Effects of finite correlation radius of dielectric-constant fluctuations for waves propagating in randomly-inhomogeneous media

V. S. Lutovinov and V. R. Chechetkin

Moscow Institute of Radio Engineering, Electronics, and Automation
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Effects of the influence of a finite longitudinal correlation radius of the dielectric-constant fluctuations on wave propagation in randomly inhomogeneous media are considered in the framework of the parabolic equation. The results are compared in detail in the Bourret and in the Markov approximations. It is shown that deviations from the Markov approximation lead to phase shifts due to fluctuations of the dielectric constant, and to a substantially stronger dependence of the effective attenuation on the transverse wave vectors. These effects can lead also to a peripheral redistribution of the intensity in axial beams and can alter the character of various asymptotic expressions.

1. INTRODUCTION

The problem of wave propagation in randomly inhomogeneous media can be generally formulated with the aid of the Helmholtz equation with a fluctuating dielectric constant

$$\Delta U+k^2(1+\varepsilon(\mathbf{r}))U=0, \tag{1.1}$$

where $k=\omega\langle\tilde\varepsilon\rangle^{1/2}/c$ is the wave vector corresponding to the average dielectric constant $\langle\tilde\varepsilon\rangle$, while $\varepsilon(\mathbf{r})=(\tilde\varepsilon(\mathbf{r})-\langle\tilde\varepsilon\rangle)/\langle\tilde\varepsilon\rangle$ describes the relative fluctuations. Equation (1.1) can be used to describe both electromagnetic and acoustic waves. Polarization effects in electromagnetic waves are neglected in the Helmholtz equation, but backward scattering is described in this approximation exactly. If the characteristics of the beam and of the medium vary slowly along the wave propagation path, Eq. (1.1) can be approximately replaced by the parabolic equation[1–3]

$$U(\rho,z)=u(\rho,z)\,e^{ikz}, \tag{1.2}$$

$$2ik\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial\rho^2}+k^2\varepsilon(\rho,z)u=0. \tag{1.3}$$

Here z is the coordinate along the propagation path, $\partial/\partial\rho$ is the gradient in the transverse direction, and $\partial^2/\partial\rho^2$ is the transverse Laplacian. Equation (1.3) neglects both polarization and backscattering effects and describes only the propagating wave. It is valid under the conditions[1–3]

$$kl_\varepsilon\gg 1,\quad (kl_\varepsilon)^3 l_\varepsilon\gg z,\quad 1\gg\pi^2k^2z\int_{2^{1/2}k}^{2k}\Phi_\varepsilon(\varkappa,0)\,\varkappa\,d\varkappa, \tag{1.4}$$

where l_ε is the characteristic spatial dimension of the fluctuations of ε, and $\Phi_\varepsilon(\varkappa,0)$ is the spectral density of the fluctuations.

Many results, formally exact as well as approximate, have been published by now both for Eq. (1.1) (see e.g., Ref. 1 and the detailed bibliography in that monograph) and for Eq. (1.3) (see, e.g., Refs. 3 and 4). On the one hand, however, the results for the Helmholtz equation must be substantially modified when applied to the propagation of axial wave beams. On the other, in specific applications of the parabolic equations the analysis is carried out only in the Markov approximation with zero correlation radius of the dielectric-constant fluctuations. In the present paper we wish to discuss the role played by a finite longitudinal correlation radius in the approximation of Eq. (3), and compare the corresponding results with the Markov limit.

The plan of the paper is the following. The general formalism is described in Sec. 2, in which equations are derived for the average correlators. The propagation of plane-wave packets is considered in Sec. 3. The fundamental role of non-Markov corrections is discussed in Sec. 4. The Appendix contains brief comments concerning the employed approximation.

2. EQUATIONS FOR THE AVERAGE CORRELATORS

We begin with a derivation of equations for the average correlators. In contrast to Refs. 3 and 4, where Fradkin's method is used, it is more convenient for our purposes to use the direct approach developed in Ref. 5. We write the formal solution of Eq. (1.3) in the form

$$u(\rho,z)=\int d^2\rho' K(\rho,z;\rho',0)\,u_0(\rho'), \tag{2.1}$$

where

$$u(\rho,z)|_{z=0}=u_0(\rho), \tag{2.2}$$

and $K(\rho,z;\rho',z')$ is the exact Green's function of the equation

$$\left[i\frac{\partial}{\partial z}+\frac{1}{2k}\frac{\partial^2}{\partial\rho^2}+\frac{k}{2}\varepsilon(\rho,z)\right]K(\rho,z;\rho',z')=\delta(\rho-\rho')\,\delta(z-z'). \tag{2.3}$$

At the same time, one can use for K a representation in the form of a path integral[5]:

$$K(\rho,z;\rho',0)=\int\mathscr{D}\rho(\zeta)\exp\left\{\frac{ik}{2}\left[\int_0^z d\zeta\left(\frac{\partial\rho(\zeta)}{\partial\zeta}\right)^2+\int_0^z d\zeta\,\varepsilon(\rho(\zeta),\zeta)\right]\right\}, \tag{2.4}$$

where the integration is over all the rays emerging from the

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point ρ' at $\zeta = 0$ and reaching the point ρ at $\zeta = z$. As seen from (2.1)–(2.4), in the case of fluctuating $\varepsilon(\rho,z)$ the problem reduces to averaging K, or more specifically, to averaging the factor

$$\exp\left[\frac{ik}{2}\int_0^z d\zeta\varepsilon(\rho(\zeta),\zeta)\right].$$

Comparing this exponential with the definition of the characteristic functional

$$Z_z\{\eta(\rho,\zeta)\} = \left\langle \exp\left[i\int_0^z d\zeta \int d^2\rho\eta(\rho,\zeta)\varepsilon(\rho,\zeta)\right]\right\rangle, \tag{2.5}$$

where $\eta(\rho,\zeta)$ is a specified determinate function, we find right away that the average

$$\left\langle \exp\left[\frac{ik}{2}\int_0^z d\zeta\varepsilon(\rho(\zeta),\zeta)\right]\right\rangle$$

corresponds to the choice

$$\eta(\rho,\zeta) = \frac{k}{2}\delta(\rho-\rho(\zeta)). \tag{2.6}$$

Using next the identity

$$\exp\left(\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\right)f(\rho) = \frac{k}{2iz}\int d^2\rho' \exp\left(\frac{ik|\rho-\rho'|^2}{2z}\right)f(\rho'), \tag{2.7}$$

it is easy to show that the final answer can be written in the form

$$\langle u(\rho,z)\rangle = \exp\left(\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\right)\hat{L}\left\langle \exp\left[\frac{ik}{2}\int_0^z dz'\hat{\varepsilon}(\rho,z')\right]\right\rangle u_0(\rho), \tag{2.8}$$

where

$$\hat{\varepsilon}(\rho,z) = \exp\left(-\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\right)\varepsilon(\rho,z)\exp\left(\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\right), \tag{2.9}$$

and the operator $\hat{L}$ is similar to the time-ordering operator (see, e.g., Refs. 6 and 7). For Gaussian fluctuations, in particular, we obtain from (2.8) (cf. Refs. 8 and 9)

$$\langle u(\rho,z)\rangle = \exp\left(\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\right) \times \hat{L}\exp\left[-\frac{k^2}{8}\int_0^z dz' \int_0^z dz''\langle\hat{\varepsilon}(\rho,z')\hat{\varepsilon}(\rho,z'')\rangle\right]u_0(\rho). \tag{2.10}$$

Since the operators $\hat{\varepsilon}(\rho,z)$ do not commute for different z, the expression for the derivative $\partial\langle u(\rho,z)\rangle/\partial z$ cannot be expressed in closed form. If, however, the inequality

$$kl_{\varepsilon\|}\int_0^\infty dz\, z\left\langle \frac{\partial\varepsilon(\rho,z)}{\partial\rho}\frac{\partial\varepsilon(\rho,0)}{\partial\rho}\right\rangle \ll 1 \tag{2.11}$$

holds, we get (cf. Refs. 8–11)

$$\hat{L}\exp\left[-\frac{k^2}{8}\int_0^z dz' \int_0^z dz''\langle\hat{\varepsilon}(\rho,z')\hat{\varepsilon}(\rho,z'')\rangle\right] \approx \exp\left[-\frac{k^2}{8}\hat{L}\int_0^z dz' \int_0^z dz''\langle\hat{\varepsilon}(\rho,z')\hat{\varepsilon}(\rho,z'')\rangle\right], \tag{2.12}$$

which is equivalent in fact to the Bourret approximation.[1)] In this case we obtain the following equation for the average field:

$$\frac{\partial}{\partial z}\langle u(\rho,z)\rangle = \frac{i}{2k}\frac{\partial^2}{\partial\rho^2}\langle u(\rho,z)\rangle - \frac{k^2}{4}\int_0^z dz'\left\langle \varepsilon(\rho,z) \times\exp\left[\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho^2}\right]\varepsilon(\rho,z') \times\exp\left[-\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho^2}\right]\right\rangle\langle u(\rho,z)\rangle. \tag{2.13}$$

We assume throughout that the inequality (2.11) holds and that the fluctuations are Gaussian.

The equations for the remaining field correlators are derived in exactly the same way, recognizing that a sum of Gaussian quantities has again a Gaussian distribution. For example, the equations for paired correlators take the form

$$\begin{aligned}
\frac{\partial}{\partial z}\langle u(\rho_1,z)u^*(\rho_1',z)\rangle
&= \frac{i}{2k}\left(\frac{\partial^2}{\partial\rho_1^2}-\frac{\partial^2}{\partial\rho_1'^2}\right)\langle u(\rho_1,z)u^*(\rho_1',z)\rangle \\
&-\frac{k^2}{4}\int_0^z dz'\left\{\left\langle \varepsilon(\rho_1,z)\exp\left[\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\right]\times\varepsilon(\rho_1,z')\exp\left[-\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\right]\right\rangle\right. \\
&+\left\langle \varepsilon(\rho_1',z)\exp\left[-\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\right]\times\varepsilon(\rho_1',z')\exp\left[\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\right]\right\rangle \\
&-\left\langle \varepsilon(\rho_1',z)\exp\left[\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\right]\times\varepsilon(\rho_1,z')\exp\left[-\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\right]\right\rangle \\
&\left.-\left\langle \varepsilon(\rho_1,z)\exp\left[-\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\right]\times\varepsilon(\rho_1',z')\exp\left[\frac{i(z-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\right]\right\rangle\right\} \\
&\times\langle u(\rho_1,z)u^*(\rho_1',z)\rangle,
\end{aligned} \tag{2.14}$$

$$\frac{\partial}{\partial z_1}\langle u(\rho_1,z_1)u^*(\rho_1',z_1')\rangle=\frac{i}{2k}\frac{\partial^2}{\partial\rho_1^2}\langle u(\rho_1,z_1)u^*(\rho_1',z_1')\rangle$$

$$-\frac{k^2}{4}\Big\{\int_0^{z_1}dz'\Big\langle\varepsilon(\rho_1,z_1)\exp\Big[\frac{i(z_1-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\Big]\times\varepsilon(\rho_1,z')\exp\Big[-\frac{i(z_1-z')}{2k}\frac{\partial^2}{\partial\rho_1^2}\Big]\Big\rangle$$

$$-\int_0^{z_1'}dz'\Big\langle\varepsilon(\rho_1,z_1)\exp\Big[-\frac{i(z_1'-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\Big]\times\varepsilon(\rho_1',z')\exp\Big[\frac{i(z_1'-z')}{2k}\frac{\partial^2}{\partial\rho_1'^2}\Big]\Big\rangle\Big\}$$

$$\times\langle u(\rho_1,z_1)u^*(\rho_1',z_1')\rangle. \qquad (2.15)$$

The intensity of the fluctuations is usually described with the aid of the spectral density

$$\langle\varepsilon(\rho_1,z_1)\varepsilon(\rho_2,z_2)\rangle=\int d^3\mathbf{q}e^{i\mathbf{q}(\mathbf{r}_1-\mathbf{r}_2)}\Phi_\varepsilon(\mathbf{q}_\perp,q_z). \qquad (2.16)$$

Latin letters will hereafter be used for three-dimensional vectors, and Greek for transverse two-dimensional vectors. It will also be assumed that

$$\Phi_\varepsilon(\mathbf{q}_\perp,q_z)=\Phi_\varepsilon(-\mathbf{q}_\perp,q_z)=\Phi_\varepsilon(\mathbf{q}_\perp,-q_z). \qquad (2.17)$$

To calculate the correlators in Eqs. (2.13)–(2.15) it is convenient to introduce formally

$$\varepsilon(\mathbf{r})=\sum_{\mathbf{q}}\varepsilon(\mathbf{q})e^{i\mathbf{q}\mathbf{r}}, \qquad (2.18)$$

$$\langle\varepsilon(\mathbf{q})\varepsilon(\mathbf{q}')\rangle=\delta_{-\mathbf{q}',\mathbf{q}}\Phi_\varepsilon(\mathbf{q}_\perp,q_z), \qquad (2.19)$$

where $\delta_{-\mathbf{q}',\mathbf{q}}$ is the Kronecker delta. Next, using the equation

$$\exp\Big(\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\Big)\exp(i\mathbf{q}\mathbf{r})\exp\Big(-\frac{iz}{2k}\frac{\partial^2}{\partial\rho^2}\Big)=\exp(i\mathbf{q}\mathbf{r})\exp\Big[\frac{z}{2k}\Big(-i\mathbf{q}_\perp^2-2\mathbf{q}_\perp\frac{\partial}{\partial\rho}\Big)\Big], \qquad (2.20)$$

we can rewrite (2.13) and (2.14) in the form

$$\frac{\partial\Gamma_{1,0}}{\partial z}=\frac{i}{2k}\frac{\partial^2\Gamma_{1,0}}{\partial\rho^2}-\frac{k^2}{4}\int_0^z dz'\int d^3\mathbf{q}\Phi_\varepsilon(\mathbf{q}_\perp,q_z)\times\exp[-iq_z(z-z')]\exp\Big[\frac{(z-z')}{2k}\Big(-i\mathbf{q}_\perp^2-2\mathbf{q}_\perp\frac{\partial}{\partial\rho}\Big)\Big]\Gamma_{1,0}, \qquad (2.21)$$

$$\frac{\partial\Gamma_{1,1}}{\partial z}=\frac{i}{2k}\Delta_{1,1}\Gamma_{1,1}-\frac{k^2}{4}\int_0^z dz'\int d^3\mathbf{q}\Phi_\varepsilon(\mathbf{q}_\perp,q_z)\times\exp[-iq_z(z-z')]\Big\{[1-\exp(-i\mathbf{q}_\perp(\rho_1'-\rho_1))]\times\exp\Big[\frac{(z-z')}{2k}\Big(-i\mathbf{q}_\perp^2-2\mathbf{q}_\perp\frac{\partial}{\partial\rho_1}\Big)\Big]$$
$$+[1-\exp(-i\mathbf{q}_\perp(\rho_1-\rho_1'))]\times\exp\Big[-\frac{(z-z')}{2k}\Big(-i\mathbf{q}_\perp^2-2\mathbf{q}_\perp\frac{\partial}{\partial\rho_1'}\Big)\Big]\Big\}\Gamma_{1,1}, \qquad (2.22)$$

where

$$\Gamma_{n,m}=\langle u(\rho_1,z)\dots u(\rho_n,z)u^*(\rho_1',z)\dots u^*(\rho_m',z)\rangle, \qquad (2.23)$$

$$\Delta_{n,m}=\frac{\partial^2}{\partial\rho_1^2}+\dots+\frac{\partial^2}{\partial\rho_n^2}-\frac{\partial^2}{\partial\rho_1'^2}-\dots-\frac{\partial^2}{\partial\rho_m'^2}. \qquad (2.24)$$

Further transformations can be carried out with the aid of the identity

$$\exp[\alpha(\partial/\partial\rho)]f(\rho)=f(\rho+\alpha), \qquad (2.25)$$

but Eqs. (2.21) and (2.22) are in fact more convenient for actual calculations.

The integrals with respect to z' converge mainly in the vicinity of z over distances of the order of $l_{\varepsilon\|}$ (where $l_{\varepsilon\|}$ is the longitudinal correlation radius). Therefore slow variations of Φ_ε with height[1,2] can be taken at the point z in investigations of wave propagation over long distances.

3. PROPAGATION OF PACKETS OF PLANE WAVES

1. Equation (2.21) can be solved in elementary fashion using a Fourier transform with respect to the transverse coordinates. We describe only the asymptotic solution for $z\gg l_{\varepsilon\|}$, when the upper limit of integration with respect to z' can be approximately replaced by infinity. Using the equation[1,6,7]

$$\int_0^\infty dz'\exp(-iq_zz')=-\hat{P}\frac{i}{q_z}+\pi\delta(q_z), \qquad (3.1)$$

where $\hat{P}$ means taking the principal value in the integration over q_z, we obtain the final answer in the form

$$\langle u(\rho,z)\rangle=\int\frac{d^2\varkappa}{(2\pi)^2}\langle u(\varkappa,0)\rangle\exp[i\varkappa\rho+i\delta k_z(\varkappa)z-Q_r(\varkappa)z], \qquad (3.2)$$

$$\langle u(\varkappa,0)\rangle=\int d^2\rho\exp(-i\varkappa\rho)\langle u(\rho,0)\rangle, \qquad (3.3)$$

$$\delta k_z(\varkappa)=-\varkappa^2/2k+Q_i(\varkappa), \qquad (3.4)$$

$$Q_i(\varkappa)=\frac{k^2}{4}\int d^2\mathbf{q}_\perp⨍dq_z\frac{\Phi_\varepsilon(\mathbf{q}_\perp,q_z)}{q_z+\mathbf{q}_\perp^2/2k+\mathbf{q}_\perp\varkappa/k}, \qquad (3.5)$$

$$Q_r(\varkappa)=\frac{\pi k^2}{4}\int d^3\mathbf{q}\delta(q_z+\mathbf{q}_\perp^2/2k+\mathbf{q}_\perp\varkappa/k)\Phi_\varepsilon(\mathbf{q}_\perp,q_z). \qquad (3.6)$$

In actual calculations with $\Phi_\varepsilon(\mathbf{q}_\perp,q_z)$ it is customary to use the Booker-Gordon approximations and a Gaussian or Karman's approximation of the Kolmogorov-Obukhov turbulent spectrum.[1-3] The corresponding dependences of $Q_i(\varkappa)$ and $Q_r(\varkappa)$ on the modulus $|\varkappa|$ of the transverse wave vector for these typical cases are shown in Fig. 1. New additional features compared with the Markov theory are the appearance of a finite phase shift as a result of the fluctuations of ε [Eq. (3.5)] and the dependence of the effective attenuation on $\varkappa$ [Eq. (3.6)]. The Markov limit is formally taken by replacing $q_z+\mathbf{q}_\perp^2/2k+\mathbf{q}_\perp\varkappa/k$ by q_z in Eqs. (3.5) and (3.6). Taking relations (2.17) into account and assuming axial symmetry, we obtain, retaining the first nonvanishing corrections with respect to the Markov approximation,

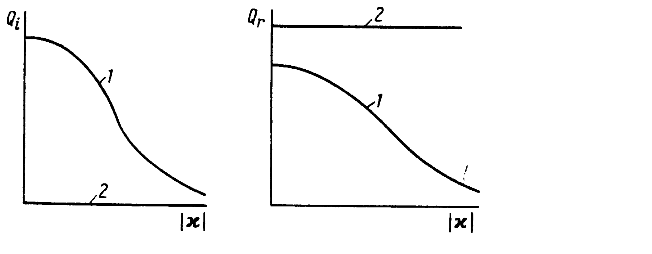

FIG. 1. Typical plots of the functions $Q_i(\varkappa)$ and $Q_r(\varkappa)$ in Eqs. (3.5) and (3.6) vs the transverse wave vector $|\varkappa|$: 1—non-Markov theory with finite longitudinal radius of fluctuations of ε, 2— Markov theory.

$$Q_i(\varkappa) = -\frac{k}{8}\int d^2\mathbf{q}_\perp \int dq_z \frac{q_\perp^2}{q_z}\frac{\partial \Phi_\varepsilon(\mathbf{q}_\perp, q_z)}{\partial q_z}, \tag{3.7}$$

$$Q_r(\varkappa) = \frac{\pi k^2}{4}\int d^2\mathbf{q}_\perp \left[\Phi_\varepsilon(\mathbf{q}_\perp, 0) + \frac{1}{2}\left(\frac{q_\perp^2}{2k}\right)^2 \frac{\partial^2\Phi_\varepsilon(\mathbf{q}_\perp, 0)}{\partial q_z^2}\right] + \frac{\pi\varkappa^2}{16}\int d^2\mathbf{q}_\perp q_\perp^2 \frac{\partial^2\Phi_\varepsilon(\mathbf{q}_\perp, 0)}{\partial q_z^2}. \tag{3.8}$$

We have used here the relation

$$\int_{-\infty}^{\infty} dq_z \frac{f(q_z)}{q_z+\alpha} = \int_{-\infty}^{\infty} dq_z \frac{f(q_z-\alpha)}{q_z} \tag{3.9}$$

and put

$$\frac{\partial^2\Phi_\varepsilon(\mathbf{q}_\perp, 0)}{\partial q_z^2} = \left.\frac{\partial^2\Phi_\varepsilon(\mathbf{q}_\perp, q_z)}{\partial q_z^2}\right|_{q_z=0}. \tag{3.10}$$

In the case of spherical symmetry, Eq. (3.7) takes the form

$$Q_i(\varkappa) = -\frac{k}{12}\int d^3\mathbf{q}\, q \frac{\partial\Phi_\varepsilon(q)}{\partial q}. \tag{3.11}$$

Since the following inequalities usually hold

$$\partial\Phi_\varepsilon(q)/\partial q < 0,\quad \partial^2\Phi_\varepsilon(\mathbf{q}_\perp, 0)/\partial q_z^2 < 0,$$

we reach several important conclusions. First, fluctuations of the dielectric constant shift the longitudinal wave vectors into the short-wave region (cf. Ref. 1, p. 408). Second, shorter waves are effectively less attenuated than longer ones. The last circumstance is physically quite obvious, since shorter waves are more weakly scattered than long ones. This effect can lead to a peripheral redistribution of the intensity in axial beams with abrupt boundaries (see Fig. 2), since the central region "consists" of harmonics with small transverse wave vectors, while waves with short harmonics are grouped near the beam boundary. In the more general case one could expect the onset of an annular structure in the cross section of the beam at large propagation distances. Estimates of the distance z at which the redistribution effects assume an important role will be given in the next section. In addition, this effect (together with the influence of self-focusing) is apparently partially responsible for formation of a peculiar filamentary structure of the wave beam at large propagation distances.

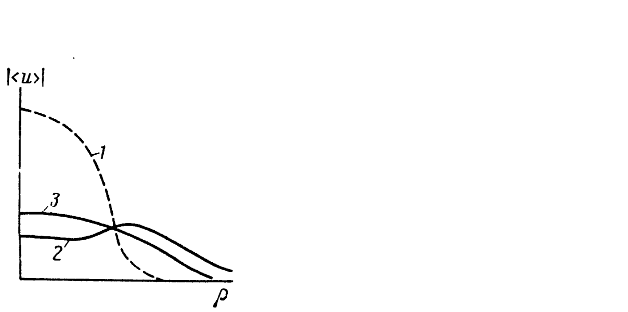

FIG. 2. Modulus $|\langle u\rangle|$ of mean-field amplitude vs the radius and the transverse viscosity: 1—distribution at $z = 0$; 2—distribution at large distances (non-Markov theory); 3—distribution at large distances (Markov theory).

We present now some specific estimates, using the Gaussian approximation for the spectral density of the fluctuations:

$$\Phi_\varepsilon(q) = \frac{\sigma_\varepsilon^2 l^3}{(2\pi)^{3/2}}\exp\left(-\frac{q^2 l^2}{2}\right), \tag{3.12}$$

where σ_ε^2 is the quadratic variance of the fluctuations of ε, and l is the characteristic length, and also the Karman approximation for the Komogorov-Obukhov spectrum:

$$\Phi_\varepsilon(q) = 0.033 C_\varepsilon^2 (q^2+q_0^2)^{-11/6}\exp(-q^2/q_m^2), \tag{3.13}$$

where q_0 and q_m represent the external and internal turbulence scales, while C_ε is a structure constant. Substitution of these spectral densities yields:

$$-\frac{1}{12}\int d^3\mathbf{q}\, q \frac{\partial\Phi_\varepsilon(q)}{\partial q} = \begin{cases} {}^1/_4\sigma_\varepsilon^2, \\ 0.19 C_\varepsilon^2 q_0^{-2/3}, \end{cases} \tag{3.14}$$

$$-\frac{\pi}{16}\int d^2\mathbf{q}_\perp q_\perp^2 \frac{\partial^2\Phi_\varepsilon(\mathbf{q}_\perp, 0)}{\partial q_z^2} = \begin{cases} \pi^{1/2}\sigma_\varepsilon^2 l/2^{5/2} \\ 0.09 C_\varepsilon^2 q_0^{-5/3}. \end{cases} \tag{3.15}$$

For atmospheric fluctuations of ε we have[1-3] $C_\varepsilon^2 \sim 10^{-14}$ $m^{-3/2}$ (in the radio and optical bands) and $q_0 \sim 1\ \mathrm{m}^{-1}$; for $k \sim 10^7\ \mathrm{m}^{-1}$ and $\varkappa \sim 0.1k$ we obtain $Q_i \sim 10^{-8}\ \mathrm{m}^{-1}$ and $\Delta Q_r \sim 10^{-3}\ \mathrm{m}^{-1}$ (where ΔQ_r corresponds to the $\varkappa$-dependent part of Q_r).

Substitution of the spectrum (3.13) in the criterion (3.11) leads to the condition

$$10^{-2}(k/q_0)C_\varepsilon^2 q_0^{-5/3} \ll 1,$$

which is satisfied with high accuracy in both the optical and in the radio bands.

2. The solution of Eq. (2.22) is much more complicated. We describe here only the propagation of plane-wave packets in the form

$$\Gamma_{1,1}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1'; z) = \int d^2\varkappa \exp[i\varkappa(\boldsymbol{\rho}_1+\boldsymbol{\rho}_1')] I_\varkappa(z), \tag{3.16}$$

where

$$I_\varkappa(z) = I_{-\varkappa}(z). \tag{3.17}$$

Substitution of this function in (2.22) yields an equation

$$\frac{\partial I_\varkappa(z)}{\partial z} = \frac{\pi k^2}{4}\int_{-\infty}^{\infty} dq_z \int d^2\mathbf{q}_\perp \int d^2\varkappa' [\delta(\mathbf{q}_\perp+\varkappa-\varkappa') \times\delta(q_z+\varkappa^2/2k-\varkappa'^2/2k) \times\Phi_\varepsilon(\mathbf{q}_\perp, q_z)(I_{\varkappa'}(z)-I_\varkappa(z)) +\delta(\mathbf{q}_\perp+\varkappa'-\varkappa)\delta(q_z+\varkappa'^2/2k-\varkappa^2/2k) \times\Phi_\varepsilon(\mathbf{q}_\perp, q_z)(I_{\varkappa'}(z)-I_\varkappa(z))], \tag{3.18}$$

that coincides formally with classical kinetic equations and describes radiation transport (cf. Refs. 1–4 and 13–15). Equations of type (3.18) were studied earlier in weak-turbulence theory (see, e.g., the review in Ref. 16). In view of this analogy it is easily seen that in the Markov limit ($q_z \pm \varkappa^2/2k \mp \varkappa'^2/2k \approx q_z$) there exists a nontrivial power-law solution of Eq. (3.18) for an inertial interval of the Kolmogorov-Obukhov spectrum:

$$\Phi_\varepsilon(q) \infty q^{-11/3}, \quad q_0 \ll q \ll q_m. \tag{3.19}$$

We demonstrate this formally using the conformal-mapping method of Katz and Kontorovich.[17] We redesignate identically the integration variables in the second term of (3.18) as $\tilde{\mathbf{q}}_\perp$ and $\tilde{\boldsymbol{\varkappa}}'$ (it should be noted that the two terms are equal, and we distinguish between them formally only for convenience) and make the change of variables

$$\tilde{\boldsymbol{\varkappa}}' = \left(\frac{\varkappa}{\varkappa'}\hat{R}\right)^2 \boldsymbol{\varkappa}', \quad \tilde{\mathbf{q}}_\perp = \left(\frac{\varkappa}{\varkappa'}\hat{R}\right)\mathbf{q}_\perp, \tag{3.20}$$

where the operation $(\varkappa\hat{R}/\varkappa')$ is defined by the condition

$$\frac{\varkappa}{\varkappa'}\hat{R}\boldsymbol{\varkappa}' = \boldsymbol{\varkappa}, \tag{3.21}$$

i.e., it corresponds to rotation of the vector $\boldsymbol{\varkappa}'$ to $\boldsymbol{\varkappa}$ and to a change of its length by a factor $\varkappa/\varkappa'$. We seek the solution in power-law form

$$I_\varkappa \infty |\boldsymbol{\varkappa}|^\alpha. \tag{3.22}$$

We obtain then after the transformations (3.22) the following expression for the term in the right-hand side of (3.18):

$$\frac{\pi k^2}{4}\int d^2\mathbf{q}_\perp \int d^2\boldsymbol{\varkappa}'\, \delta(\mathbf{q}_\perp + \boldsymbol{\varkappa} - \boldsymbol{\varkappa}')\Phi_\varepsilon(\mathbf{q}_\perp, 0)$$
$$\times (I_{\varkappa'} - I_\varkappa)(1 - (\varkappa/\varkappa')^{\alpha + 1/3}). \tag{3.23}$$

Consequently, the distribution

$$I_\varkappa \infty \varkappa^{-1/3} \tag{3.24}$$

is a solution, which does not depend on z, of (3.18). It is easy to verify that this solution is local and is therefore physically admissible (note that the influence of e boundaries in momentum space must be taken into account even for local distributions[18]). The distribution (3.24) corresponds to constant transfer of intensity in the transverse direction as a result of wave scattering by Kolmogorov vortices. The transfer is directed from the shorter to the longer waves (in a direction opposite to the energy transfer in the Kolmogorov turbulence theory).

Note that in the Markov limit Eq. (3.18) can be written in coordinate space also in the form

$$\frac{\partial}{\partial z}\Gamma_{1,1}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_1', z) = -\frac{\pi k^2}{4}H(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_1')\Gamma_{1,1}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_1', z), \tag{3.25}$$

where $H(\boldsymbol{\rho} - \boldsymbol{\rho}_1')$ is defined in Eq. (4.10) below. It is obvious therefore that for the regularized spectrum (3.13) the right-hand side of (3.25) cannot vanish exactly. The distribution (3.24) must therefore be understood as a spectral density that minimizes the Fourier transform of the right-hand side of (3.25) in the case when the inertial spectra interval predominates. Although the deviation from a purely power-law spectrum (3.19) [cf. (3.13)] can play an important role, it is well known[16] that distributions of the type (3.24) can be really observed asymptotically within the limits of a certain spectral interval.

4. DIFFERENTIAL EQUATIONS FOR THE AVERAGE CORRELATORS WITH ALLOWANCE FOR THE FIRST NONVANISHING CORRECTIONS FOR THE LONGITUDINAL CORRELATION RADIUS

1. We discuss in this section the principal influence exerted on the wave propagation by finite corrections with respect to the longitudinal correlation radius of the fluctuations. The corresponding differential equation for the mean field is of the form

$$\frac{\partial\langle u\rangle}{\partial z} = \frac{i}{2k}\frac{\partial^2\langle u\rangle}{\partial\rho^2} - \frac{k^2\tilde{A}_0}{8}\langle u\rangle + iQ_i(0)\langle u\rangle - D\frac{\partial^2\langle u\rangle}{\partial\rho^2}, \tag{4.1}$$

where

$$\tfrac{1}{8}k^2\tilde{A}_0 = Q_r(0), \tag{4.2}$$

while $Q_r(0)$, $Q_i(0)$, and D are defined respectively by Eqs. (3.8), (3.11), and (3.15). Since usually $D > 0$, the term with real diffusion leads to a radial redistribution of the intensity $|\langle u\rangle|^2$ over sufficiently long propagation distances. This effect is due to the weaker effective attenuation of the shorter waves (see Sec. 3.1). We estimate the characteristic length z over which the redistribution effects become important, using a Gaussian beam as an example.

Let the initial distribution in the $z = 0$ plane be

$$\langle u_0(\boldsymbol{\rho})\rangle = u_0 \exp[-(1/a_0^2 + ik/2R_0)\rho^2] \equiv u_0 \exp(-\tfrac{1}{2}k\alpha\rho^2), \tag{4.3}$$

$$\alpha = \alpha_r + i\alpha_i. \tag{4.4}$$

Here a_0 is the width of the beam and R_0 the curvature radius of the front (for wide beams this corresponds to focusing of the beam in the plane ($z = R_0$). Solution of Eqs. (4.1)–(4.4) leads to the form

$$\langle u(\boldsymbol{\rho}, z)\rangle = \frac{i\tilde{k}u_0}{(i\tilde{k} - \alpha kz)} \times \exp\left[-\frac{i\rho^2\alpha k\tilde{k}}{2(i\tilde{k} - \alpha kz)} + iQ_i(0)z - \frac{k^2\tilde{A}_0}{8}z\right], \tag{4.5}$$

where

$$\tilde{k} = k/(1 + 2iDk). \tag{4.6}$$

Determining $|\langle u\rangle|$ from (4.5), we easily see that the redistribution effects become important starting with $z > z^*$, where

$$z^* = \alpha_r/2Dk(\alpha_r^2 + \alpha_i^2). \tag{4.7}$$

The approximation (4.1) no longer holds at $z > z^*$ and it is necessary to retain in the non-Markov corrections additional terms of the expansion. At $\alpha_i \gg \alpha_r$ we get $z^* \sim R_0^2/Dk^2a_0^2$. For atmospheric turbulence (in the optical and radio bands) $D \sim 10^{-15}$ m [see Eq. (3.15)], which yields in the optical band at $k \sim 10^7$ m^{-1} an estimate $z^* \sim 10(R_0^2/a_0^2)$ m. In the case of accurately collimated beams, when $\alpha_i = 0$, we get $z^* \sim a_0^2/D$. The effect will therefore be much more strongly pronounced for focused beams than for collimated ones.

The corresponding differential equation for the coherence function takes the form

$$\frac{\partial \Gamma_{1,1}}{\partial z}=\frac{i}{2k}\Delta_{1,1}\Gamma_{1,1}-\frac{\pi k^2}{4}\tilde{H}(\rho_1-\rho_1')\Gamma_{1,1}$$
$$-\frac{ik}{8}\left(\frac{\partial}{\partial \rho_1}-\frac{\partial}{\partial \rho_1'}\right)F(\rho_1-\rho_1')\left(\frac{\partial}{\partial \rho_1}+\frac{\partial}{\partial \rho_1'}\right)\Gamma_{1,1}$$
$$-\frac{\pi}{16}\left(\frac{\partial}{\partial \rho_1}-\frac{\partial}{\partial \rho_1'}\right)G(\rho_1-\rho_1')\left(\frac{\partial}{\partial \rho_1}-\frac{\partial}{\partial \rho_1'}\right)\Gamma_{1,1}$$
$$-\frac{\pi}{8}\left(D_{\alpha\beta}(0)-D_{\alpha\beta}(\rho_1-\rho_1')\right)\left(\frac{\partial^2}{\partial \rho_{1\alpha}\partial \rho_{1\beta}}+\frac{\partial^2}{\partial \rho_{1\alpha}'\partial \rho_{1\beta}'}\right)\Gamma_{1,1}, \tag{4.8}$$

where

$$\tilde{H}(\rho_1-\rho_1')=H(\rho_1-\rho_1')+\Delta H(\rho_1-\rho_1'), \tag{4.9}$$

$$H(\rho_1-\rho_1')=2\int d^2q_\perp\, \Phi_\varepsilon(q_\perp,0)\,[1-\cos(q_\perp(\rho_1-\rho_1'))], \tag{4.10}$$

$$\Delta H(\rho_1-\rho_1')=\int d^2q_\perp\left(\frac{q_\perp^2}{2k}\right)^2\frac{\partial^2\Phi_\varepsilon(q_\perp,0)}{\partial q_z^2}\,[1-\cos(q_\perp(\rho_1-\rho_1'))], \tag{4.11}$$

$$F(\rho_1-\rho_1')=\int d^2q_\perp\int dq_z\frac{1}{q_z}\frac{\partial\Phi_\varepsilon(q_\perp,q_z)}{\partial q_z}\exp[iq_\perp(\rho_1-\rho_1')], \tag{4.12}$$

$$D_{\alpha\beta}(\rho_1-\rho_1')=\frac{\partial^2}{\partial\rho_{1\alpha}\partial\rho_{1\beta}}\int d^2q_\perp\frac{\partial^2\Phi_\varepsilon(q_\perp,0)}{\partial q_z^2}\exp[iq_\perp(\rho_1-\rho_1')], \tag{4.13}$$

$$G(\rho_1-\rho_1')=-\mathrm{Tr}\,D_{\alpha\beta}(\rho_1-\rho_1')=-D_{\alpha\alpha}(\rho_1-\rho_1'). \tag{4.14}$$

Summation is carried out over repeated Greek indices ($\alpha=1$ corresponds to the coordinate x and $\alpha=2$ to the coordinate y). Also assumed are axial symmetry of the fluctuations and satisfaction of the condition $z\gg l_{\varepsilon\parallel}$. In the Markov theory, ΔH, F, G, and $D_{\alpha\beta}$ are equal to zero.

We use Eqs. (4.8)–(4.14) to determine the evolution of the beam width in the case of prolonged averaging. The corresponding width is determined from Ref. 2

$$\langle\rho_L^2\rangle=\int d^2\rho\,\Gamma_{1,1}(\rho,\rho;z)\rho^2, \tag{4.15}$$

where the following normalization is assumed:

$$\int d^2\rho\,\Gamma_{1,1}(\rho,\rho;z)=1.$$

All the mean values are determined next by the equation

$$\langle\hat{O}\rangle=\int d^2\rho\langle u^*(\rho,z)\hat{O}u(\rho,z)\rangle, \tag{4.16}$$

where $\hat{O}$ is in the general case a certain operator.

It is convenient to introduce the operator

$$\hat{\theta}=-\frac{i}{k}\frac{\partial}{\partial\rho}, \tag{4.17}$$

which can be approximately interpreted as the operator of the arrival angles. In fact, in the case of a plane wave, $u\propto\exp(i\varkappa\cdot\rho)$, we get $\hat{\theta}=\varkappa/k$. For Gaussian beams

$$\Gamma_{1,1}(\rho_1,\rho_1';0)=\frac{k\alpha_r}{\pi}\exp\left[-\frac{1}{2}(k\alpha)\rho_1^2-\frac{1}{2}(k\alpha^*)\rho_1'^2\right], \tag{4.18}$$

however, calculation yields

$$\langle\hat{\theta}^2\rangle=(\alpha_r^2+\alpha_i^2)/k\alpha_r, \tag{4.19}$$

so that at $\alpha_i\gg\alpha_r$ we have $\langle\hat{\theta}^2\rangle\sim a_0^2/2R_0^2$.

With the aid of the operator θ we can write the equation for $\langle\rho_L^2\rangle$ in the form

$$\frac{\partial\langle\rho_L^2\rangle}{\partial z}=\langle\hat{\theta}\rho_L+\rho_L\hat{\theta}\rangle, \tag{4.20}$$

$$\frac{\partial}{\partial z}\langle\hat{\theta}\rho_L+\rho_L\hat{\theta}\rangle=2\langle\hat{\theta}^2\rangle+\frac{1}{2}f-\frac{\pi}{8}g\langle\hat{\theta}\rho_L+\rho_L\hat{\theta}\rangle, \tag{4.21}$$

$$\frac{\partial\langle\hat{\theta}^2\rangle}{\partial z}=-\frac{3\pi}{8}g\langle\hat{\theta}^2\rangle+\frac{\pi}{4}\tilde{h}, \tag{4.22}$$

where

$$f=\frac{\partial^2}{\partial\rho_1^2}F(\rho_1-\rho_1')\,|_{\rho_1=\rho_1'},\quad g=\frac{\partial^2}{\partial\rho_1^2}G(\rho_1-\rho_1')\,|_{\rho_1=\rho_1'}, \tag{4.23}$$

$$\tilde{h}=\frac{\partial^2}{\partial\rho_1^2}\tilde{H}(\rho_1-\rho_1')\,|_{\rho_1=\rho_1'}. \tag{4.24}$$

In typical situations, all the constants f, g, and $\tilde{h}$ are positive. From (4.9)–(4.14) we find that the non-Markov corrections can change the character of the asymptotic relations, since we have $z\gg g^{-1}$

$$\langle\rho_L^2\rangle=\frac{8}{\pi g}\left(\frac{1}{2}f+\frac{4}{3}\frac{\tilde{h}}{g}\right)z, \tag{4.25}$$

whereas the Markov theory yields

$$\langle\rho_L^2\rangle=\frac{\pi}{12}hz^3. \tag{4.26}$$

Such regimes, however will never be reached in atmospheric turbulence, for under these conditions $g\sim0$, $1C_\varepsilon^2q_m^{1/3}$, which yields $g\sim10^{-14}$ m^{-1} for $C_\varepsilon^2\sim10^{-14}$ m$^{-2/3}$ and $q_m\sim10^4$ m^{-1}. For media with higher optical density, however, these deviations can in principle be appreciable. For acoustic waves we have $C_\varepsilon^2\sim10^{-6}$ m$^{-3/2}$, so that $g\sim10^{-6}$ m^{-1}. Since $g\sim h$, the influence of the non-Markov corrections can alter slightly the statistics of the arrival angles at $z\ll g^{-1}$, when

$$\langle\hat{\theta}^2\rangle=\langle\hat{\theta}^2\rangle_0\left(1-\frac{3\pi}{8}gz\right)+\frac{\pi}{4}\tilde{h}z, \tag{4.27}$$

where $\langle\hat{\theta}^2\rangle_0$ corresponds to the value of $\langle\hat{\theta}^2\rangle$ at $z=0$. If $\langle\hat{\theta}^2\rangle_0$ is not too small, the non-Markov corrections alter also the constant in the asymptotic relation (4.26) at $z\ll g^{-1}$. Another interesting application of Eq. (4.8) is to effects of redistribution of the intensity $\langle|u|^2\rangle$ along the propagation path.

3. We present here also, to complete the picture, the corresponding equation for the correlator $\Gamma_{2,2}$, in view of its importance for effects connected with intensity fluctuations:

$$\frac{\partial\Gamma_{2,2}}{\partial z}=\frac{i}{2k}\Delta_{2,2}\Gamma_{2,2}-\frac{\pi k^2}{4}\Big[\sum_{m=1}^{2}\sum_{n=1}^{2}\tilde H(\rho_m-\rho_n')$$

$$-\tilde H(\rho_1-\rho_2)-\tilde H(\rho_1'-\rho_2')\Big]\Gamma_{2,2}$$

$$-\frac{ik}{8}\Big[\sum_{m=1}^{2}\sum_{n=1}^{2}\Big(\frac{\partial}{\partial\rho_m}-\frac{\partial}{\partial\rho_n'}\Big)F(\rho_m-\rho_n')\cdot$$

$$\times\Big(\frac{\partial}{\partial\rho_m}+\frac{\partial}{\partial\rho_n'}\Big)-\Big(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big)F(\rho_1-\rho_2)\Big(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big)$$

$$+\Big(\frac{\partial}{\partial\rho_1'}-\frac{\partial}{\partial\rho_2'}\Big)F(\rho_1'-\rho_2')\Big(\frac{\partial}{\partial\rho_1'}-\frac{\partial}{\partial\rho_2'}\Big)\Big]\Gamma_{2,2}$$

$$-\frac{\pi}{16}\Big[\sum_{m=1}^{2}\sum_{n=1}^{2}\Big(\frac{\partial}{\partial\rho_m}-\frac{\partial}{\partial\rho_n'}\Big)G(\rho_m-\rho_n')\Big(\frac{\partial}{\partial\rho_m}-\frac{\partial}{\partial\rho_n'}\Big)$$

$$-\Big(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big)G(\rho_1-\rho_2)\Big(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big)$$

$$-\Big(\frac{\partial}{\partial\rho_1'}-\frac{\partial}{\partial\rho_2'}\Big)G(\rho_1'-\rho_2')\Big(\frac{\partial}{\partial\rho_1'}-\frac{\partial}{\partial\rho_2'}\Big)\Big]\Gamma_{2,2}$$

$$-\frac{\pi}{8}\sum_{m=1}^{2}\tilde D_{\alpha\beta}(\rho_m)\frac{\partial^2\Gamma_{2,2}}{\partial\rho_{m\alpha}\partial\rho_{m\beta}}-\frac{\pi}{8}\sum_{m=1}^{2}\tilde D_{\alpha\beta}'(\rho_m')\frac{\partial^2\Gamma_{2,2}}{\partial\rho_{m\alpha}'\partial\rho_{m\beta}'},\qquad(4.28)$$

where

$$\tilde D_{\alpha\beta}(\rho)=D_{\alpha\beta}(\rho-\rho_1)+D_{\alpha\beta}(\rho-\rho_2)-D_{\alpha\beta}(\rho-\rho_1')-D_{\alpha\beta}(\rho-\rho_2').$$

$$\tilde D_{\alpha\beta}'(\rho')=D_{\alpha\beta}(\rho'-\rho_1')+D_{\alpha\beta}(\rho'-\rho_2')-D_{\alpha\beta}(\rho'-\rho_1)-D_{\alpha\beta}(\rho'-\rho_2).\qquad(4.29)$$

$$z\gg l_{\varepsilon\parallel},\quad kl_{\varepsilon\perp}^2\gg l_{\varepsilon\parallel},\quad akl_{\varepsilon\perp}\gg l_{\varepsilon\parallel},\qquad(4.30)$$

The remaining notation is the same as in Eqs. (4.9)–(4.14).

5. CONCLUSION

It follows from the results of the present paper that non-Markov corrections can play in principle an important role in the redistribution of the intensity in the beam cross section for waves propagating in randomly inhomogeneous media. These corrections can also change the asymptotic dependences of a number of average characteristics.

We present in conclusion criteria for the applicability of the Markov approximation. In all cases it is necessary to satisfy the inequalities

$$z\gg l_{\varepsilon\parallel},\quad kl_{\varepsilon\perp}^2\gg l_{\varepsilon\parallel},\quad akl_{\varepsilon\perp}\gg l_{\varepsilon\parallel},$$

where $l_{\varepsilon\parallel}$ and $l_{\varepsilon\perp}$ are respectively the longitudinal and transverse correlation radii of the fluctuations of ε, and a is the beam width. The other inequalities depend on the specific correlators. For $\Gamma_{1,0}$ under condition (2.11), for example, it is necessary to satisfy the inequality $z\ll z^*$ [see Eq. (4.7)], for $\Gamma_{1,1}$ we must have $z\ll g^{-1}$ [see Eqs. (4.14) and (4.24)], and so on. These criteria differ in part from those given in Refs. 1 and 3. Furthermore, it is necessary to satisfy the criterion (1.4) in order to ensure applicability of the parabolic equation.

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APPENDIX

We wish to make here a few comments concerning the Bourret approximation [Eq. (2.12)]. We elucidate the main ideas using the elementary one-dimensional equation

$$\frac{\partial u(z)}{\partial z}=\frac{ik}{2}\varepsilon(z)u(z)\qquad(A.1)$$

with Gaussian correlations

$$\langle\varepsilon(z)\varepsilon(z')\rangle=B_\varepsilon(z-z').\qquad(A.2)$$

There is no noncommutativity problem here, and (2.12) is an exact equality.[3,19] We write down, however, the solution of Eq. (A.1) in terms of an $\hat L$-ordered exponential

$$u(z)=\hat L\exp\Big[\frac{ik}{2}\int_0^z dz'\,\varepsilon(z')\Big]u_0,\qquad(A.3)$$

$$\hat L\exp\Big[\frac{ik}{2}\int_0^z dz'\,\varepsilon(z')\Big]=1+\frac{ik}{2}\int_0^z dz_1\,\varepsilon(z_1)+\Big(\frac{ik}{2}\Big)^2\int_0^z dz_1\,\varepsilon(z_1)\int_0^{z_1}dz_2\,\varepsilon(z_2)+\dots\,.\qquad(A.4)$$

We consider next, for example the fourth-order term after the averaging

$$\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\int_0^{z_3}dz_4\langle\varepsilon(z_1)\varepsilon(z_2)\varepsilon(z_3)\varepsilon(z_4)\rangle$$

$$=\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\int_0^{z_3}dz_4[\langle\varepsilon(z_1)\varepsilon(z_2)\rangle\langle\varepsilon(z_3)\varepsilon(z_4)\rangle$$

$$+\langle\varepsilon(z_1)\varepsilon(z_3)\rangle\langle\varepsilon(z_2)\varepsilon(z_4)\rangle+\langle\varepsilon(z_1)\varepsilon(z_4)\rangle\langle\varepsilon(z_2)\varepsilon(z_3)\rangle].\qquad(A.5)$$

Let the function $B_\varepsilon(z-z')$ have a maximum at $z=z'$ and let it tend rapidly monotonically to zero at $|z-z'|\gtrsim l_\varepsilon$. At $z\gg l_\varepsilon$ we obtain then the estimates

$$\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\int_0^{z_3}dz_4\,B_\varepsilon(z_1-z_2)B_\varepsilon(z_3-z_4)$$

$$\sim B_\varepsilon(0)l_\varepsilon\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\,B_\varepsilon(z_1-z_2)$$

$$\sim B_\varepsilon(0)l_\varepsilon\int_0^z dz_1\int_0^{z_1}dz_2\,z_2B_\varepsilon(z_1-z_2)\sim z^2B_\varepsilon^2(0)l_\varepsilon^2,\qquad(A.6)$$

$$\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\int_0^{z_3}dz_4\,B_\varepsilon(z_1-z_3)B_\varepsilon(z_2-z_4)$$

$$\sim l_\varepsilon\int_0^z dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\,B_\varepsilon(z_1-z_3)$$

$$\times B_\varepsilon(z_2-z_3)\sim l_\varepsilon^2B_\varepsilon(0)\int_0^z dz_1\int_0^{z_1}dz_2\,B_\varepsilon(z_1-z_2)\sim zl_\varepsilon^3B_\varepsilon^2(0).\qquad(A.7)$$

The estimate (A.7) holds also for the last term in (A.5). Thus, the terms of type (A.6) predominate at $z \ll l_\varepsilon$. Summation of such terms, however, corresponds in fact to the Bourret approximation. Actually, since we deal with summation of infinite series, we must check that the corrections to the Bourret approximation are small. This is the nub of the criterion (2.11) (it is easy to verify that for the Gaussian distribution (3.12) the criterion (2.11) agrees with that obtained in p. 412 of Ref. 1). In diagram language this means selection of nonintersecting diagrams at large distances for $z \gg l_{\varepsilon\|}$ (in the Fourier wave-vector representation they correspond to the most singular diagrams at small q_z). The real situation is not so simple, since Eq. (1.3) corresponds formally to a two-dimensional Schrödinger equation with a random potential, and in its investigation we encounter the same problems as in the theory of Anderson localization (cf. Ref. 20; concerning the effects of Anderson localization for wave propagation in randomly inhomogeneous media see, e.g., Refs. 21–23). If the results of Ref. 20 are used, we get as the lower estimates of the distances at which localization effects come into play

$$z \gtrsim g^{-1} \exp (\alpha/a^2 kg), \qquad \text{(A.8)}$$

where g is defined in (4.24) and α is a numerical coefficient of order unity. For the atmosphere, in view of the smallness of g (see Sec. 4.2), these distances are practically indistinguishable, but for a number of model media[21–23] it is possible to reach a point where the localization effects become observable.

[1)]Ideas close to the Bourret approximation were advanced already earlier in Ref. 12.

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