar{J}=\delta^{F_4}\bar{J}-\frac{i}{2}\{T,\bar{J}\}. \tag{30}$$

For each of the 27-plets, there is a cubic invariant that is analogous to the invariant (24) in the F_4 algebra: $I_3=[J,J,J]$, $\bar{I}_3=J,\bar{J},\bar{J}]$. In addition, there is the mass term $27\times\overline{27}$,

$$I_2=\langle J,\bar{J}\rangle=\mathrm{Tr}\{J,\bar{J}\}/2. \tag{31}$$

An invariant of degree four can be constructed on representations 27 and $\overline{27}$. Let d_{lmn} (d^{lmn}) be a symmetric tensor that determines the form of the cubic invariant $I_3(\bar{I}_3)$. The invariant tensor of rank four then has the form $d_{mnl}d^{pqr}\delta^l_r$. The invariant can be written in terms of the matrices J as follows:

$$\langle 27,27,\overline{27},\overline{27}\rangle=\langle J_1,J_2,\bar{J}_1,\bar{J}_2\rangle=\mathrm{Tr}\,\{(J_1\times J_2),(\bar{J}_1\times\bar{J}_2)\}. \tag{32}$$

We shall now show that the E_6 algebra is, in general, again broken down to SO_8. We shall first verify this formally by considering the expansions (27) and (29). Using the previously derived formulas (19) and (20) for the F_4 algebra, we have

$$[E_6]=[SO_8]+3\cdot[8]+3\cdot[8]+2\cdot[1], \tag{33}$$

$$\begin{aligned}\widetilde{27}&=[\tilde{1}]+3\cdot[8]+2\cdot[1],\\ \overline{27}&=[1]+3\cdot[8]+2\cdot[1].\end{aligned} \tag{34}$$

The representations 8 in (33) and (34) are related to the matrix elements P in (28) by the off-diagonal matrix elements T in (28) and the off-diagonal elements of the matrix J from the 27-plets. The two singlets in (33) are the diagonal elements of the matrix T, and the three singlets in (34) are the diagonal elements of the matrix J. As usual, heavy fields in representations 8 are paired as well. Moreover, the diagonal elements of the matrix T ($2\cdot[1]$ in (33)) do not conserve the traces of the matrices J and $\bar{J}$. These singlets therefore are also heavy. Only the gauge field from the subgroup SO_8 and two singlets from each of 27 and $\overline{27}$ remain light.

In the space of representation 27, there is a subspace H that consists of the diagonal matrices to which an arbitrary matrix from the multiplet can be reduced:

$$H=\{\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3)\}. \tag{35}$$

Almost every vector from H has SO_8 as its stationary subgroup. When some of the α_j coincide, the stationary subgroup expands. For example, when two of the α_j coincide, the stationary group turns out to be $SO(10)$, and, when all three α_j coincide, the stationary is F_4. As in the last section, it can be shown that these regimes are unstable.

The factor group $\mathscr{N}(H)/(G'=SO_8)$ is generated by the permutations of α_j and the group of diagonal matrices of the form

$$\left\{\exp iT=\mathrm{diag}(e^{i\varphi_1},e^{i\varphi_2},e^{i\varphi_3});\ \sum_{j=1}^{3}\varphi_j=0\right\}. \tag{36}$$

The polynomials on H (35) that are invariant under the factor group correspond to the four invariants constructed above. They can be expressed in terms of chiral superfields:

$$\{\alpha_j=\lambda_j e^{i\psi_j}\}\in 27,\quad \{\beta_j=\lambda_j e^{-i\psi_j}\}\in\overline{27}\quad (j=1,2,3),$$

$$I_2=\sum_j\lambda_j^2,\quad I_3=\lambda_1\lambda_2\lambda_3e^{i\psi},\quad \bar{I}_3=\lambda_1\lambda_2\lambda_3e^{-i\psi}\quad \left(\psi=\sum_j\psi_j\right),$$

$$I_4=(\alpha_1\bar{\alpha}_2\beta_1\bar{\beta}_2+\alpha_1\bar{\alpha}_3\beta_1\bar{\beta}_3+\alpha_2\bar{\alpha}_3\beta_2\bar{\beta}_3)+\text{c.c.}$$

They form the four light singlets that remain in the intermediate model. It is clear that the new dynamic variables can be λ_1, λ_2, and λ_3 and the common phase ψ.

7. THE GROUP E_7

The 133 generators of the E_7 algebra can be split into the sum of four terms[11]:

$$E_7=E_6\oplus U_1\oplus\mathfrak{M}\oplus\overline{\mathfrak{M}}. \tag{37}$$

where $\mathfrak{M}$ are the matrices from the 27-plets of the E_6 subalgebra. Correspondingly,

$$133=78+1+27+\overline{27}. \tag{38}$$

The fundamental representation of E_7 has the dimension 56. It is pseudoreal so that we need two 56-plets to construct the mass term. An arbitrary vector from the 56-plet Ψ can be resolved in accordance with the representations of the E_6 algebra:

$$\begin{aligned}&\Psi=(\xi,\eta,X,Y),\\ &56=1+1+27+\overline{27}.\end{aligned} \tag{39}$$

where ξ and η are singlets, and X and y are 27-plets of the E_6 subalgebra. Transformations from E_7 operate in this basis in accordance with the rules

$$\begin{gathered}\delta^{E_6}\Psi=(0;\,0;\,\delta^{E_6}X;\,\delta^{E_6}Y),\\ \delta_\varphi^{U_1}\Psi=(+3\varphi\xi;\,-3\varphi\eta;\,-\varphi X;\,+\varphi Y),\\ \delta_{J_1}\Psi=\left(\frac{1}{2}\langle J_1,Y\rangle;\,-\frac{1}{2}\langle J_1,X\rangle;\,\frac{1}{2}\eta J_1-J_1\times Y;\,-\frac{1}{2}\xi J_1+J_1\times X\right),\\ \delta_{J_2}\Psi=\left(-\frac{i}{2}\langle J_2,Y\rangle,\,-\frac{i}{2}\langle J_2,X\rangle;\,\frac{i}{2}\eta J_2-iJ_2\times Y;\,-\frac{i}{2}\xi J_2-iJ_2\times X\right),\\ J_1\in\mathfrak{M},\quad J_2\in\overline{\mathfrak{M}}.\end{gathered} \tag{40}$$

As already noted, because of the pseudoreal condition, we need two 56-plets. The following mass term can be constructed from them:

$$\langle\Psi_1, \Psi_2\rangle=\xi_1\eta_2-\xi_2\eta_1+[\langle X_1, Y_2\rangle-\langle X_2, Y_2\rangle]/2. \quad (41)$$

For each of the 56-plets, there is an invariant of degree four:

$$\langle\Psi, \Psi, \Psi, \Psi\rangle=\langle X\times X, Y\times Y\rangle-\xi\langle X\times X, X\rangle -\eta\langle Y\times Y, Y\rangle-(\langle X, Y\rangle-\xi\eta)^2/2. \quad (42)$$

The introduction of the two 56-plets into the supersymmetric E_7-gluodynamics leads to the spontaneous breaking of the gauge group:

$$E_7\to SO_8. \quad (43)$$

According to Ref. 10, this breaking is general in character (see Appendix). To verify this, consider the expansions for the adjoint and fundamental representations of E_7 relative to the subgroup SO_8:

$$[E_7]=[SO_8]+12\cdot[8]+9\cdot[1],$$
$$56=6\cdot[8]+8\cdot[1], \quad 56=6\cdot[8]+8\cdot[1]. \quad (44)$$

All the fields belonging to the 8-plets are paired and heavy. Moreover, nine of the SO_8-singlet fields become heavy. The supersymmetric SO_8-gluodynamics with seven light singlet superfields that correspond to seven invariants formed from the two 56-plets survives in the low-energy limit. In particular, these invariants include the three indicated explicitly above: (41) and (42).

It follows from (44) that dim $H=16$. Moreover, the continuous part of the factor algebra $\mathcal{N}(H)/G'$ is identical with $SU_2\oplus SU_2\oplus SU_2$.

Note that the Lie algebras $\mathcal{N}(H)/G'$ and G' are identical with the Lie algebras that parametrize magic squares.[12] The magic square can be used to obtain a unified description of the breaking of gauge symmetries for all the groups that enter into it, with the exception of E_8.

APPENDIX

We shall now reproduce the results taken from the paper by Elashvili.[10] They contain the answer to the following question. Suppose we have a group G and matter fields in the representation M. The question is: to which subgroup G' is G broken by vacuum expectation values of these fields in the general position? Let us enumerate all the representations M of the groups $Sp(N)$, G_2, F_4, E_6, E_7, for which G' is nontrivial, i.e., $T_M<T_G$. In the case of unitary and orthogonal groups, the number of different possibilities is very large, and is reviewed in Ref. 10. Here, we indicate the dimensions of the representations M of the above groups and of the weights on the Dynkin diagram. We shall write out the expansions of M and of the adjoint representation of G over the representations of G'. We shall indicate the form of the invariant subspace H (Ref. 10) (in those cases where it is known), together with the values of T_G and T_M and the Lie algebra of the factor group $\mathcal{N}(H)/G'$.

FIG. 1. The group $Sp(N)$. Weights of the representations $[2N]$.

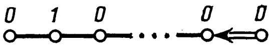

FIG. 2. The group $Sp(N)$. Weights of the representations $[2N^2-N-1]$.

The group $G=Sp(N)$: $T_G=N+1$, dim $G=N(2N+1)$

(1) The representation $M=[2N]$. Its weights in the Dynkin scheme are indicated in Fig. 1. The group G breaks down to $G'=Sp(N-1)$, and

$$M=[2(N-1)]+2\cdot[1], \quad [G]=[G']+2\cdot[2(N-1)]+3\cdot[1],$$
$$H=(\xi^1, \xi^2, 0,\ldots,0), \quad T_M=1/2, \quad \mathcal{N}(H)/G'=SU(2).$$

(2) The representation $M=[2N^2-N-1]$ in the space of the traceless symmetric quaternion matrices of order N is determined by the weights indicated in Fig. 2. In this case,

$$G'=SU(2)\oplus\ldots\oplus SU(2),$$
$$M=[(2)\oplus(2)\oplus(1)\oplus\ldots\oplus(1)]$$
$$+[(2)\oplus(1)\oplus(2)\oplus\ldots\oplus(1)]+\ldots$$
$$\ldots+[(1)\oplus\ldots\oplus(1)\oplus(2)\oplus(2)]+(N-1)[(1)\oplus\ldots\oplus(1)],$$
$$[G]=[G']+[(2)\oplus(2)\oplus(1)\oplus\ldots\oplus 1]+\ldots$$
$$\ldots+[(1)\oplus\ldots\oplus(1)\oplus(2)\oplus(2)]+(N-1)[(1)\oplus\ldots\oplus(1)],$$

where H is the space of the diagonal matrices and $T_M=(N-1)/2$, $\mathcal{N}(H)/G'=0$.

(3) For the reducible representation $M=m[2N]$ $(1<m\leqslant 2N+1)$, we have $G'=Sp(L)$, $L=N-[(m+1)/2]$, $\mathcal{N}(H)/G'=SU(2)$.

The group $Sp(3)$

This group has special representations that are not present in the case of $Sp(N)$:

(1) The representation $M=[14]$ (see Fig. 3). Here $G'=SU(3)$, $M=[6]+[\bar{6}]+[1]+[1]$, $[G]=[G']+[6]+[\bar{6}]+[1]$, $T_M=20$, $\mathcal{N}(H)/G'=U(1)$.

(2) For the reducible representation $M=[6]+[14]$ we have $G'=SU(2)$, $\mathcal{N}(H)/G'=SU(2)$.

The group G_2: dim $G_2=14$, $T_G=4$

(1) The representation $M=[7]$ (see Fig. 4). $G'=SU(3)$, $M=[3]+[\bar{3}]+[1]$, $[G]=[G']+[3]+[\bar{3}]$, $H=(\xi, 0,\ldots,0)$, $T_M=1$, $\mathcal{N}(H)/G'=0$.

(2) The reducible representation $M=[7]+[7]$. $G'=SU(2)$, $\mathcal{N}(H)/G'=SU(2)$.

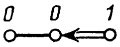

FIG. 3. The group $Sp(3)$. Weights of the representation [14].

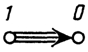

FIG. 4. The group G_2. Weights of the representation [7].

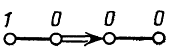

FIG. 5. The group F_4. Weights of the representation [26].

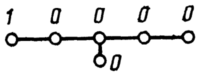

FIG. 6. The group E_6. Weights of the representation [27].

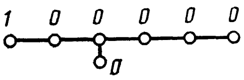

FIG. 7. The group E_7. Weights of the representation [56].

Group F_4: $\dim G = 52$, $T_G = 9$

(1) The representation $M = [26]$ [see Fig. 5 and (16)], $G' = SO(8)$, $M = [8_V] + [8_S^L] + [8_S^R] + 2\cdot[1]$, $[G] = [G'] + [8_V] + [8_S^L] + [8_S^R]$. The space H is given by (25); $T_M = 3$, $\mathcal{N}(H)/G' = 0$.

(2) The reducible representation $M = [26] + [26]$, $G' = SU(3)$, $\mathcal{N}(H)/G' = SU(3)$.

Group E_6: $\dim G = 78$, $T_G = 12$.

(1) The representation $M = [27]$ [see Fig. 6 and (28)], $G' = F_4$, $M = [26] + [1]$, $[G] = [G'] + [26]$. The space H is given by (35); $T_M = 3$, $\mathcal{N}(H)/G' = 0$.

(2) The reducible representation $M = [27] + [27]$, $G' = SO(8)$, $\mathcal{N}(H)/G' = U(1) \oplus U(1)$.

(3) The reducible representation $M = [27] + [\overline{27}]$, $G' = SO(8)$, $\mathcal{N}(H)/G' = U(1) \oplus U(1)$.

(4) The reducible representation $M = m[27] + n[\overline{27}]$ $(m + n = 3)$, $G = SU(3)$, $\mathcal{N}(H)/G' = SU(3) \oplus SU(3)$.

Group E_7: $\dim G = 133$, $T_G = 18$

(1) The representation $M = [56]$ [see Fig. 7 and (39)], $G' = E_6$, $M = [27] + [\overline{27}] = 2\cdot[1]$, $[G] = [G'] + [27] + [\overline{27}] + [1]$, $T_M = 6$, $\mathcal{N}(H)/G' = U(1)$.

(2) The reducible representation $M = [56] + [56]$, $G' = SO(8)$, $\mathcal{N}(H)/G' = SU(2) \oplus SU(2) \oplus SU(2)$.

We note that the Dynkin indices of different nontrivial representations can readily be found using the terminology of instanton calculus. The Dynkin index T_M is one-half the number of fermionic zero modes in the representation M if the instanton corresponds to the embedding $SU_2 \to G$ with minimum topologic charge. We can readily find the chain $SU_2 \to \ldots \to G' \to G$ and use it to expand M into a sum of irreducible representations of SU_2 (they are labeled by the half-integer spin j):

$$M = \sum_j a_j[2j+1].$$

The instanton field has $2j(j+1)(2j+1)/3$ fermionic zero modes in the representation $[2j+1]$. Hence,

$$T_M = \sum_j \{a_j j(j+1)(2j+1)/3\}.$$

For example, for the adjoint representation of G_2, $14 = 8 + 3 + \bar{3}|_{SU(3)}$ $= 3 + 2 + 2 + 1 + 2 + 1 + 2 + 1|_{SU(2)}$ and $T_{14} = 4$. Similarly, $7 = 3 + 3 + 1|_{SU(3)} = 2 + 1 + 2 + 1 + 1|_{SU(2)}$ and $T_7 = 1$. Of course, we need not come right down to SU_2 itself, but can stop with the group with known values of the Dynkin indices. For example,

$$\text{for } F_4, 52=28+8+8'+8''|_{SO(8)} \to T_{52}=6+3\cdot 1=9,$$

$$26=8+8'+8''+1+1|_{SO(8)} \to T_{26}=3\cdot 1=3;$$

$$\text{for } E_6, 78=52+26|_{F_4} \to T_{78}=9+3=12,$$

$$27=26+1|_{F_4} \to T_{27}=T_{26}=3;$$

$$\text{for } E_7, 133=78+27+\overline{27}+1|_{E_6} \to T_{133}=12+2\cdot 3=18,$$

$$56=27+\overline{27}+1+1|_{E_6} \to T_{56}=2\cdot 3=6;$$

$$\text{for } E_8, 248=133+56+56+1+1+1|_{E_7} \to T_{248}=18+2\cdot 6=30.$$

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