Exact solutions for moving matter in general relativity

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Solutions are obtained using the Hilbert principle and a reference frame with comoving time in which the velocity field is identified with certain components of the metric. A general solution is found for the spherically symmetric motion of charged dust, defined by three arbitrary functions. Synchronized solutions of the Einstein–de Sitter–Friedmann type are examined. Matter does not collapse to a point, but each layer in its interior is compressed down to only the classical electromagnetic radius of matter. A subclass of axisymmetric motions is obtained for rotating charged and uncharged dustlike matter with a Kerr-Newman metric in the external region. A general solution is also found for the spherically symmetric motion of a photon gas (with a nonconserved number of particles) and for a gas-dust mixture.

1. HILBERT PRINCIPLE AND THE REFERENCE FRAME WITH COMOVING TIME

One way of obtaining solutions in GTR is to use Hilbert's principle[2] which states that the consequence of invariance under arbitrary coordinate transformations is that, out of all the equations describing a set of fields in GTR, four are satisfied identically when all the others are satisfied. This principle is used in the Rainich geometrodynamics[2] in which Maxwell's equations are a consequence of the equations of gravitation. The principle can be used much more effectively in hydrodynamics, where the field variables are the metric tensor and the velocity field: the latter can be identified with certain components of the metric tensor by a special choice of the coordinate frame, i.e., a system with comoving time (SCT) in which space coordinates are arbitrary and variable gravitational fields are described by a metric tensor of spatial separations γ_{ij} (the space part of the tetrad η_i^α, $\alpha, i = 1,2,3$), and the time part of the tetrad is directly related to the velocity field in the medium:

$$\eta_0{}^0=u^0\equiv 1/f,\qquad \eta_0{}^i=u^i\equiv v^i/f,\qquad \eta_\alpha{}^0=0, \tag{1}$$

which leads to the Arnowitt-Deser-Misner (ADM) metric[2]

$$ds^2=(f^2-\gamma_{ij}v^iv^j)dt^2+2\gamma_{ij}v^idx^jdt-\gamma_{ij}dx^idx^j. \tag{2}$$

In this metric, the only nonzero components of the energy-momentum tensor of a Pascal fluid are[3]

$$T_0{}^0=\varepsilon,\qquad T_0{}^i=(\varepsilon+p)v^i,\qquad T_j{}^i=-p\delta_j{}^i. \tag{3}$$

In the spherically-symmetric case, the motion is radial ($u^r\equiv a$):

$$ds^2=(f^2-b^2a^2)dt^2+2b^2adrdt-b^2dr^2-r^2d\omega^2,$$
$$d\omega^2=d\theta^2+\sin^2\theta d\varphi^2. \tag{4}$$

The components of the Einstein tensor can be expressed in terms of f, b, a:

$$G_0{}^0=[r(1+a^2-b^{-2})]'/r^2, \tag{5}$$

$$G_1{}^1=\frac{2\dot a}{fr}+\frac{1}{r^2}\left(1+a^2-b^{-2}+2raa'-\frac{2f'r}{b^2f}\right), \tag{6}$$

$$G_1{}^0=\frac{2}{fbr}[\dot b+a(fb)']=0. \tag{7}$$

In the case of pressureless charged matter (dust) the transformation of an arbitrary metric g^{ij} in SCT requires the solution of the Hamilton-Jacobi equation that defines the comoving time in terms of the old coordinates:

$$g^{ij}=(\partial\tau/\partial x^i+qA_i)(\partial\tau/\partial x^j+qA_j)=1, \tag{8}$$

where q is the specific charge of the particles and A_i is the vector potential of the electromagnetic field. The covariant components of the velocity 4-vector are then

$$u_i=\partial\tau/\partial x^i+qA_i. \tag{9}$$

For example, let us transform the Reissner-Nordstrom metric to the SCT:

$$ds^2=Udt^2-U^{-1}dr^2-r^2d\omega^2,\qquad U=1-2M/r+Q^2/r^2. \tag{10}$$

The Hamilton-Jacobi equation is stationary:

$$U^{-1}\dot\tau^2-U(\tau')^2=1,\quad \dot\tau=h,\quad \tau'=u/U,\quad u^2=h^2-U, \tag{11}$$

and h is the specific energy of the particles. Hence

$$f=1,\quad a=-u=\mp(h^2-U)^{1/2},\quad b^2=h^{-2}. \tag{12}$$

In particular, for $h = 1$,

$$ds^2=(1-a^2)d\tau^2+2adrd\tau-dr^2-r^2d\omega^2,\quad a=\pm(2M/r-Q^2/r^2)^{1/2} \tag{13}$$

and we obtain one of the simplest representations of the Reissner-Nordstrom (Schwarzschild) metric with a flat spatial projection.

When the motion of matter is described, the "redundant" equations (in the sense of the Hilbert principle) are usually taken to be the equations describing the dynamics of the medium. However, greater simplification can often be achieved by solving the dynamical equations, which leads to the solution of a smaller number of Einstein equations. In particular, when the motion of dust is described, the trajectories of the dust particles can be determined as trajectories in curved space produced by internal layers. In the absence of gravitational radiation, the dust-particle trajectories are the characteristics of the entire set of equations, and the constants of integration of a given layer are constant on characteristics that determine the nonstationary solution during the motion of the particles. We shall use this method in Sec. 2 to find the general solution for spherically-symmetric motion, and in Sec. 3 for the subset of the axisymmetric (Kerr) motion of charged dust. In Sec. 4, we shall find the general solution for the spherically symmetric motion of a photon gas and a photon gas mixed with dust.

2. GENERAL SOLUTION FOR SPHERICALLY SYMMETRIC MOTION OF CHARGED DUST

Consider a spherical cloud of charged dust, compressed by gravitational forces. Let us follow the motion of the outer-

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most, infinitely thin, layer of mass dm, specific charge q, and specific energy h. The fall of the layer occurs in the Reissner-Nordstrom metric with two parameters (M and Q) and the electrostatic field $E = Q/r^2$. The Lagrangian and the specific energy of the particles in the layer are respectively given by

$$\frac{dZ}{dm} = -(U-U^{-1}\dot{r}^2)^{1/2} - \frac{qQ}{r}, \quad h=U(U-U^{-1}\dot{r}^2)^{-1/2} + \frac{qQ}{r}. \tag{14}$$

Hence

$$(U-U^{-1}\dot{r}^2)^{-1/2} = U^{-1}\left(h - \frac{qQ}{r}\right) = \frac{dt}{d\tau} = u^0, \quad u_0 = h - \frac{qQ}{r}, \tag{15}$$

where τ is the proper time and

$$(dr/d\tau)^2 = (h^2-1) + 2(M-hqQ)/r - (1-q^2)Q^2/r^2. \tag{16}$$

The component T_0^0 of the energy-momentum tensor is given by

$$T_0{}^0 = (\rho u^0)u_0 + \frac{E^2}{8\pi} = \frac{1}{8\pi r^2}\left[2\frac{\partial m}{\partial r}\left(h - \frac{qQ}{r}\right) + \frac{Q^2}{r^2}\right], \tag{17}$$

since

$$\partial m/\partial r = 4\pi r^2 \rho u^0. \tag{18}$$

From the Einstein equations[4]

$$8\pi r^2 T_0{}^0 = [r(1-U)]' = 2M' - 2QQ'/r + Q^2/r^2 \tag{19}$$

we find that the charge in the vacuum constants on the layer is determined by the energy and charge of the layer:

$$dM = h\,dm, \quad dQ = q\,dm; \quad h = dM/dm, \quad q = dQ/dm. \tag{20}$$

All that remains is to intergrate (16). Let

$$h^2 - 1 \equiv A, \quad M - hqQ \equiv B, \quad (1-q^2)Q^2 \equiv C, \quad 1 + ACB^{-2} \equiv k^2. \tag{21}$$

The constant A determines the type of the trajectory:

$A = 0$:

$$\tau = \tau_0 + \frac{1}{3B^2}(2Br - C)^{1/2}(Br + C); \tag{22}$$

$A > 0$:

$$r = \frac{B}{A}(k \operatorname{ch}\chi - 1), \quad \tau = \tau_0 + BA^{-3/2}(k \operatorname{sh}\chi - \chi); \tag{23}$$

$A < 0$:

$$r = \frac{B}{|A|}(1 - k\cos\chi), \quad \tau = \tau_0 + B|A|^{-3/2}(\chi - k\sin\chi). \tag{24}$$

The trajectories of the motion in proper time, and the local metric in the given layer, satisfy the Einstein equations and the law of motion of the particles. We can therefore remove the assumption that the layer is external, i.e., the presence of layers outside the given layer has no effect on either the gravitational or the electromagnetic field in the interior.

Each layer can be labeled by the amount of matter, m, in its interior. If we arbitrarily specify the distributions of total charge and total energy among the layers (as functions of mass), $Q(m)$ and $M(m)$, and also the intergration constant $\tau_0(m)$, which can assume different values in different layers and which defines for each layer the instant of time at which the layer reaches its minimum radius, then (22)–(24) give the mass $m(r,\tau)$ as a function of radius and proper time and, hence, $M(r,\tau)$, $Q(r,\tau)$, which determine the metric.

The Hamilton-Jacobi equation is

$$U^{-1}\left(h - \frac{qQ}{r}\right)^2 - U\left(\frac{\partial\tau}{\partial r}\right)^2 = 1, \quad \frac{\partial\tau}{\partial r} = \pm\frac{a}{U},$$

$$a^2 = \left(h - \frac{qQ}{r}\right)^2 - U = A + \frac{2B}{r} - \frac{C}{r^2}, \tag{25}$$

and, in accordance with (12), gives

$$g_{00} = Uh^{-2}, \quad g_{01} = -ah^{-2}, \quad g_{11} = -\frac{1}{Uh^2}\left(1 - \frac{2B}{r} + \frac{C}{r^2}\right),$$

$$(-g)^{1/2} = \frac{r^2}{h}\sin\theta, \tag{26}$$

where M and Q are no longer constants, but turn out to be functions of radial distance and comoving time. The general solution is determined by the arbitrary functions $M/(m)$, $Q(m)$, and $\tau_0(m)$.

For $Q(m) = 0$, we obtain the Tolman solutions.[5]

Special cases

1. The Einstein-de Sitter solution is $h = 1$, $\tau_0 = 0$, const $= q^2 < 1$. In terms of the new variables

$$mq^2/r \equiv y, \quad 3\tau^2(q^{-2}-1)/r^2 \equiv x^2 \tag{27}$$

equation (22) is a cubic in y:

$$y^3 + 3y(x^2-1) - 2 = 0, \tag{28}$$

and the corresponding discriminant is nonnegative:

$$D = x^2(x^4 - 3x^2 + 3) \equiv w^2 \tag{29}$$

so that the solution is unique

$$m(r, \tau) = rq^{-2}[(1+w)^{1/3} + (1-w)^{1/3}]. \tag{30}$$

For $\tau < 0$, all the layers move toward the center and instantaneously stop at $\tau = 0$, so that $m(r,0) = 2rq,^{-2}$ and the density increases toward the center as r^{-2}, with a singularity at the center, but with zero mass concentrated upon it; each layer comes to rest at the classical electromagnetic radius of the medium in the interior: $r_0 = Q^2/(2m) = q^2m/2$. Expansion begins for $\tau > 0$.

For $\tau \neq 0$, we find that near the center $[x \gg 1, r^2 \ll \tau^2(\rho^{-2} - 1)]$ we have $m \sim r^3\tau,^{-2}\rho =$ const, and the effect of the electric charge is insignificant because of its quadratic contribution to the metric, so that the particles in this region execute the usual homogeneous Einstein-de Sitter collapse with a deceleration $(1 - q^2)^{1/2}$. However, as τ tends to zero, the homogeneous region shrinks to zero. Conversely, as time increases, an increasing amount of mass departs well beyond the electromagnetic radius and its motion is the same as in the absence of the charge. It is clear from the above construction that m cut off at some upper value m_0 and the solution is joined to the metric (10). For $q^2 < 1$, the end of the compression state at the electromagnetic radius of the charged cloud occurs under both gravitational radii of the Nordstrom metric[6] in the region where $g_{00} > 0$. When compression stops inside the medium, we have the Minkowskii metric.

2. Systems whose dynamics consist in a synchronous change in the general scale (Robertson-Walker) constitute an analog of the Friedmann solutions. We shall confine our attention to the closed case (24). The condition for synchronous motion of the layers is

$$k^2 = 1 - |A|CB^{-2} = \text{const}, \quad B|A|^{-3/2} = T = \text{const}. \tag{31}$$

The function χ is then common to all layers, and depends only on time. We then have

$$r/\tau=|A|^{1/2}f(\chi),\quad |A|=1-h^2=[r/R(\tau)]^2=x^2, \tag{32}$$

$$R(\tau)=T(1-k\cos\chi),\quad \tau=T(\chi-k\sin\chi). \tag{33}$$

The last equations describe the change in the scale with time. The quantity x is constant in a layer. Next,

$$B=Tx^3=M-hqQ=M-(1-x^2)\frac{d}{dM}\left(\frac{Q^2}{2}\right),$$

$$C=(1-k^2)T^2x^4=(1-q^2)Q^2=2\left(\frac{Q^2}{2}\right)-(1-x^2)\left[\frac{d}{dM}\left(\frac{Q^2}{2}\right)\right]^2. \tag{34}$$

The solution of these equations is

$$M/T=x^3+\varepsilon x(1-x^2),\quad (Q/T)^2=\varepsilon^2x^2(1-{}^3/_2x^2)+{}^3/_2\varepsilon x^4, \tag{35}$$

where ε determines k:

$$k^2=(1-\varepsilon)(1-\varepsilon/2),\quad 0<\varepsilon<1,\quad 2<\varepsilon<3. \tag{36}$$

The spatial part of the metric is determined by (26):

$$1-\frac{2B}{r}+\frac{C}{r^2}=1-2\frac{T}{R}x^2+x^2\left(\frac{T}{R}\right)^2\frac{\varepsilon}{2}(3-\varepsilon),$$

$$U=1-2\frac{T}{R}[\varepsilon+x^2(1-\varepsilon)]+\left(\frac{T}{R}\right)^2\varepsilon\left[\varepsilon+\frac{3}{2}x^2(1-\varepsilon)\right]. \tag{37}$$

For $x\to 1$, we have $b^2\to(1-x^2)^{-1}$, as for a sphere, but, for $x\to 0$, the spatial part of the metric has a singularity associated with the singular density of matter at this point. Since the metric involves only Q^2, the solution in the second "hemisphere" is analogous to that given above, except for the replacement $Q\to -Q$ which ensures that the electrostatic field varies correctly across the equator.

3. COLLAPSE OF CHARGED ROTATING DUST

The solutions for the spherically symmetric case are simple because electromagnetic and gravitational radiation are automatically absent, i.e., the dynamic variables of the system are due to the moving matter alone. However, motion without radiation is also possible in the axisymmetric case if the internal metric joins externally to the Kerr-Newman metric[6] which we shall write in the form

$$ds^2=\frac{\Delta}{\rho^2}\overline{dt}^2-\frac{\rho^2}{\Delta}dr^2-\rho^2(d\theta^2+\sin^2\theta\,\overline{d\varphi}^2),$$
$$\rho^2=r^2+a^2\cos^2\theta,\quad \Delta=r^2-2Mr+Q^2+a^2, \tag{38}$$

where $\overline{dt}$ and $\overline{d_\chi}$ are the differential 1-forms due to the increase dt, $d\phi$ in the coordinates, produced by the application of a unitary matrix:

$$\begin{pmatrix}\overline{dt}\\ \overline{d\varphi}\end{pmatrix}=\begin{pmatrix}1 & -a\sin^2\theta\\ -a/\rho^2 & (a^2+r^2)/\rho^2\end{pmatrix}\begin{pmatrix}dt\\ d\varphi\end{pmatrix}. \tag{39}$$

The corresponding coforms can be expressed in terms of the inverse matrix:

$$\left(\frac{\partial}{\overline{\partial t}},\frac{\partial}{\overline{\partial\varphi}}\right)=\left(\frac{\partial}{\partial t},\frac{\partial}{\partial\varphi}\right)\begin{pmatrix}(a^2+r^2)/\rho^2 & a\sin^2\theta\\ a/\rho^2 & 1\end{pmatrix}. \tag{40}$$

The electromagnetic potential has the single component $A_{\bar t}=Qr/\rho^2$. The equation of motion of the quasispherical charged layer will be determined from the Hamilton-Jacobi equation[6]

$$\frac{\partial\tau}{\partial t}=-h,\quad \frac{\partial\tau}{\partial\varphi}=l,$$

$$\Delta^{-1}[(a^2+r^2)h-qQr-al]^2-\Delta\left(\frac{\partial\tau}{\partial r}\right)^2-r^2$$
$$=\left(\frac{\partial\tau}{\partial\theta}\right)^2+a^2\cos^2\theta+\left(\frac{l}{\sin\theta}-ah\sin\theta\right)^2. \tag{41}$$

The presence of the term $l/\sin\theta$ for $l\neq 0$ ensures that the region is bounded in the angle θ, which prevents motion of the dust as one whole layer that is homeomorphic to the sphere with $l\neq 0$. It follows that $l=0$.

We shall confine our attention to the simplest case $h=1$, i.e., to a layer falling from infinity with zero kinetic energy. We then have $\partial\tau/\partial\theta=0$ for each particle in the layer, where θ is a constant of motion. Despite the fact that $l=0$, the layer rotates and contributes to the overall angular momentum of the system.

The change in the layer radius r in proper time is determined as for a test particle:

$$dr/d\tau=-u\rho^{-2},\quad u^2=2(M-qQ)r(r^2+a^2)-Q^2[r^2(1-q^2)+a^2]. \tag{42}$$

The dependence on θ shows that the layer is deformed as it falls. In the general case, the solution of (42) is expressed in terms of an elliptic integral.

Let us first investigate the motion of uncharged dust ($Q=0$, $q=0$):

$$d\tau=\frac{r^2+a^2\cos^2\theta}{[2Mr(r^2+a^2)]^{1/2}}dr, \tag{43}$$

$$(2M)^{1/2}(\tau-\tau_0)=\frac{2}{3}[r(r^2+a^2)]^{1/2}$$
$$+\left(\cos^2\theta-\frac{1}{3}\right)\int_0^{r/a}\frac{dx}{[x(x^2+1)]^{1/2}}. \tag{44}$$

The last integral is finite as $r/a\to\infty$ and determines the relative shifts in the layer. In each layer, we have $M=$ const, $\tau_0=$ const, and $a=$ const. A layer is best observed by choosing particles with $\cos^2\theta=1/3$. The solution for such particles takes the form of an elementary function:

$$9M(\tau-\tau_0)^2=2r(r^2+a^2). \tag{45}$$

Having specified the arbitrary functions $\tau_0(M)$, $a(M)$, we find $M(r,\tau)$, and this determines the solution.

We now take the simplest (synchronized) case $\tau_0=0$, $a=\alpha M$, $\alpha=$ const, $\alpha^2<1$; $M(r,\tau)$ is then determined from the quadratic equation

$$9M\tau^2=2r(r^2+\alpha^2M^2). \tag{46}$$

When $\tau=0$, all matter has $r=0$, but this manifold is not a point but a disk of radius a. Each layer is thus flattened into a pancake of its own radius. The density of matter becomes infinite, but the singularity is not a point singularity. For $\tau\neq 0$, the distribution of mass near the center ($r\ll|\tau|\alpha^{-1/2}$) is

$$M\approx\frac{2r^3}{9\tau^2}\left(1+\frac{4\alpha^2r^4}{81\tau^4}+\dots\right). \tag{47}$$

Near the center, the mass, and consequently, the rotation parameter a, are small, and we have uniform motion.

In the case of charged dust, the most characteristic feature is the stopping of each layer on a surface $r=r_0$ that is specific to that layer. From (42) we have

$$[2(M-qQ)]^{1/2}a^{-1/2}(\tau-\tau_0)=\int_{x_0}^{1/\varepsilon}[P(x)]^{-1/2}(x^2+\cos^2\theta)\,dx, \tag{48}$$

$$P(x)=x(x^2+1)-A[x^2(1-q^2)+1],\quad A=Q^2/2(M-qQ)a.$$

If we specify $x_0(M)$ (or $\varepsilon=1/x_0$) for given $Q(M)$, $q(M)=dQ/dM$, this determines $a(M)$:

$$a=\frac{\varepsilon Q^2(1-q^2+\varepsilon^2)}{2(M-qQ)(1+\varepsilon^2)}. \tag{49}$$

In particular, for $q=\text{const}$, $Q=qM$, $\varepsilon=1/x_0=\text{const}$, the quantity a is also a linear function of M and, when the layer stops, the mass is a linear function of the radial distance. At this time we have $\partial\tau/\partial r=0$ and, since we have $\partial\tau/\partial\phi=0$, $\partial\tau/\partial t=1$, the transformation matrix between the Kerr-Newman coordinates and the comoving coordinates is a unit matrix, so that the metric becomes identical with (38), but

$$\rho^2=r^2(1+\varepsilon^2\cos^2\theta),\quad \Delta=r^2\frac{(1+\varepsilon^2)(1-q^2+\varepsilon^2)^2}{(1-q^2-\varepsilon^2)^2}. \tag{50}$$

4. SPHERICALLY SYMMETRIC MOTION OF A PHOTON GAS AND OF A MIXTURE

Models that can be solved exactly include symmetric systems with the function f in (4) always equal to unity. For a Pascal fluid in which the number of particles is conserved, this corresponds to dust alone. The equation $f=1$ signifies that the chemical potential is equal to zero and, if pressure is present, it describes systems in which the number of particles is not conserved. In general, processes involving the creation and annihilation of particles in macroscopic motion give rise to pressure anisotropy.

In the case of spherical symmetry, the stress tensor then has two components:

$$T_0^{\ 0}=\varepsilon,\quad T_0^{\ 1}=a(\varepsilon+p),\quad T_1^{\ 1}=-p,\quad T_2^{\ 2}=T_3^{\ 3}=-p_\perp. \tag{51}$$

The relationship between them can be determined from the Hilbert-Einstein identity:

$$\nabla_i T_1^{\ i}=0,\quad \partial_1 p+(\varepsilon+p)(\Gamma_{01}^{\ \ 0}+a\Gamma_{11}^{\ \ 0})-2\Gamma_{12}^{\ \ 2}(p-p_\perp)=0. \tag{52}$$

Direct substitution yields

$$p_\perp=p+rp'/2=(pr^2)'/2r. \tag{53}$$

In particular, $p_\perp=0$ for

$$p=\text{const}\,r^{-2}. \tag{54}$$

For a complete thermodynamic description of the system, we must establish the relationship between ε and p. If we set the trace of the energy-momentum tensor equal to zero, this model will approximately represent the gravidynamics of a photon gas or a high-temperature "soup" of particles and antiparticles:

$$\varepsilon-p-2p_\perp=0,\quad \varepsilon=3p+rp'=(r^3p)'/r^2. \tag{55}$$

Actually, in the homogeneous case $p'=0$ holds, and (53) yields $p_\perp=p$, $\varepsilon=3p$. Conversely, is there is a radial flux of pure radiation then, as in the case of a plane wave, $\varepsilon=p$, $p_\perp=0$, and (54) signifies simply a radial expansion of this flux.

If we substitute (55) in (5), and assume the absence of the singularity at the center, we find that

$$p=(1+a^2-b^{-2})/(8\pi r^2) \tag{56}$$

Substituting this in Einstein's equations (6) and (7) with $f=1$, we obtain the set of equations

$$\dot b+ab'=0,\quad \dot a+aa'+(1+a^2-b^{-2})/r=0. \tag{57}$$

In the case of dust ($p=0, f=1$), this system becomes

$$\dot b+ab'=0,\quad \dot a+aa'+(1+a^2-b^{-2})/(2r)=0 \tag{58}$$

which differs only by the factor in the last term. Both sets of equations can be solved by the method of characteristics, and the solution of (58) is the Tolman solution.[5] The set of equations given by (57) reduces to the form

$$\dot b+ab'=0,\quad \dot w+aw'=0,\quad w=r^2(a^2-h),\quad h=b^{-2}-1. \tag{59}$$

The quantities b and w are constants on the characteristics and, once the relationship $F(h,w)=0$ has been specified for the layers, the trajectories determine $w(r,t)$ and $b(r,t)$:

$$\left(\frac{dr}{dt}\right)^2=h+\frac{w}{r^2},\quad \frac{h^2(t-t_0)^2}{w}-\frac{hr^2}{w}=1. \tag{60}$$

The trajectories are second-degree curves that do not extend over the entire time axis. To reconstruct the solution, we must specify one further arbitrary function: $F_1(h,w,t_0)=0$. In the homogeneous case ($t_0=0$, $h=0$), the solutions of (59) are the Friedmann solutions with the compressibility law $\varepsilon=3p$.

It is possible to construct a composite model consisting of a photon gas and dust in which radiation is at rest and interacts only through the gravitational field. The dust energy, $\varepsilon_n=M'/(4\pi r^2)$, must then be added to the radiation energy density, which corrects the expression for the pressure given by (56):

$$p=(a^2-h-2M/r)/(8\pi r^2),\quad h=b^{-2}-1. \tag{61}$$

The desired set of equations then follows from (6) and (7):

$$\dot h+ah'=0,\quad \dot M+aM'=0,\quad \dot w+aw'=0,$$
$$w=r^2(a^2-h)-2Mr\geqslant 0. \tag{62}$$

The general solution is determined by three arbitrary functions: $f_1(h, M, w, t_0)=0$ where t_0 is, as before, the layer asynchrony function in the solution that is formally identical with (22)–(24) if we substitute $A=-h$, $B=M$, and $c=-w$.

Special cases

1. $w=0$, Tolman solution for dust.
2. $M=0$, the solution for pure radiation.
3. Homogeneous models.

Despite the formal similarity with the solution for charged dust, the situation is now quite different and there are Friedmann-type homogeneous solutions. For synchronization, we must have

$$Mh^{-3/2}=T=\text{const},\quad k^2=1+hwM^{-2}=\text{const}. \tag{63}$$

It then follows from (24) that

$$r=h^{1/2}T(1-k\cos\chi)=h^{1/2}R(t), \tag{64}$$

where

$$R(t)=T(1-k\cos\chi),\quad t=T(\chi-k\sin\chi) \tag{65}$$

are the equations of scale dynamics and

$$h=[r/R(t)]^2=b^{-2}-1,\quad b^{-2}=1-r^2/R^2 \tag{66}$$

determines the metric of a sphere of radius $R(t)$ varying in accordance with (65) [compare with Ref. 7.] From (63) we have

$$M=T(r/R)^3, \quad w=(k^2-1)T^2(r/R)^4,$$
$$4\pi p=w/r^4=(k^2-1)T^2/R^4. \tag{67}$$

The pressure is uniform and varies as R^{-4}.

Solutions for open models are similarily obtained from (23).

4. Plane solutions: $h = 0$. From (22), we have

$$3M^2(t-t_0(M))=(2Mr+w)^{1/2}(Mr-w). \tag{68}$$

This class of solutions is determined by two functions, namely, $t_0(M)$ and $w(M)$. For example, the solutions are self-similar for $w(M) = \beta M^{\alpha}$, $t_0(M) = \gamma M^{(3a-4)/2}$:

$$y=M^{\alpha-1}/r, \quad x=tr^{(2-1,5\alpha)/(\alpha-1)},$$
$$3(y^{0,5/(\alpha-1)}x-\gamma y^{3/2})=(2+\beta y)^{1/2}(1-\beta y). \tag{69}$$

5. CONCLUSION

The utility of our solutions is not confined to the simulation of different astrophysical and cosmological situations. Solutions with charge are found to smear out the singularity when it collapses, and enable us to examine in greater detail the collapse problem. The end of compression always occurs in the region $g_{00} > 0$. After compression, the system begins to expand in the same space. If the specific energy of the particles is less than unity, the system undergoes repeated compression and expansion, covering the region $g_{00} < 1$ in finite proper-time intervals. Of course, an external observer cannot see even a single traversal of the gravitational radius.

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