

# ***N*-modulation signals in a single-mode optical waveguide under nonlinear conditions**

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The Darboux representations in combination with numerical methods are used to obtain *N*-modulation solutions of the nonlinear Schrödinger equation describing the process of transformation of a continuous signal with a weak initial modulation into a periodic sequence of pulses. The optimal parameters for such a transformation are found, as well as the conditions under which the off-duty factor of the pulses is maximal. It is shown that in the optimal case the off-duty factor and the peak value of the power in a pulse are proportional to the square of the number *N* of the modulation harmonics in the input signal. Examples are given of numerical calculations of the profiles of the resultant pulses. A numerical analysis is made of the evolution of a spectrum of *N*-modulation signals. It is shown that a suitable selection of the parameters makes it possible to use such signals also for effective transformation of the spectra.

## **1. INTRODUCTION**

We shall consider the possibility of generation of a periodic sequence of picosecond light pulses in an optical fiber due to the process of self-modulation of continuous laser radiation.<sup>1–3</sup> The first experimental observation of the splitting into pulses as a result of spontaneous self-modulation was reported in Ref. 4. However, from the point of view of generation of pulses it is more convenient to utilize a process in which the output pulses can be controlled<sup>2,3</sup> by preliminary initial modulation of a continuous signal. The first experiment of this type was described in Ref. 5. The question arises as to whether it is possible to increase the off-duty factor in such a process without altering the repetition period. In other words, whether it is possible to compress additionally the pulses in each period of a sequence in the same way as in the case of nonlinear compression of single (multi-soliton) pulses in an optical fiber.<sup>6</sup> We shall show that, within the limits of the limitations imposed by the validity of the nonlinear Schrödinger equation, the degree of compression may reach any desired value and the process can be controlled by imposing a spectrum of weak initial modulation on a continuous signal at the entry to a fiber. In reality, this means that the pulse duration can be reduced to femtosecond values when pulses are separated by picosecond intervals. Moreover, we shall consider the problem of transformation of the signal spectrum as a result of its nonlinear evolution.

Propagation of signals in an optical fiber at wavelengths 1.55–1.65  $\mu\text{m}$  in the negative group dispersion range, where there are strong nonlinear effects, is described by the nonlinear Schrödinger equation. The exact solutions of this equation were obtained in Ref. 3 (see also Refs. 7 and 8) and they describe the process of self-modulation of waves in an optical fiber. By analogy with *N*-soliton solutions,<sup>9</sup> we can call the new solutions “*N*-modulation.” Both classes of solutions are separatrix solutions of the nonlinear Schrödinger equation, but in the former case they are periodic in respect of one variable ( $\xi$ ) and in the latter case they are periodic in another variable ( $\tau$ ). Such *N*-modulation solutions were obtained in Ref. 3 and analyzed for the cases *N* = 1 and 2. However, if  $N \geq 3$ , these solutions acquire completely new

features. In particular, only such solutions can ensure a high degree of compression of the pulses. However, if  $N \geq 3$ , the solutions become so cumbersome that it is pointless to obtain them in an analytic form and different approaches are needed. Direct numerical modeling of these solutions by the existing methods<sup>10,11</sup> fails to give the desired results because the initial conditions for their realization are not known and also because errors accumulate rapidly in the course of calculations when rapidly varying functions are employed. Numerical methods based on the inverse scattering problem<sup>12</sup> have not yet been applied to periodic solutions.

A hierarchy of solutions of the nonlinear Schrödinger equation of a certain class was obtained in Ref. 13 by developing Darboux transformations which however have not yet been used for practical purposes and this applies also to finite-band theories<sup>11</sup> (Refs. 14 and 15). The only exception seems to be the work reported in Ref. 17, but even there the regular calculation procedure is not described. We shall apply the Darboux transformations to obtain *N*-modulation solutions of the nonlinear Schrödinger equation by numerical methods and we shall analyze these solutions up to *N* = 10 inclusive. In contrast to the evolution methods,<sup>10,11</sup> we can use the proposed method to find the solution directly for any section of an optical fiber and this shortens greatly the calculation time, so that for relatively low values of *N* the calculations are practically instantaneous. This will enable us to derive the main relationships governing the process of self-modulation of waves in a fiber in the case when  $N > 2$ , to determine the dependences of the off-duty factor of the pulses and other characteristics of *N*-modulation signals on the value of *N*, and to identify the initial conditions under which the process of transformation of a continuous signal into a periodic sequence of pulses is proceeding in an optimal manner. Moreover, we shall investigate the process of transformation of a signal spectrum as a function of the number of harmonics in the input signal, which is equal to the value of *N*. We shall show that the Darboux transformations are the most convenient method for numerical determination of such solutions.

The present paper is organized as follows. In the second section we shall formulate the problem and give the general

Darboux transformation formulas for arbitrary solutions. In the third section we shall find specific formulas for  $N$ -modulation solutions. In the fourth section we shall report the results of numerical calculations of a signal at the exit from a fiber and in the fifth section we shall analyze the initial conditions needed to ensure that pulses of the optimal profile are formed. In the sixth section we shall carry out a spectral analysis of the signals from the point of view of the most convenient utilization of an optical fiber as a device for the transformation of spectra. In the seventh section we shall summarize the results obtained.

## 2. FORMULATION OF THE PROBLEM AND THE DARBOUX TRANSFORMATION FORMULAS

It is known<sup>1-3,6</sup> that the propagation of signals in an optical fiber in the range of frequencies corresponding to negative dispersion, where nonlinear properties of the fiber material are important, is described by the following nonlinear Schrödinger equation

$$i\psi_s + \psi_{\tau\tau}/2 + |\psi|^2\psi = 0, \quad (1)$$

where

$$\xi = q^2 x/\lambda, \quad \tau = q(-\lambda k'')^{-1/2}(t - x/v), \quad \psi = (\pi n_2)^{1/2} \varphi/q, \quad (2)$$

$$\lambda = 2\pi c/\omega, \quad k'' = \partial^2 k/\partial \omega^2, \quad v = \partial \omega/\partial k,$$

$\varphi$  is the envelope of the optical field,  $x$  is the longitudinal coordinate,  $t$  is time,  $n_2$  is the nonlinear refractive index of quartz,  $\omega$  is the frequency,  $\lambda$  is the wavelength,  $k$  is the wave vector, and  $q$  is a normalization factor governing the relationship between the signal amplitude and the characteristic changes of the field with time and distance in the fiber.

The Darboux transformation method<sup>13</sup> for deriving a hierarchy of solutions with an increasing number of parameters is based on the ability to represent the nonlinear Schrödinger equation for a function  $\psi(\xi, \tau)$  in the form of the compatibility condition of the following system of linear equations

$$R_\tau = UR + UR, \quad R_\xi = (I^2 J + IU + V/2)R, \quad (3)$$

where

$$R = \begin{pmatrix} r \\ s \end{pmatrix}, \quad U = \begin{pmatrix} 0 & i\psi^* \\ i\psi & 0 \end{pmatrix}, \quad (4)$$

$$V = \begin{pmatrix} -i|\psi|^2 & \psi_{\tau\tau}^* \\ -\psi_{\tau\tau} & i|\psi|^2 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$l$  is an arbitrary constant.

For each solution  $\psi(\xi, \tau)$  of the nonlinear Schrödinger equation there is a pair of functions  $r$  and  $s$  which depend not only on  $l$ , but also on two arbitrary integration constants (we shall call them  $C$  and  $D$ ). A set of these constants together with  $l$  will be denoted by  $\sigma = \{l, C, D\}$ . If we know a certain (initial) solution of the nonlinear Schrödinger equation  $\psi = \psi_1(\xi, \tau)$ , then in the first stage of obtaining new solutions the functions  $r_1(\sigma)$  and  $s_1(\sigma)$  can be found only by direct solution of the system (3). Therefore, the solution  $\psi_1(\xi, \tau)$  should be relatively simple. We shall assume that the set of the parameters in these functions is fixed so that  $\sigma = \sigma_1$ . Beginning from  $\psi_1$ ,  $r_1$ , and  $s_1$ , we can obtain a new solution of the nonlinear Schrödinger equation by means of the formula

$$\psi_{II}(\xi, \tau) = \psi_I(\xi, \tau) + \frac{2(l_1^* - l_1)r_1^*s_1}{|r_1|^2 + |s_1|^2}. \quad (5)$$

The solution  $\psi_{II}$  is determined not only by those parameters which influence  $\psi_I$ , but also by new parameters  $\sigma_1$ . The power of numerical methods used in the Darboux transformation approach is such that the functions  $r_{II}$  and  $s_{II}$  for the next step, corresponding to the solution of Eq. (5), can be found without solving Eq. (3) but simply using an operator  $M$  given below and capable of yielding new functions  $R_{II}$ , employing solely arithmetic transformations:

$$R_{II}(\sigma_1, \sigma_2) = M[R_I(\sigma_1), R_I(\sigma_2)], \quad (6)$$

where  $\sigma_2$  is a second fixed set of parameters. The full relationships represented by Eq. (6) can be written as follows:

$$r_{II}(\sigma_1, \sigma_2) = \Delta[(l_1^* - l_1)s_1^*(\sigma_1)r_1(\sigma_1)s_1(\sigma_2) + (l_2 - l_1)|r_1(\sigma_1)|^2r_1(\sigma_2) + (l_2 - l_1^*)|s_1(\sigma_1)|^2r_1(\sigma_2)], \quad (7)$$

$$s_{II}(\sigma_1, \sigma_2) = \Delta[(l_1^* - l_1)s_1(\sigma_1)r_1^*(\sigma_1)r_1(\sigma_2) + (l_2 - l_1)|s_1(\sigma_1)|^2s_1(\sigma_2) + (l_2 - l_1^*)|r_1(\sigma_1)|^2s_1(\sigma_2)],$$

where  $\Delta = (|r_1(\sigma_1)|^2 + |s_1(\sigma_1)|^2)^{-1}$ . In numerical calculations the factor  $\Delta$  does not affect the form of the new solutions of the nonlinear Schrödinger equation and it can be dropped. This process can be continued and new functions  $R_N$  consisting of  $N$ -functions  $\psi_i(\sigma_i)$  are obtained at each step  $N$  ( $i = 1, 2, \dots$ ). The recurrence formula for this procedure is

$$R_N(\sigma_1, \dots, \sigma_N) = M[R_{N-1}(\sigma_1, \dots, \sigma_{N-1}), R_{N-1}(\sigma_1, \dots, \sigma_{N-2}, \sigma_N)], \quad (8)$$

where instead of  $l_1$  and  $l_2$  in Eq. (7) we have to substitute  $l_{N-1}$  and  $l_N$ , respectively. Then, for  $N = 2$ , Eq. (8) reduces to Eq. (7).

The new solution of the nonlinear Schrödinger equation found at each stage of the calculations can be deduced from the formula

$$\psi_{N+1}(\xi, \tau) = \psi_N(\xi, \tau) + \frac{2(l_N^* - l_N)s_N r_N^*}{|r_N|^2 + |s_N|^2}. \quad (9)$$

The selection of the initial solution  $\psi_1(\sigma_1)$  and of the constants  $\sigma_1$  is determined by the actual physical problem to be solved as well as by the initial and boundary conditions, for example, those applied to obtain  $N$ -soliton solutions  $\psi_1(\sigma_1) = 0$ .

## 3. PROBLEM OF THE MODULATION INSTABILITY

We shall assume that the initial solution of the nonlinear Schrödinger equation is

$$\psi_I = \exp(i\xi + i\theta). \quad (10)$$

An arbitrary phase factor  $e^{i\theta}$  in Eq. (10) will be omitted. The functions  $r_I$  and  $s_I$  corresponding to the solution (10) are readily found by integration of Eq. (3):

$$r_I = \{C \exp[i(2\chi + \kappa_1\tau + l_1\kappa_1\xi)/2] - D \exp[-i(2\chi + \kappa_1\tau + l_1\kappa_1\xi)/2]\} \exp(-i\xi/2), \quad (11)$$

$$s_I = \{C \exp[i(-2\chi + \kappa_1\tau + l_1\kappa_1\xi)/2] + D \exp[-i(-2\chi + \kappa_1\tau + l_1\kappa_1\xi)/2]\} \exp(i\xi/2),$$

where  $\kappa_1 = 2(1 + l_1^2)^{1/2}$  and  $\cos 2\chi = \kappa_1/2$ . We shall assume that the parameter  $l_1$  is purely imaginary, so that

$l_1 = i\nu_1$ , and we shall replace the constants  $C$  and  $D$  with real parameters  $\tau_{01}$  and  $\xi_{01}$ :

$$C = \exp \{ [\delta_1 \xi_{01} - i\kappa_1 (\tau_{01} + \pi/2)] / 2 \}, \quad (12)$$

$$D = \exp \{ [-\delta_1 \xi_{01} + i\kappa_1 (\tau_{01} + \pi/2)] / 2 \},$$

where  $\delta_1 = \kappa_1 \nu_1$ . Then Eq. (11) can be written in the form

$$\begin{aligned} r_1 &= \{ \exp [ (2i\chi + i\kappa_1 (\tau - \tau_{01} - \pi/2) - \delta_1 (\xi - \xi_{01})) / 2 ] \\ &- \exp [ (-2i\chi - i\kappa_1 (\tau - \tau_{01} - \pi/2) + \delta_1 (\xi - \xi_{01})) / 2 ] \} \exp (-i\xi/2), \\ s_1 &= \{ \exp [ (-2i\chi + i\kappa_1 (\tau - \tau_{01} - \pi/2) - \delta_1 (\xi - \xi_{01})) / 2 ] \\ &+ \exp [ (2i\chi - i\kappa_1 (\tau - \tau_{01} - \pi/2) + \delta_1 (\xi - \xi_{01})) / 2 ] \} \exp (i\xi/2). \end{aligned} \quad (13)$$

In this case the solution  $\psi_1(\sigma_1)$  obtained with the aid of Eq. (5) is

$$\begin{aligned} \psi_1(\xi, \tau) &= \left\{ 1 - \frac{\kappa_1^2 \operatorname{ch} \delta_1 (\xi - \xi_{01}) + 2i\kappa_1 \nu_1 \operatorname{sh} \delta_1 (\xi - \xi_{01})}{2 [\operatorname{ch} \delta_1 (\xi - \xi_{01}) - \nu_1 \cos \kappa_1 (\tau - \tau_{01})]} \right\} \\ &\times \exp [i(\xi + \pi)]. \end{aligned} \quad (14)$$

Apart from the relabeling of the constant  $\nu$ , it is identical with the solution describing the modulation instability of a wave with a continuous amplitude when the initial modulation is due to one harmonic [Eq. (9) in Ref. 3]. Then, the quantity  $\kappa_1$  ( $0 < \kappa_1 < 2$ ) determines the frequency of the initial ( $\xi \rightarrow -\infty$ ) modulation of the continuous signal, whereas

$$\delta_1 = \kappa_1 (1 - \kappa_1^2/4)^{1/2} \quad (15)$$

is a growth increment of the modulated signal instability. The dependence of  $\delta_1$  on  $\kappa_1$  is plotted in Fig. 1. The increment is real within the range  $0 < \kappa_1 < 2$  and its maximum corresponds to  $\kappa_1 = 2^{1/2}$ .

The selection of the parameters  $\sigma_j$  in the next stages of obtaining the hierarchy of the solutions of the nonlinear Schrödinger equation is made in the same way as in the case of Eq. (12):

$$\begin{aligned} l_N &= i\nu_N, \quad \kappa_N = 2(1 - \nu_N^2)^{1/2}, \quad \delta_N = \kappa_N \nu_N, \\ C &= \exp \{ [\delta_N \xi_{0N} - i\kappa_N (\tau_{0N} + \pi/2)] / 2 \}, \\ D &= \exp \{ [-\delta_N \xi_{0N} + i\kappa_N (\tau_{0N} + \pi/2)] / 2 \}. \end{aligned} \quad (16)$$

The periodicity of the solution in respect of the variable  $\tau$  is retained if we assume that  $\kappa_N = N\kappa_1$ , but  $0 < \kappa_N < 2$  ( $\kappa_1 < 2/N$ ) so as to ensure that the growth increment of the  $N$ th harmonic  $\delta_N = \kappa_N (1 - \kappa_N^2/4)^{1/2}$  remains real (Fig. 1). When  $N = 2$  the solution can still be written in an analytic form. However, the next steps corresponding to  $N \geq 3$  can be made only by numerical methods.

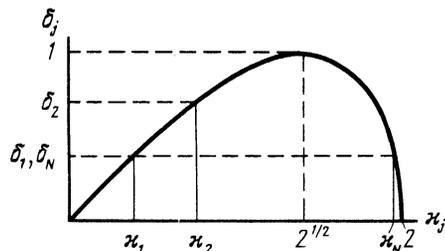


FIG. 1. Instability growth rates of harmonics during the initial state of formation of an  $N$ -modulation signal.

#### 4. NUMERICAL CALCULATIONS

We investigated  $N$ -modulation signals in an optical fiber by obtaining the exact solutions of the nonlinear Schrödinger equation up to  $N = 10$ . For these values of  $N$  it is possible to reveal all the characteristic features of the solutions, as well as their asymptotic behavior when  $N \rightarrow \infty$ . It should be pointed out that up to  $N = 5$  inclusive the calculations can be carried out to one significant place, but in order to avoid accumulation of rounding errors in the range  $5 < N \leq 10$ , we need accuracy to within two significant figures. In the calculations we selected the fundamental frequency  $\kappa_1$  so that the growth increment for the first and  $N$ th harmonics are the same (Fig. 1). Using Eq. (15), we can readily find this value:  $\kappa_1 = 2(N^2 + 1)^{-1/2}$ . We then have  $\delta_1 = \delta_N = 2N/(N^2 + 1)$ . This condition guarantees that the growth increments of all the harmonics are real and are approximately of the same order of magnitude. Moreover it is found that this selection of  $\kappa_1$  makes it possible to optimize the profile of the pulses obtained at the exit from an optical fiber.

If all the values of  $\xi_{0j}$  are different and the differences between them are very large, we obtain a sum of elementary solutions. Therefore, generation of sharp pulses can be expected only for values of  $\xi_{0j}$  close to one another. The numerical calculations demonstrate that the best results are obtained when all  $\xi_{0j}$  are the same. If moreover, all of them obey  $\xi_{0j} = 0$ , then the complete solution becomes symmetric:  $\psi(-\xi) = \psi^*(\xi)$ .

Numerical calculations also demonstrate that for the value  $\kappa_1$  selected above and zero values of  $\xi_{0j}$  the largest amplitude and off-duty factor of the pulses in a periodic signal are obtained for  $\xi = 0$  provided all  $\tau_{0j}$  are assumed to be zero. Figure 2 shows the signal profiles within each period for the cases when  $N = 3, 4$ , or 5. It is clear from this figure that for the selected parameters the pulses appear sharply

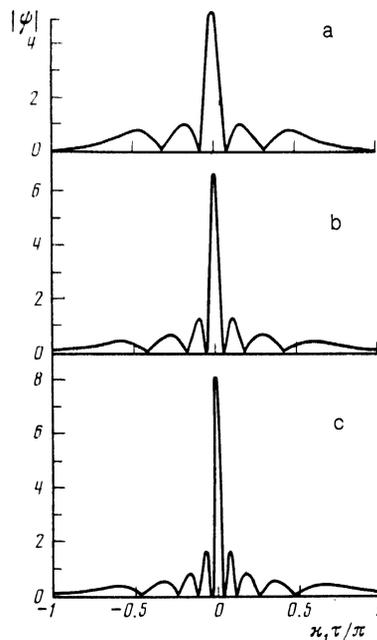


FIG. 2. Profiles of pulses with the maximum off-duty factor at the point  $\xi = 0$  for  $N = 3$  (a), 4 (b), and 5 (c).

TABLE I. Parameters of a periodic sequence of pulses of optimal profile

$N$	$\mu$	$Q$	$\eta$	$N$	$\mu$	$Q$	$\eta$
1	2.4142	2.00	0.950	6	9.5760	40.83	0.803
2	3.6833	5.45	0.867	7	11.1139	55.53	0.801
3	5.0790	11.03	0.831	8	12.6601	71.95	0.799
4	6.5463	18.82	0.815	9	14.2119	91.18	0.798
5	8.0507	28.81	0.807	10	15.7676	112.7	0.798

against a small-signal background. An increase in  $N$  increases the pulse amplitude and reduces its width. The dependence of this process on  $N$  will be described by introducing the following parameters. We shall use  $\psi_{\max}$  to denote the peak value of the amplitude for a pulse. If we bear in mind that in the asymptotic case corresponding to  $\xi \rightarrow -\infty$  the signal amplitude is  $\psi_0 = 1$ , then this quantity in fact determines the relative increase in the pulse amplitude  $\mu = \psi_{\max}/\psi_0$  in the course of evolution. Moreover, we shall use  $\tau_0$  to denote that zero which is close to the maximum of the function  $\psi(\xi, \tau)$  and, for the sake of simplicity, we shall define the off-duty factor by  $Q = T/2\tau_0$ , where  $T = 2\pi/\kappa_1$  is the pulse repetition period. At high values of  $N$  the pulses assume a certain asymptotic profile and, if we define the off-duty factor in the usual way as the ratio of the repetition period to the width (duration) of a pulse at half the peak power, calculations show that for  $N > 5$  this factor is  $\sim 2.96Q$ . The relative fraction of the energy concentrated in a pulse can be calculated from

$$\eta = \frac{I}{I_0} = \int_{-\tau_0}^{\tau_0} |\psi|^2 d\tau \left( \int_{-\tau/2}^{\tau/2} |\psi|^2 d\tau \right)^{-1}. \quad (17)$$

All three above parameters describing a periodic sequence of pulses, calculated for different values of  $N$ , are listed in Table I. Using the results of this table, we find that the off-duty factor  $Q$  obtained for large numbers  $N$  is described by an asymptotic formula

$$Q \approx 2^{-1/2} \pi (N^2 + 1) \approx 2^{-1/2} \pi N^2, \quad (18)$$

from which we can see that the factor rises quadratically on increase in  $N$ . The increase in the maximum amplitude of the pulses can also be described by an asymptotic formula:

$$\mu = a(N^2 + 1)^{1/2} + b, \quad (19)$$

where  $a = 1.564 \dots$ ,  $b = 0.046 \dots$ . The peak value of the power carried by a pulse is  $\sim \mu^2$  and, like the off-duty factor, rises almost quadratically. The fraction of the energy carried by the main peak decreases slightly and for  $N \rightarrow \infty$  it approaches a certain constant  $\eta \sim 0.797 \dots$ , so that the maximum theoretical efficiency of the process of transformation into pulses is fairly high. It should be noted that the values  $\kappa_1 = 2(N^2 + 1)^{-1/2}$  selected initially are optimal for ensuring the best parameters of the pulses if  $N$  is fixed. Variation of  $\kappa_1$  to the right or left of this optimal value increases strongly the background of the signal and the parameter  $\eta$  deteriorates.

As already mentioned, the case when all of  $\tau_{0j}$  vanish is optimal, but this is not the only way of generating compressed pulses. For example, if  $N = 3$ , we can also obtain narrow pulses for  $\tau_{01} = 0$ ,  $\tau_{02} = 0$ , and  $\tau_{03}\kappa_3/\pi = 1$ . Such a

combination of phases will be denoted by  $\{\tau_{0j}\} = \{0, 0, 1\}$ . The maximum value of the field in a pulse is then  $\mu = 3.81$  and the off-duty factor is  $Q = 7.86$ . Figure 3 shows the pulse profiles for  $N = 3, 4$ , and  $5$ , respectively when the phases are  $\{\tau_{0j}\} = \{0, 0, 1\}$ ,  $\{0, 0, 0, 1\}$  and  $\{0, 0, 0, 0, 1\}$ . A comparison of Figs. 2 and 3 show that the energy in a pulse decreases and at the same time the background increases compared with the case of zero phases. An increase in  $N$  increases the number of possible combinations of the phases which can provide pronounced pulses. Table II gives characteristic parameters of the pulses obtained for  $N = 3, 4$ , and  $5$  in the course of a gradual shift away from the optimally selected phases  $\{\tau_{0j}\}$ . We can see that for  $N = 5$  it is possible to obtain pronounced pulses for ten different phase relationships between the harmonics. In experiments the relationships between the phases can be set by selecting the phases of the harmonics of the initial modulation of a continuous signal.

### 5. AMPLITUDES OF INITIAL MODULATION OF A CONTINUOUS SIGNAL

An arbitrary  $N$ -modulation signal can be represented, in the asymptotic case  $\xi \rightarrow \infty$ , in the form of a wave of constant amplitude on which the following periodic perturbation is superimposed:

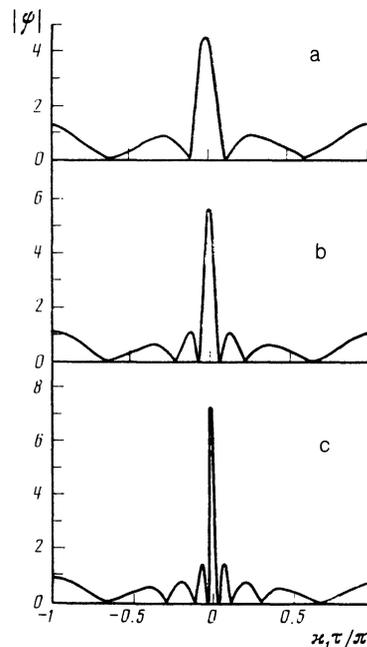


FIG. 3. profiles of pulses of an  $N$ -modulation signal at a point  $\xi = 0$  calculated for  $\tau_{0N}\kappa_N/\pi = 1$  ( $\tau_{0j} = 0$  when  $j \neq N$ ) for the cases  $N = 3$  (a), 4 (b), and 5 (c).

TABLE II. Parameters of a periodic sequence of pulses obtained for nonoptimal phase relationships

$(\tau_{0j})$	$\mu$	$Q$	$\eta$	$(\tau_{0j})$	$\mu$	$Q$	$\eta$
$N=3$				$N=5$			
0, 0, 0	5.08	11.0	0.831	0, 0, 0, 0, 0	8.05	28.8	<b>0.808</b>
0, 0, 1	3.81	7.86	0.640	0, 0, 0, 0, 1	7.27	25.8	<b>0.731</b>
1, 1, 1	3.08	6.40	0.515	1, 1, 1, 1, 1	6.05	21.5	<b>0.670</b>
$N=4$				0, 0, 0, 1, 0	5.56	19.4	<b>0.566</b>
0, 0, 0, 0	6.54	18.8	0.815	1, 1, 1, 1, 0	5.26	18.4	<b>0.534</b>
0, 0, 0, 1	5.58	15.7	0.700	0, 0, 1, 0, 0	4.82	16.7	<b>0.491</b>
0, 0, 1, 0	3.78	10.2	0.490	0, 0, 0, 1, 1	4.78	16.3	<b>0.492</b>
1, 1, 1, 0	3.57	9.69	0.460	0, 1, 0, 0, 0	4.37	15.1	<b>0.446</b>
0, 0, 1, 1	2.82	6.68	0.392	1, 0, 0, 0, 0	4.12	14.3	<b>0.420</b>
				0, 1, 0, 0, 1	3.58	11.9	<b>0.373</b>

$$\psi(\xi, \tau) = \left[ 1 + \sum_{j=1}^N \rho_j(\xi) \exp(i\alpha_j) \cos \kappa_j(\tau - \tau_{0j}) \right] \exp[i(\xi + \varphi)], \quad (20)$$

where  $\varphi$  is the total phase of the signal,  $\alpha_j$  are the relative phases of the harmonics given by  $\tan \alpha_j = 2\delta_j/\kappa_j^2$ , and the quantities

$$\rho_j(\xi) = A_j \exp(\delta_j \xi) \quad (21)$$

are the initial amplitudes of the harmonics. The constants  $A_j$  corresponding to low values of  $N$  can be found by linearization of the complete solution. For example, if  $N=1$ , the amplitude is  $A_1 = 2$ , whereas for  $N=2$ , it is  $A_1 = A_2 = 8/3$ . If  $N \geq 3$ , we can find linear amplitudes  $A_j$  by numerical methods. Table III lists the asymptotic values of  $A_j$  for  $N=3, 4$ , and 5. However, it should be pointed out that these asymptotic values correspond to much larger values of  $\xi$  than those in the cases  $N=1$  or 2, where  $\rho_j(\xi)$  become between three and five orders of magnitude smaller than unity. Moreover, if  $N \geq 3$ , some second-order terms may become comparable with linear terms in the expansion of Eq. (20). Bearing this point in mind, we used calculation methods to find the coefficients  $\rho_j(\xi)$  for relatively small values of  $\xi$ . The phases  $\alpha_j$  then differ from the values determined from the formula  $\tan \alpha_j = 2\delta_j/\kappa_j^2$ .

Table IV lists the values of  $\rho_j$ ,  $\alpha_j$ , and  $A_j$  calculated using the Fourier transformation formulas given below. The values of  $\xi$  were selected so that the depth of modulation at the entry to a fiber was 2–3%. A comparison of the values of  $A_j$  in Tables III and IV shows that they agree for  $j=1$  and  $j=N$ , when  $\delta_j$  are the smallest. In the case of intermediate values of  $j$  the difference can be very large and these  $A_j$  depend on  $\xi$ , which is related to the contribution made to  $\rho_j(\xi)$  by terms of the second and higher orders of the type  $\exp(2\delta_j \xi)$ , etc.

It therefore follows that in experimental realization of  $N$ -modulation signals in an optical fiber the amplitudes and phases of the harmonics of the initial modulation must be calculated for each given length of a fiber and for each intensity of the continuous signal at the entry. We shall now obtain some estimates. We shall assume that  $N=5$  and that the pulse repetition period is 6 ps. Then, the duration of the pulses at half the peak power is  $\sim 70$  fs. Using Eq. (2) we find that  $q \sim 4.7 \times 10^{-4}$  and the length of the fiber necessary to achieve this process is  $\sim 100$  m (for  $\xi = -15$ ). Then, for a fiber of  $\sim 20 \mu\text{m}^2$  cross section the average power of the continuous signal at the fiber entry should be  $\sim 25$  W. The depth of modulation of the first harmonic can be seen from Table II to amount to  $\sim 1.74\%$ . If we allow for the contribution of the other harmonics, then the total depth of modulation is  $\sim 2.5\%$ .

## 6. EVOLUTION OF SPECTRA OF $N$ -MODULATION SIGNALS

All the functions considered above are even within a period if  $\tau_{0j} \kappa_j / \pi = 0$  or 1. Therefore, in an analysis of the spectra we can represent the solutions of the nonlinear Schrodinger equation in the form of the cosine Fourier expansion:

$$\psi(\xi, \tau) = f_0(\xi) + 2 \sum_{n=1}^{\infty} f_n(\xi) \cos n\kappa\tau, \quad (22)$$

where the coefficients are given by

$$f_0(\xi) = \frac{\kappa_1}{2\pi} \int_{-\pi/\kappa_1}^{\pi/\kappa_1} \psi(\xi, \tau) d\tau, \quad (23)$$

$$f_n(\xi) = \frac{\kappa_1}{2\pi} \int_{-\pi/\kappa_1}^{\pi/\kappa_1} d\tau \psi(\xi, \tau) \cos n\kappa\tau, \quad n=1, 2, \dots \quad (24)$$

The squares of the moduli of the Fourier coefficients for any  $\xi$  satisfy the relationship

TABLE III. Asymptotic values of the amplitudes of harmonics of initial modulation

$N$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
3	3.570	8.151	2.622	—	—
4	4.554	18.913	17.744	2.428	—
5	5.576	36.612	63.726	33.586	2.223

TABLE IV. Fourier coefficients and phases in expansion of Eq. (20), and values of amplitudes  $A_j$  calculated using Eq. (21) for finite  $\xi$

$j$	$\rho_j(\xi)$	$\alpha_j, \text{rad}$	$A_j$
$N=3$ <span style="float: right;"><math>\xi=-9</math></span>			
1	$0.1611 \cdot 10^{-1}$	-1.894	3.568
2	$0.1022 \cdot 10^{-2}$	-2.460	6.904
3	$0.1183 \cdot 10^{-2}$	-2.821	2.618
4	$0.1354 \cdot 10^{-3}$	0.785	-
5	$0.7351 \cdot 10^{-5}$	0.453	-
6	$0.3685 \cdot 10^{-4}$	0.323	-
7	$1 \cdot 10^{-6}$	-2.623	-
$N=4$ <span style="float: right;"><math>\xi=-12</math></span>			
1	$0.1607 \cdot 10^{-1}$	-1.816	4.554
2	$0.4634 \cdot 10^{-3}$	-2.235	12.225
3	$0.1449 \cdot 10^{-3}$	3.121	23.104
4	$0.8562 \cdot 10^{-2}$	3.386	2.427
5	$0.9767 \cdot 10^{-4}$	0.784	-
6	$0.2471 \cdot 10^{-5}$	0.523	-
7	$0.5212 \cdot 10^{-6}$	0.073	-
$N=5$ <span style="float: right;"><math>\xi=-15</math></span>			
1	$0.1741 \cdot 10^{-1}$	4.515	5.576
2	$0.3629 \cdot 10^{-3}$	4.087	18.223
3	$0.2801 \cdot 10^{-4}$	3.881	44.235
4	$0.8838 \cdot 10^{-4}$	2.536	192.722
5	$0.6940 \cdot 10^{-2}$	3.339	2.223
6	$0.8852 \cdot 10^{-4}$	0.797	-
7	$0.1403 \cdot 10^{-5}$	0.558	-

$$|f_0|^2 + 2 \sum_{n=1}^{\infty} |f_n(\xi)|^2 = 1, \quad (25)$$

which describes conservation of the total energy of the field during propagation along a fiber.

The Fourier coefficients can be found analytically only for  $N = 1$ , when the solution is relatively simple and is given by Eq. (14) [see Ref. 7]. In all other cases we have to find the Fourier coefficients by numerical methods. A full picture of the evolution of the signal spectrum in the optimal case, when all  $\tau_{0j}$  vanish, is plotted in Fig. 4 for the cases  $N = 3, 4, 5$ . Since the process is symmetric relative to  $\xi = 0$ , this figure gives only the values in the range  $\xi < 0$ . For  $\xi \rightarrow -\infty$  obviously the whole energy of the signal is represented by the carrier frequency and we have  $|f_0|^2 = 1$ . In the process of evolution the energy is distributed between side bands and for  $\xi = 0$  there is an approximately uniform distribution between the harmonics, because the signal itself is nearly in the form of the  $\delta$  function.

We also investigated the spectra of  $N$ -modulation signals from the point of view of maximum transformation into one of the side bands. The problem was then to find such a relationship between the initial phases  $\tau_{0j}$  for which the carrier energy at some point  $\xi$  is transferred mainly into some specific side band. For example, if  $N = 2$  and  $\{\tau_{0j}\} = \{0, 1\}$ , almost all the energy of the carrier at the point  $\xi = 0$  is transferred symmetrically to the first side band  $|f_1|^2 = 0.465$ . The total energy of the first side band (on the left or right) is  $\sim 93\%$  of the whole initial energy. At high values of  $N$  the energy may be transferred to further side bands. Table V lists all possible cases investigated by us (in the range  $N \leq 5$ ) in which a high degree of conversion into any one of the side bands is achieved. We can see from this table that in none of these cases is the conversion efficiency greater than 0.4. However, it should be pointed out that we have ignored the values of the phases  $\kappa_j \tau_{0j} / \pi$  which are not integers when the

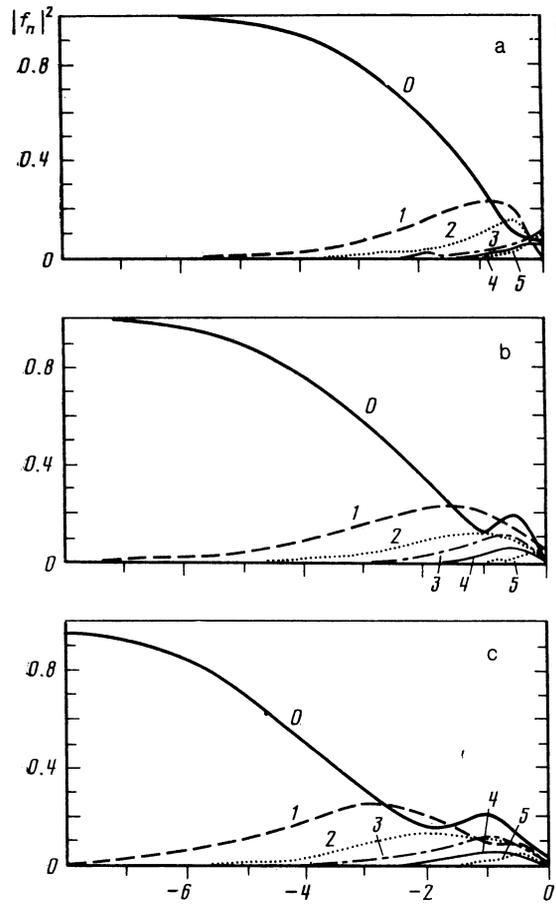


FIG. 4. Evolution of the spectrum of  $N$ -modulation signals with the maximum off-duty factor, shown for  $N = 3$  (a), 4 (b), and 5 (c). The numbers alongside the curves give the harmonic numbers.

spectrum becomes asymmetric. In this case the efficiency of conversion to a single side band may exceed 0.5. However, this is a subject of a separate communication.

## 7. CONCLUSIONS

We considered the process of self-modulation of a continuous signal in an optical fiber due to a nonlinear dependence of the refractive index of the fiber on the wave field. The Darboux transformation method used in the above analysis, together with numerical calculations, allowed us to carry out a complete analysis of the solutions of the nonlinear

TABLE V. Relationships between the phases  $\{\tau_{0j}\}$  of  $N$ -modulation signals corresponding to maximum energy transfer to one of the side bands.

$\{\tau_{0j}\}$	$ f_j ^2$
$\{0, 1\}, \{1, 1\}$	$N=2$ $ f_1 ^2 = 0.4650$
$\{1, 1, 0\}, \{0, 1, 1\}$	$N=3$ $ f_1 ^2 = 0.3876$
$\{1, 0, 0\}, \{0, 0, 1\}$	$ f_2 ^2 = 0.2017$
$\{1, 1, 0, 0\}, \{0, 1, 1, 0\}$	$N=4$ $ f_1 ^2 = 0.3883$
$\{1, 0, 0, 1\}, \{0, 0, 1, 1\}$	$ f_2 ^2 = 0.3567$
$\{1, 1, 1, 0\}, \{0, 1, 0, 0\}$	$ f_3 ^2 = 0.2469$
$\{1, 1, 0, 0, 1\}, \{0, 1, 1, 0, 0\}$	$N=5$ $ f_1 ^2 = 0.3775$
$\{1, 0, 0, 1, 0\}, \{0, 0, 1, 1, 1\}$	$ f_2 ^2 = 0.3508$
$\{1, 1, 1, 0, 0\}, \{0, 1, 0, 0, 1\}$	$ f_3 ^2 = 0.2438$

Schrodinger equation representing such self-modulation. It was found that the degree of compression of a pulse in each repetition period can be increased to very high values. This effect can be useful in the construction of master oscillators (lasers) for optical communication lines with high transmission rates because it is so far the only method for generating pulses of femtosecond duration with a repetition frequency of the order of thousands or more of gigahertz. Obviously, the generation of pulses with a high off-duty factor considered here cannot be realized by spontaneous self-modulation as in Ref. 4. One requires an imposed initial modulation of the signal and this has to be done by several harmonics. Such modulation is relatively easily realized by means of semiconductor lasers with an output in the form of continuous radiation representing several longitudinal modes with strongly locked phases.

The effect under consideration also provides new opportunities for the transformation of a signal spectrum in a fiber. It is clear from our results that the efficiency of conversion into side bands can be increased up to 46% and one can select the situation when the transfer of energy is to a given side band. The Darboux transformation method once again provides the best approach for a rapid and complete investigation of the effect and for determination of the parameters of the signal at the exit from a fiber.

We shall conclude by noting that  $N$ -modulation solutions can be constructed also in the case of the sine-Gordon equation. An analysis based on the theory of dynamic systems<sup>18</sup> makes it possible to predict some features of such multipass solutions which have a number of properties in common with  $N$ -soliton solutions.

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<sup>1)</sup>An explanation of this situation can be found in Ref. 16.

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