Rotation of the plane of polarization of a photon in a Petrov type $\mathscr{D}$ space-time

N. Yu. Gnedin and I. G. Dymnikova

A. F. Ioffe Physicotechnical Institute, USSR Academy of Sciences
(Submitted 9 July 1987)
Zh. Eksp. Teor. Fiz. **94**, 26–31 (January 1988)

A detailed study is made of the rotation of the plane of polarization of a photon under the influence of the gravitational field of a rotating source that possesses in the general case electric and magnetic charges. Observational manifestations of the rotation are also pointed out. An exact expression is obtained for the angle of rotation of the plane of polarization of a photon propagating along a geodesic of arbitrary form in a Petrov type $\mathscr{D}$ space-time. It is shown that observation of rotation of the plane of polarization could be a new test of general relativity as well as a method for determining the physical characteristics of a rotating and charged black hole or neutron star.

A possible new test of general relativity would be observation of rotation of the plane of polarization of a photon under the influence of the gravitational field of a rotating body. This effect, which is a consequence of the dragging of the inertial reference frames by the gravitational field of a rotating source, has been investigated in the literature in the weak field approximation for the case of a photon traveling parallel to the rotation axis from $-\infty$ to $+\infty$ or traveling along the axis from the rotating object. The results obtained by different methods in Refs. 1–6 do not agree with each other. One of the reasons for the discrepancy is an insufficiently correct choice of the employed approximations. This was shown in Ref. 7, in which the case of a photon leaving a radiating object radially was also considered in the weak field approximation.[1)] A second reason is the absence of a sufficiently clear and unambiguous definition of a method for measuring the angle of rotation of the plane of polarization of a photon propagating in a curved space-time.

In x-ray astronomy, a new trend has now been successfully developed. It is x-ray polarimetry, and there is hope that in the not too distant future the effect of rotation of the plane of polarization in the gravitational field of a rotating object will be measured. It is clear that this effect must be maximal in the strong gravitational field near a neutron star or a rotating black hole, where the weak-field approximation is insufficient. It is therefore necessary to consider it in more detail.

In the present paper, we propose an unambiguous definition of the method for measuring the angle and we obtain an exact expression for the angle of rotation of the plane of polarization of a photon propagating along a geodesic of arbitrary form in a Petrov type $\mathscr{D}$ space-time, and we also point out possible observational manifestations of the effect. We use the geometrical system of units with $c = G = 1$ and signature $+2$ of the metric.

For a photon propagating in a curved space-time, the problem of defining the rotation of its plane of polarization arises even in the simplest case when the photon is emitted (for example, at $z = -\infty$) and detected (at $z = +\infty$) in flat space-time. Indeed, to specify the polarization vector at $z = -\infty$ one constructs a tetrad of basis vectors at the point $\mathscr{P}_1$, at which the polarization vector $\bar{f}$ is specified. This tetrad is then transported parallel to itself along two paths: a) along the isotropic geodesic that is the trajectory of the considered photon, b) along the $r = \infty$ path in asymptotically flat space (see Fig. 1). As a result, two tetrads are obtained at $z = +\infty$ at the point $\mathscr{P}_2$—the one "transported geodesically" along path a and the one with "flat transport" along path b. They are rotated relative to each other in the plane orthogonal to the unit time vector $\bar{e}_t$ and the vector $\bar{k}$ tangent to the geodesic through an angle $\Delta\Phi_0$. In addition, the vector $\bar{f}$ is rotated with respect to the geodesically transported tetrad through an angle $\Delta\Psi$ in the same plane. Then the complete rotation of the polarization vector of the photon with respect to the "flat-transported" tetrad, measured by the observer at infinity, is

$$\Delta\Phi = \Delta\Psi + \Delta\Phi_0.$$

In some of the studies quoted above, the rotation angle of the plane of polarization is taken to be the angle $\Delta\Psi$, and this leads to discrepancy with the results obtained by other authors. This can be avoided if as the original tetrad one chooses an isotropic canonical tetrad of the Newman-Penrose formalism, which is allowed naturally by the very structure of the space-time. Since this tetrad is transported parallel to itself, it is not rotated with respect to the flat-transported tetrad, i.e., in this case $\Delta\Phi_0 = 0$. This property of it means that one can also unambiguously define the angle of rotation of the plane of polarization in the general case when the photon is emitted or detected in the immediate proximity of a field source, i.e., in truly curved space-time. Since now $\Delta\Phi = \Delta\Psi$, the problem reduces to calculating the rotation angle of the photon polarization vector with respect to the

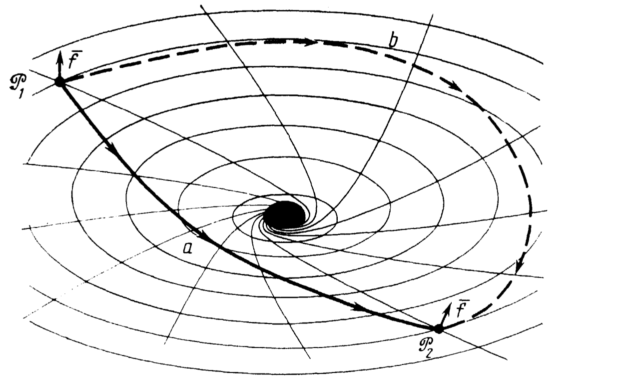

FIG. 1. Rotation of the polarization vector of a photon propagating in curved space-time; here, a is an isotropic geodesic, and b is the path of transport of the tetrad of basis vectors along the $r = \infty$ path in flat space.

 0038-5646/88/010013-03$04.00

isotropic tetrad in the Newman-Penrose formalism.

In a Petrov type $\mathscr{D}$ space-time there exists an integral of the motion for isotropic geodesics; this was obtained in Ref. 8 and is

$$K_s=\{(\bar{k}\bar{l})(\bar{f}\bar{n})-(\bar{k}\bar{n})(\bar{f}\bar{l})-(\bar{k}\bar{m})(\bar{f}\bar{m}^*)+(\bar{k}\bar{m}^*)(\bar{f}\bar{m})\}\Psi_2^{-1/3}. \quad (1)$$

Here, $\bar{l}$ and $\bar{n}$ are real and $\bar{m}$ and $\bar{m}^*$ complex isotropic basis vectors satisfying the conditions

$$\bar{l}\bar{m}=\bar{l}\bar{m}^*=\bar{n}\bar{m}=\bar{n}\bar{m}^*=0, \quad \bar{l}\bar{n}=-1, \quad \bar{m}\bar{m}^*=1,$$

Ψ_2 is, in the considered case, the only nonvanishing tetrad complex scalar component of the Weyl tensor describing the gravitational field outside the source, $\bar{k}$ is the tangent vector to the geodesic, $\bar{k}^2=0$, and $\bar{f}$ is a spacelike vector orthogonal to $\bar{k}$ and transported parallel to itself along $\bar{k}$, so that

$$\bar{f}\bar{k}=0, \quad \nabla_{\bar{k}}\bar{f}=0.$$

The vector $\bar{f}$ can be identified with the photon's polarization vector.

We introduce two real vectors:

$$\bar{a}=(\bar{k}\bar{l})\bar{n}-(\bar{k}\bar{n})\bar{l}, \quad i\bar{b}=(\bar{k}\bar{m}^*)\bar{m}-(\bar{k}\bar{m})\bar{m}^*,$$

and two real scalar quantities:

$$A=\bar{a}\bar{f}, \quad B=\bar{b}\bar{f}.$$

It is easy to show that

$$K_s=(A+iB)\Psi_2^{-1/3}.$$

The vectors $\bar{a}$ and $\bar{b}$ satisfy the relation

$$\bar{a}^2=\bar{b}^2=2(\bar{k}\bar{l})(\bar{k}\bar{n})=2|\bar{k}\bar{m}|^2\geqslant 0. \quad (2)$$

Equality is achieved when $\bar{k}$ lies on a principal null congruence, i.e., when $\bar{k}\sim\bar{l}$ or $\bar{k}\sim\bar{n}$). This case is degenerate, and we therefore consider first the more general case of a vector $\bar{k}$ that does not lie on a principal null congruence. The vectors $\bar{a}$ and $\bar{b}$ possess the properties

$$\bar{a}\bar{k}=0, \quad \bar{b}\bar{k}=0, \quad \bar{a}\bar{m}=0, \quad \bar{b}\bar{l}=0,$$
$$\bar{a}\bar{b}=0, \quad \bar{a}\bar{m}^*=0, \quad \bar{b}\bar{n}=0.$$

Thus, they are mutually orthogonal and each of them lies in the plane defined by a pair of isotropic basis vectors; in addition, they are always orthogonal to $\bar{k}$ (if $\bar{k}$ does not lie on a principal null congruence this condition is not trivial). Therefore, they can be used as polarization unit vectors of the photon. Then

$$\Delta\Phi=\arg\{(\bar{a}+i\bar{b})\bar{f}\}|_{\mathscr{P}_1}^{\mathscr{P}_2}=\arg(A+iB)|_{\mathscr{P}_1}^{\mathscr{P}_2}$$

for propagation of the photon from the point $\mathscr{P}_1$ to the point $\mathscr{P}_2$ of the space-time. Since $K_s=\text{const}$, we have

$$\arg K_s=\arg(A+iB)+\arg(\Psi_2^{-1/3})=\text{const}$$

and, therefore,

$$\Delta\Phi={}^1/_3\arg\Psi_2|_{\mathscr{P}_1}^{\mathscr{P}_2}. \quad (3)$$

It follows from continuity considerations that this formula is also true for principal null congruences.

The Schwarzschild and Reissner-Nordström metrics are characterized by a real scalar tetrad component Ψ_2. Therefore, as one would expect, for a nonrotating source there is no rotation of the photon's plane of polarization.

In the Kerr metric,

$$\Psi_2=M(r-ia\cos\vartheta)^{-3},$$

where M is the mass and a is the specific dimensionless angular momentum of the gravitating body; Boyer-Lindquist coordinates, which coincide asymptotically with ordinary spherical coordinates in flat space, are used here. In this case[2)]

$$\Delta\Phi=\varepsilon\,\text{arctg}\left(\frac{a}{r}\cos\vartheta\right)\Big|_{\mathscr{P}_1}^{\mathscr{P}_2},$$

where

$$\varepsilon=\text{sign}\{(r(\mathscr{P}_2)-r(\mathscr{P}_1))(\vartheta(\mathscr{P}_2)-\vartheta(\mathscr{P}_1))\}.$$

The signum ε arises because when the photon approaches the source and leaves it there is a change in the sign of the components k^r and k^ϑ of the geodesic vector, these being determined by the equations (see, for example, Ref. 9)

$$k^r=\pm(\mathscr{P}(r))^{1/2}\rho^{-2}, \quad k^\vartheta=\pm(\Theta(\vartheta))^{1/2}\rho^{-2}, \quad (4)$$

where

$$\mathscr{P}(r)=r^4+(a^2-\xi^2-\eta)r^2+2M(\eta+(\xi-a)^2)r-a^2\eta,$$
$$\Theta(\vartheta)=\eta+a^2\cos^2\vartheta-\xi^2\,\text{ctg}^2\vartheta,$$
$$\rho^2=r^2+a^2\cos^2\vartheta, \quad \xi=L_zE^{-1}, \quad \eta=QE^{-2}.$$

Here, ξ and η are integrals of the geodesic motion, L_z and E are the orbital angular momentum and the energy of the photon, and Q is Carter's integral (see Ref. 9).

For a photon traveling parallel to the rotation axis of the source,

$$\Delta\Phi=-2\,\text{arctg}\left(\frac{a}{r}\cos\vartheta\right)\Big|_{r=\infty}^{\mathscr{P}_i},$$

where $\mathscr{P}_i$ is the point of closest approach of the photon to the deflecting body:

$$P(R)=0, \quad R=r(\mathscr{P}_i), \quad \vartheta_i=\vartheta(\mathscr{P}_i).$$

In the first order in a/R and M/R, we obtain

$$\Delta\Phi=-2\,\text{arctg}\left(\frac{a\cos\vartheta_i}{R}\right)\approx-2\frac{a}{R}\cos\vartheta_i.$$

The values of ϑ_i and R are obtained by solving Eqs. (4). For a photon with arbitrary initial value of the polar angle ϑ_0,

$$\cos\vartheta_i\approx-\frac{2M}{R}\cos\vartheta_0, \quad \Delta\Phi\approx\frac{4Ma}{R^2}\cos\vartheta_0.$$

For $\vartheta_0=\pi$, the angle $\Delta\Phi\approx-4Ma/R^2$. The minus sign reflects the rule for measuring the angle, namely, if we look after the photon along the direction $\bar{k}$, then the anticlockwise direction will be positive (as argument of a complex number).

In the case of the Kerr-Newman metric, which describes the gravitational field of a rotating charged body, the component of the Weyl tensor has the form

$$\Psi_2=M(r-ia\cos\vartheta)^{-3}\left\{1-\frac{P^2+Q^2}{M(r+ia\cos\vartheta)}\right\}, \quad (5)$$

where Q and P are, respectively, the electric and magnetic charges of the source. Calculating the rotation angle in accordance with Eq. (3), we find

$$\Delta\Phi=\varepsilon\left\{\text{arctg}\left(\frac{a\cos\vartheta}{r}\right)+\frac{1}{3}\text{arctg}\left(\frac{a\cos\vartheta}{r+M\rho^2(Q^2+P^2)^{-1}}\right)\right\}\Big|_{\mathscr{P}_1}^{\mathscr{P}_2}. \quad (6)$$

Far from the source, the rotation angle, calculated to terms of second order in a/R and M/R, has the form

$$\Delta\Phi\approx\varepsilon\frac{a\cos\vartheta}{r}\left\{1+\frac{Q^2+P^2}{3Mr}\right\}\Big|_{\mathcal{P}_i}^{\mathcal{P}_s}. \qquad (7)$$

In this case, as follows from the expressions we have obtained, measurement of the rotation angle of the photon's plane of polarization makes it possible to measure the electric and magnetic charges of the black hole.

In conclusion, we list briefly the possible observational manifestations of the considered effect. In Ref. 10 there is a calculation of the degree of polarization and of the position angle of the radiation of the accretion disk around a rotating black hole as functions of the photon energy, i.e., effectively as functions of the distance in the disk to the black hole. The calculation takes into account only the geometrical factors associated with the relativistic bending of the accretion disk near the rotating hole. Allowance for this effect leads to an additional rotation of the plane of polarization compared with the results of Ref. 10. For distances of ~ 10 gravitational radii, the magnitude of this effect is small and amounts to a few degrees. Therefore, the effect can be discovered only in future polarimetric observations. However, searches for this effect are important, since they would permit determination of the physical characteristics of the black hole, for example, its electric and magnetic charges.

A second possible observational manifestation is the following. It is well known that by virtue of symmetry the integrated polarization of the radiation of a spherical star is zero although the radiation of each of its part is polarized, i.e., there is complete cancellation of the polarization from the individual sections of the surface of the star. The effect we have considered leads to a breakdown of the cancellation, since the effect depends strongly on the position on the surface of the star. As a result, we obtain an integrated polarization along the rotation axis of the star:

$$P_l={}^1/_3\varepsilon(a/R_s)^2,$$

where $\varepsilon = 100\%\tau$ for an optically thin and $\varepsilon = 11.7\%$ for an optically thick radiating region (τ is the optical thickness). It is possible to express a in terms of the angular velocity of the rotation of the star. For this, it is necessary to know the expression for the relativistic moment of inertia of the star. In the case of a neutron star, one can obtain a relationship between a and Ω by matching the Kerr metric to the metric within the star. To obtain an estimate, we assume that the star rotates as a rigid body. Then, using the expression for the metric within the star in the form of (5.111) from Ref. 11, we find

$$a\approx(R_s-r_g)^4\Omega/R_s r_g.$$

For a neutron star of radius $R_8 = 10$ km and $M = M_\odot$, we obtain

$$P_l\approx 0.2\varepsilon\,(v_e/c)^2,$$

where v_e is the equatorial rotation velocity of the star. It is proposed to make detailed calculations of this effect in a separate paper. In this case, observation of the rotation of the plane of polarization under the influence of the gravitational field of a rotating source will not only serve as a test of general relativity but may also give a new independent relation connecting the angular momentum, mass, and radius of a neutron star.

We take this opportunity to express our great thanks to Yu. N. Gnedin, who drew our attention to the observational manifestations of the rotation of the photon plane of polarization, and also M. Demianski and A. I. Tsygan for helpful discussions.

1) We note that Ref. 7 also gives a general expression for the angle of rotation of the plane of polarization of a photon traveling parallel to the axis. This is given in the form of an integral which, however, is calculated under the assumption that the photon travels absolutely straight without undergoing deflections.

2) *Translator's Note*: The Russian notation for the trigonometric, inverse trigonometric, hyperbolic functions, etc., is retained here and throughout the article in the displayed equations.

1. G. V. Skrotskii, Dokl. Akad. Nauk SSSR **114**, 73 (1957) [Sov. Phys. Dokl. 2, 226 (1958)].
2. N. L. Balazs, Phys. Rev. **110**, 236 (1958).
3. J. Plebanski, Phys. Rev. **118**, 1396 (1960).
4. B. B. Godfrey, Phys. Rev. D **1**, 2721 (1970).
5. S. Pineault and R. C. Roeder, Astrophys. J. **212**, 541 (1977).
6. F. S. O. Su and R. L. Mallett, Astrophys. J. **238**, 1111 (1980).
7. F. Fayos and L. Llosa, Gen. Rel. Grav. **14**, 865 (1982).
8. M. Walker and R. Penrose, Commun. Math. Phys. **18**, 265 (1970).
9. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973) [Russ. transl., Mir, Moscow, 1977], §33.5.
10. P. A. Connors, T. Piran, and R. F. Stark, *Preprint of Center for Relativity* (University of Texas at Austin, 1979).
11. M. Demianski, *Relativistic Astrophysics* (PWN, Warsaw, 1985).

Translated by Julian B. Barbour