Exact solution of a model of one-dimensional fermions with $SU(N)\times SU(M)$ symmetry

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Equations describing the thermodynamics of an integrable (1 + 1)-dimensional model of fermions, the interaction of which is attractive in one channel and repulsive in the other, are derived. The excitation spectrum of the system consists of Abelian and non-Abelian goldstone bosons and excitations whose energy is separated by a gap from the ground state. The latter are parafermions; an expression for their statistics is obtained. The asymptotic forms of certain correlation functions of the theory are found.

INTRODUCTION

In the present paper we give the exact solution of a model of one-dimensional fermions with orbital degeneracy and interaction of the Hund-coupling type. Both these factors should play an important role in the formation of the spectrum of the electronic excitations in such quasi-one-dimensional compounds as, e.g., polyynes—materials containing carbon-atom chains =C=C=... (Ref. 1). In these materials two electrons of the outer shell of the carbon atom are in strongly hybridized s–p_z states and form a valence band, while the other two electrons, in the p_x and p_y states, form a conduction band lying above the valence band. Hund coupling facilitates the formation of an $S = 1$ spin state of the electrons, while hopping between sites hinders this. Both these processes, which are important for the formation of the spectrum, are taken into account in the following exactly solvable model:

$$H=\int dx\left[-\frac{1}{2m}\psi_{\alpha n}^{+}\left(\frac{d^2}{dx^2}+p_F^2\right)\psi_{\alpha n}-v_F c\psi_{\alpha n}^{+}(x)\cdot\psi_{\alpha k}(x)\psi_{\beta k}^{+}(x)\psi_{\beta n}(x)\right],\quad \alpha,\beta=1,\ldots,N,\quad n,k=1,\ldots,M. \tag{1}$$

The Hamiltonian is symmetric under the replacement $N\to M$ and the simultaneous replacement $c\to -c$. Without loss of generality we can assume $c>0$. After this, N and M are no longer on the same footing, The Hamiltonian (1) is invariant under the action of the tensor group product $SU(N)\times SU(M)$. In the above-mentioned case of the polyynes, $N=M=2$.

An analogous model, invariant, instead, under the tensor group product $O(2N)\times O(M)$, was formulated in Ref. 2 to describe the behavior of a gapless semiconductor in a strong magnetic field. In this case, N is the number of valleys of the spectrum and M is proportional to the magnetic field.

The model (1) has applications in quantum field theory. In the limit $c\ll 1$ and $M\to\infty$ it is equivalent to the model of the principal chiral field on the group $SU(N)$. This fact was used in the derivation of the exact solution of the chiral-field model.[3]

The limit $c\ll 1$ is the most interesting one from a physical point of view, since in this case the excitation spectrum is broken down into several independent sectors—sectors corresponding to Abelian and non-Abelian goldstone fields, and a sector of gap excitations. This decoupling of the different degrees of freedom makes it possible to obtain a number of quantitative results pertaining to the Green's functions and to the statistics of the excitations.

The presence of a non-Abelian goldstone mode in the spectrum of the model (1) is an important feature that distinguishes the model from all the one-dimensional integrable models that have been considered hitherto. We assume that such a mode is present in any theory which is invariant under a tensor product of groups and in which one of the channels is attractive while the other is repulsive. As will be seen below, in our case the non-Abelian goldstone field possesses the symmetry of the group $SU(M)$. According to Witten,[4] the action of such a field can be written in the form of the Wess-Zumino action:

$$S=\frac{N}{16\pi}\left\{\int d^2x\,\mathrm{Sp}\,(\partial_\mu g\partial_\mu g^{-1})+\frac{2}{3}\int_\Omega d^2x\,d\tau\cdot\varepsilon_{\alpha\beta\gamma}\,\mathrm{Sp}\,(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g)\right\},\quad g\in SU(M),\quad x_1=x,\quad x_2=v_F t, \tag{2}$$

where Ω is the three-dimensional hemisphere whose boundary is the two-dimensional space (x,t). The value of the integral over this hemisphere is determined by the value of the field on the boundary: $g(x,t,\tau=0)\equiv g(x,t)$, since the integrand is the Jacobian of the transformation from the coordinates of the three-dimensional space to the group coordinates. Therefore, the model (2) corresponds to a two-dimensional theory. The action (2) is symmetric under left and right translations on the group $SU(M)$:

$$g\to gV,\quad g\to Ug\quad (V,U\in SU(M)).$$

This symmetry is a consequence of the mutual independence of excitations with left-handed and right-handed helicities; the latter follows from the definition of a goldstone mode.

The model (2) possesses conformal symmetry.[5] In principle, this property makes it possible to calculate all the correlation functions. The two-frequency correlators of conformal theories decay as powers of the arguments at $T=0$

$$\langle O(x,t)O(0,0)\rangle=(x+iv_F t)^{-\Delta}(x-iv_F t)^{-\tilde{\Delta}}.$$

The power exponents Δ and $\tilde{\Delta}$ are called conformal dimensions. The complete set of these dimensions is deter-

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mined by the central charge C of the conformal theory. According to the data of Ref. 5, for the model (2) we have

$$C_2=(M^2-1)N/(M+N). \tag{3}$$

The physical meaning of the central charge becomes clear if we take into account the fact that the specific heat of the conformal theory is proportional to it[6]:

$$C_v/L=\pi TC/6 \tag{4}$$

(L is the length of the system.)

Thus, the central charge indicates the number of degrees of freedom per unit cell of momentum space ($Lp/2\pi$). The fact that the central charge (3) is not an integer shows that a certain boundary condition is imposed on the free fields in the theory under consideration (on this question, see, e.g., Ref. 7). In the original theory (1) the corresponding number of degrees of freedom is $C_1 = MN$. The Abelian goldstone field takes one degree of freedom, and the non-Abelian goldstone takes C_2 degrees of freedom. The number of degrees of freedom that remain for the gap excitations is

$$C_3=C_1-C_2-1=M(N^2-1)/(M+N). \tag{5}$$

The nonintegral nature of C_3 indicates that certain restrictions are also imposed on the gap-sector excitations. These lead, in particular, to a very curious formula describing the statistics of these excitations:

$$n(\varepsilon)=[1+\sigma^{-1}\exp(\varepsilon/T)]^{-1},$$
$$\sigma=\frac{\operatorname{sh}(NH/2T)}{\operatorname{sh}(H/2T)}\frac{\sin[\pi N/(N+M)]}{\sin[\pi/(N+M)]}. \tag{6}$$

Thus, only the Abelian particles are particles in the usual sense.

The plan of the paper is as follows. In Sec. 1 we derive by the Bethe-ansatz method the equations determining the spectrum of the model (1). In Sec. 2 we discuss the properties of the ground state. In Sec. 3 we analyze the thermodynamic equations in the limit in which the bare coupling constant is small ($c\ll 1$). In the Conclusion we discuss the behavior of certain Green's functions of the theory.

1. EXACT SOLUTION

The model (1), as shown in Ref. 8, admits an exact solution by the Bethe-ansatz method. The two-particle S-matrix of the model has the form

$$S(k,p)=\frac{k-p-ic\hat{P}_N}{k-p-ic}\otimes\frac{k-p+ic\hat{P}_M}{k-p+ic}, \tag{7}$$

where $\hat{P}_N$ is the permutation operator acting on the indices α, and $\hat{P}_M$ is the permutation operator acting on the indices m.

The S-matrix (7) satisfies the Yang-Baxter equation, and, consequently, the Hamiltonian (1) is fully integrable. The Bethe equations have the form

$$\exp(ic\lambda_\alpha^{(0)}L)=\prod_{\beta=1}^{m^{(1)}}e_1(\lambda_\alpha^{(0)}-\lambda_\beta^{(1)})\prod_{\gamma=1}^{l^{(1)}}e_{-1}(\lambda_\alpha^{(0)}-\mu_\gamma^{(1)}), \tag{8a}$$

$$\prod_{\tau=\pm1}\prod_{\beta=1}^{m^{(j+\tau)}}e_1(\lambda_\alpha^{(j)}-\lambda_\beta^{(j+\tau)})=\prod_{\beta=1}^{m^{(j)}}e_2(\lambda_\alpha^{(j)}-\lambda_\beta^{(j)}),$$
$$j=1,\ldots,N-1, \tag{8b}$$

$$\prod_{\tau=\pm1}\prod_{\beta=1}^{l^{(j+\tau)}}e_1(\mu_\alpha^{(j)}-\mu_\beta^{(j+\tau)})=\prod_{\beta=1}^{l^{(j)}}e_2(\mu_\alpha^{(j)}-\mu_\beta^{(j)}),\quad \{\mu^{(0)}\}\equiv\{\lambda^{(0)}\},$$
$$j=1,\ldots,M-1, \tag{8c}$$

where $e_n(x)=(x-in/2)/(x+in/2)$.

The energy of the system is

$$E=c^2\sum_{\alpha=1}^{N_0}\lambda_\alpha^{(0)^2}-N_0p_F^2, \tag{9}$$

where N_0 is the number of particles. The number of particles of a given "color" is

$$n_j=m_{j-1}-m_j,$$

the number of particles of a given "flavor" is

$$A_p=l_{p-1}-l_p,$$

and L is the size of the system.

We shall consider Eqs. (8), (9) in the thermodynamic limit, when the number of excitations of all kinds is large. The rapidities can form complexes. Namely, the sets of rapidities

$$\lambda_{\alpha_1}^{(0)},\ldots,\lambda_{\alpha_p r_p}^{(0)},\quad \lambda_{\alpha_1}^{(1)},\ldots,\lambda_{\alpha_{(p-1)}r_p}^{(1)}$$
$$\lambda_{\alpha_1}^{(p-1)},\ldots,\lambda_{\alpha_{r_p}}^{(p-1)}$$

(p can take values from 1 to N) form bound states:

$$\lambda_{\alpha+q}^{(j)}=\Lambda_\alpha^{(p)}+i\left(\frac{p-j+1}{2}-q\right)+\gamma_{\alpha+q}^{(j,p)},$$
$$q=1,\ldots,p-j,\quad \gamma\sim\exp(-\text{const}\,N_0). \tag{10}$$

The rapidities $\lambda_\alpha^{(j)}$ that do not appear in the complexes (10) also form bound states, as also do the rapidities $\mu_\beta^{(k)}$:

$$\lambda_{\alpha+k}^{(j)}=\lambda_{\alpha,n}^{(j)}+i\left(\frac{n+1}{2}-k\right)+\gamma_{\alpha+k}^{(j,n)}. \tag{11}$$

The quantities γ are exponentially small in the number of excitations.

Omitting the details, we write the integral equations for the particle densities:

$$cp/2\pi=\tilde{\rho}_p+G_{pq}*\rho_q+a_n*\sigma_p^{(n)}-s*A_{pn}*\Sigma_1^{(n)}, \tag{12}$$
$$p,\ q=1,\ldots,N\quad(\sigma_N^{(n)}\equiv0),$$

$$a_n*\rho_\alpha=\tilde{\sigma}_\alpha^{(n)}+A^{nm}*C_{\alpha\beta}*\sigma_\beta^{(m)},\quad \alpha,\beta=1,\ldots,N-1, \tag{13}$$
$$n,\ m=1,2,\ldots,$$

$$\delta_{p1}s*A_{nm}*\rho_m=\Sigma_p^n+A^{nm}*C_{pq}*\Sigma_q^{(k)},$$
$$p,\ q=1,\ldots,M-1,\quad n,\ m=1,2,\ldots, \tag{14}$$

$$\frac{E}{L}=\sum_{p=1}^{N}\int_{-\infty}^{+\infty}\varepsilon_p^{(0)}(\Lambda)\rho_p(\Lambda)\,d\Lambda. \tag{15}$$

The symbol * denotes the convolution

$$f*g(\lambda)=\int_{-\infty}^{+\infty} f(\lambda-\lambda')g(\lambda')d\lambda'.$$

The Fourier transforms of the kernels are

$$a_n(\omega)=\exp(-n|\omega|/2),$$
$$s(\omega)=(2\operatorname{ch}\omega/2)^{-1},$$
$$C_{nm}(\omega)=\delta_{n,m}-s(\omega)(\delta_{n,m-1}+\delta_{n,m+1}),$$
$$A_{nm}(\omega)=\operatorname{cth}(|\omega|/2)(a_{|n-m|}(\omega)-a_{n+m}(\omega)),$$
$$G_{nm}=(1+a_2)^{-1}*A_{nm},$$
$$\varepsilon_p^{(0)}(\Lambda)=c^2p(\Lambda^2-(p^2-1)/12)-pp_F^2.$$

In Eqs. (12)–(15) ρ_p are the densities of the centers of the complexes (10), a tilde indicates densities of holes, $\sigma_\alpha^{(n)}$ are the distribution functions of strings of length n and color α ($n=1,2,\ldots;\alpha=1,\ldots,N-1$), and $\Sigma_p^{(n)}$ are the densities of the centers of the strings in μ ($n=1,2,\ldots;p=1,\ldots,M-1$).

The transition to the equations for the free energy is carried out in the standard manner[9]:

$$\varepsilon_p^{(0)}-T\ln(1+e^{\varepsilon_p/T})+G_{pq}*T\ln(1+e^{-\varepsilon_q/T})$$
$$-a_n*T\ln(1+\exp\{-\varepsilon_p^{(n)}/T\})$$
$$-Ts*A_{np}*\ln(1+\exp\{-\varkappa_1^{(n)}/T\})=\sum_{j=1}^{p}h_j, \qquad (16)$$

$$-T\ln(1+\exp\{\varepsilon_\alpha^{(n)}/T\})+A^{nm}*C_{\alpha\beta}$$
$$*T\ln(1+\exp\{-\varepsilon_\beta^{(m)}/T\})+a_n*T\ln(1$$
$$+\exp\{-\varepsilon_\alpha/T\})=-n(h_{\alpha+1}-h_\alpha), \qquad (17)$$

$$-T\ln(1+\exp\{\varkappa_p^{(n)}/T\})+A^{nm}*C_{pq}*$$
$$*T\ln(1+\exp\{-\varkappa_q^{(m)}/T\})-\delta_{p1}s*A_{qn}*$$
$$*T\ln(1+\exp\{-\varepsilon_q/T\})=-n(H_{p+1}-H_p), \qquad (18)$$

$$\frac{F}{L}=-\frac{c}{2\pi}\sum_{p=1}^{N}pT\int_{-\infty}^{+\infty}d\Lambda\ln\left(1+\exp\left\{\frac{-\varepsilon_p(\Lambda)}{T}\right\}\right), \qquad (19)$$

where h and H are the corresponding "magnetic" fields.

2. THE GROUND STATE; THE GELL-MANN–LOW FUNCTION

For $T=0$ and $H_p=\mathscr{H}_p=0$ in Eqs. (16)–(19) the quantities $\varepsilon_N\equiv\zeta$ and $\varkappa_p^{(n)}$ ($n=1,\ldots,N;p=1,\ldots,M-1$) do not vanish. The equations for these functions have the form

$$\varepsilon_N^{(0)}=\zeta^{(+)}-G_{NN}*\zeta^{(-)}-s*A_{Nn}*\varkappa_1^{(n)}, \qquad (20)$$

$$\delta_{p1}s*A_{nN}*\zeta^{(-)}=A_{nm}*C_{pq}*\varkappa_q^{(m)}, \qquad (21)$$

where

$$\zeta^{(\pm)}(\Lambda)=\theta(\pm\zeta(\Lambda))\zeta(\Lambda).$$

By inverting the kernels in Eq. (21) and substituting the result into Eq. (20), we obtain

$$\varepsilon_N^{(0)}(\Lambda)=\zeta(\Lambda)+\int_{-Q}^{Q}\mathscr{F}(\Lambda-\Lambda')\zeta(\Lambda')d\Lambda' \qquad \zeta(\pm Q)=0,$$
$$\mathscr{F}(\omega)=(e^{-N|\omega|}-e^{-M|\omega|})/(1-e^{-N|\omega|}), \qquad (22)$$

where $\zeta(\Lambda)$ is the energy of the Abelian goldstone field. Near $\Lambda=\pm Q$ the spectrun of this mode is linear:

$$\zeta(p)=c^*|p|. \qquad (23)$$

In order to derive Eq. (23) we shall consider, together with Eq. (22) for the energy, the equation for the density:

$$\rho(\Lambda)+\int_{-Q}^{Q}\mathscr{F}(\Lambda-\Lambda')\rho(\Lambda')d\Lambda'=Nc/2\pi. \qquad (24)$$

The momentum of an excitation is

$$P(\Lambda)=2\pi\int_{0}^{\Lambda}\rho(\Lambda')d\Lambda'.$$

For $\Lambda=Q+\delta\Lambda$ ($|\delta\Lambda|\ll Q$),

$$P(\Lambda)=P_0+2\pi\rho(Q)\delta\Lambda, \quad \zeta(\Lambda)=\delta\Lambda(\partial\zeta/\partial\Lambda|_Q). \qquad (25)$$

From this we find

$$c^*=\frac{1}{2\pi\rho(Q)}\frac{\partial\zeta}{\partial\Lambda}\Big|_{\Lambda=Q}$$

This is the phase velocity of the charge-density wave, i.e., of the Abelian goldstone field. In the theory it plays the role of the velocity of light. Below we shall see that there exist excitations with the dispersion law $\varepsilon=(p^2c^{*2}+\Delta_j^2)^{1/2}$, where

$$\Delta_j=\Delta_0\sin\frac{\pi j}{N}, \qquad \Delta_0=\frac{1}{N}\int_{|\Lambda|>Q}e^{-2\pi\Lambda/N}\zeta(\Lambda)d\Lambda. \qquad (26)$$

We shall find the dependence $\Delta(c)$. This will enable us to determine the Gell-Mann–Low function $\beta(c)$:

$$\int_{c}^{\infty}dc'/\beta(c')=\ln[p_Fv/\Delta_0(c)]. \qquad (27)$$

For $Q\gg1$ (small c),

$$Q+\frac{N-M}{2\pi}\ln Q=\frac{p_F}{c}\qquad\left(p_F=\frac{\pi N_0}{LNM}\right),$$
$$\Delta_0=c(\text{const}+O(1/Q))\exp(-2\pi Q/N) \qquad (28)$$

and, accordingly, the β-function is

$$\beta(c)=-\frac{Nc^2}{2\pi}+\frac{NM}{4\pi^2}c^3+\ldots \qquad (29)$$

which is the perturbation-theory result.

For $Q\ll1$ (large c),

$$\Delta_0=\text{const}\,c+O(1),$$
$$\beta(c)=-c+O(1). \qquad (30)$$

3. THE RELATIVISTIC LIMIT

As already stated in the Introduction, in the limit of small coupling constant $c\ll1$ (which corresponds to $Q\gg1$ in the thermodynamic equations (22) and (24)] there exists a region of temperatures (as will be seen from the following, the region $T\ll c^{1/2}\varepsilon_F$) in which the energies of the excitations are divided into three independent groups. We now prove this.

We rewrite the integral equations (16)–(18) by inverting the matrices $\hat{A}$ and $\hat{C}$ in them:

$$T\ln(1+e^{-\varepsilon_p^{(n)}/T})-\mathscr{A}_{pq}^{(M)}*C^{nm}*T\ln(1+e^{\varkappa_q^{(m)}/T})$$
$$=\mathscr{A}_{p1}^{(M)}*s*T\ln(1+e^{-\varepsilon_n/T}), \qquad (31)$$

where the Fourier transform of the kernel is

$$\mathscr{A}_{pq}^{(M)}(\omega)=2\operatorname{cth}\frac{\omega}{2}\operatorname{sh}\left\{(M-\max(p,q))\frac{\omega}{2}\right\}\operatorname{sh}\left\{\min(p,q)\frac{\omega}{2}\right\}\Big/\operatorname{sh}\frac{M\omega}{2}.$$

The functions ε_p $(p=1,\ldots,N-1)$ are present only in the first $N-1$ equations of the system (31). For $|\Lambda|\ll Q$ (only this region is important for the determination of the functions ε_p) we shall prove that we can consider these first $N-1$ equations separately from the others. In fact, of the other $\varkappa_q^{(m)}$ $(m\geqslant N)$, only the energy $\varkappa_q^{(N)}$ is present in the first equations (the matrix $\hat{C}$ acts only within the limits of the nearest neighbors!). We shall estimate the effect of the term with $\varkappa_q^{(N)}$ from its value for $T\to 0$. In this case, from (31) we have

$$\varkappa_p^{(N)}=\mathscr{A}_{p1}^{(M)}*s*\zeta^{(-)}. \tag{32}$$

Consequently, for $|\Lambda|\ll Q$, where the functions ε_p $(p=1,\ldots,N-1)$ have a minimum (see the figure), we have $\varkappa_p^{(N)}\sim-\varepsilon_F$, and the influence of the latter functions in Eqs. (31) with $p\leqslant N-1$ can be neglected.

For $|\Lambda|\gg Q$, it follows from (32) that the functions $\varkappa_p^{(N)}\sim-\exp(-2\pi|\Lambda|/M)$ are exponentially small, whereas $\varepsilon_p\sim\varepsilon_F$ $(p\neq N)$, and in this case, in the system (31), we must retain in the right-hand side only the term with $\varepsilon_N\equiv\zeta$. Therefore, we can write separately the equations for the functions $\varkappa_p^{(n)}$ for $|\Lambda|\ll Q$ and for $|\Lambda|\gg Q$. The intermediate region will not give a contribution to the free energy, since in this region we have ε_p, $-\varkappa_q^{(N)}\sim\varepsilon_F$ and

$$T\ln(1+\exp\{-\varkappa_p^{(n)}/T\})-\mathscr{A}_{pq}^{(M)}*C^{nm}*T\ln(1+\exp\{\varkappa_q^{(m)}/T\})$$
$$=\mathscr{A}_{p1}^{(M)}*s*T\ln(1+e^{-\varepsilon_n/T})$$
$$(|\Lambda|\ll Q);\ n,\ m=1,\ldots,N-1;\ p,\ q=1,\ldots,M-1. \tag{33}$$

Since $\varkappa(\Lambda)=\varkappa(-\Lambda)$ it is sufficient to write the equation for $\Lambda\to+\infty$:

$$\varphi_p^{(n)}(\Lambda)\equiv\varkappa_p^{(n)}(\Lambda+Q),$$
$$T\ln(1+\exp\{-\varphi_p^{(n)}(\Lambda)/T\})$$
$$-\mathscr{A}_{pq}^{(M)}*C^{nm}*T\ln(1+\exp\{\varphi_q^{(m)}(\Lambda)/T\})$$
$$=-2M^{-1}\sin(\pi p/M)A\exp(-2\pi\Lambda/M)\delta_{nN},$$
$$\lim_{n\to\infty}\varphi_p^{(n)}/n=H_p,$$
$$A=2\pi\int_{-2Q}^{0}e^{2\pi\Lambda/M}\zeta(\Lambda+Q)\,d\Lambda. \tag{34}$$

By comparing Eqs. (34) and Eqs. (14) we see that for $|\Lambda|\gg Q$, where $\rho_p=0$ $(p\neq N)$, the following remarkable relations are fulfilled:

$$\Sigma_p^{(n)}(\Lambda)=\frac{1}{2\pi v}\left|\frac{\partial\varphi_p^{(n)}(\Lambda)}{\partial\Lambda}\right|n(\varphi_p^{(n)}(\Lambda)),$$
$$\Sigma_p^{(n)}(\Lambda)=\frac{1}{2\pi v}\left|\frac{\partial\varphi_p^{(n)}(\Lambda)}{\partial\Lambda}\right|(1-n(\varphi_p^{(n)}(\Lambda))), \tag{35}$$

where

$$n(\varphi)=(\exp(\varphi/T)+1)^{-1},$$
$$v=\frac{1}{2\pi}\int e^{2\pi\Lambda/M}\frac{\partial\zeta}{\partial\Lambda}d\Lambda\Big/\left[\int e^{2\pi\Lambda/M}\rho(\Lambda)\,d\Lambda\right].$$

The validity of these relations implies that the excitation spectrum of the theory described by Eqs. (34) is linear: $\varepsilon(p)=v|p|$. It can be shown that, to within terms of order c/p_F, $v=c^*$.

Equations (34) and (35) describe the thermodynamics in the sector of the non-Abelian goldstone field. The field theory corresponding to this goldstone meson is conformally invariant. Its central charge can be calculated from the specific heat (4).

In the Appendix it is shown that $C_1=N(M^2-1)/(N+M)$, and, consequently, the non-Abelian Goldstone mode can be described by the Wess-Zumino Lagrangian (2).

We now turn to the series of massive excitations. Making use of Eq. (31), we rewrite Eq. (16) in the form

$$\varepsilon_p^{(0)}=T\ln(1+\exp(\varepsilon_p/T))-D*G_{pq}*T\ln(1+\exp(-\varepsilon_q/T))$$
$$-a_n*T\ln(1+\exp(\varepsilon_p^{(n)}/T))$$
$$-s*\mathscr{A}_{1n}^{(M)}*T\ln(1+\exp(\varkappa_n^{(p)}/T)), \tag{36}$$

where

$$D=1-\mathscr{A}_{1,1}^{(M)}\frac{a_2}{1+a_2}.$$

We shall invert the kernel DG_{NN} in Eq. (36) with $p=N$, and substitute the resulting expression for $T\ln(1+e^{-\zeta/T})$ into the remaining equations. We obtain

$$s_p*T\ln(1+\exp(\zeta/T))=T\ln(1+\exp(\varepsilon_p/T))-\tilde{D}*\mathscr{A}_{pq}^{(N)}$$
$$*T\ln(1+\exp(-\varepsilon_q/T))-a_n*T\ln(1+\exp(-\varepsilon_p^{(n)}/T))-s$$
$$*\mathscr{A}_{1l}^{(M)}*T\ln(1+\exp(\varkappa_l^{(p)}/T)), \tag{37}$$

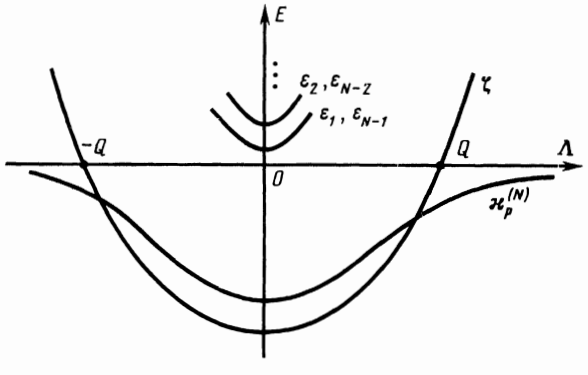

FIG. 1. Diagram classifying Eqs. (16)–(18). Schematic dependence of the energy on the rapidity for different branches of excitations. Only those energies which do not vanish as $T\to 0$ are shown in the figure.

where

$$\tilde{D}=(1+a_2)^{-1}*D,\ s_p(\omega)=\mathrm{sh}((N-p)\omega/2)/\mathrm{sh}\,N\omega/2.$$

In the derivation of Eq. (37) we make use of the relation

$$\mathscr{A}_{pq}^{(N)}=(1+a_2)^{-1}*(G_{pq}-G_{pN}{}^{-1}*G_{pN}*G_{Nq}).$$

In the left-hand side, for $Q\gg 1$ we can make the substitution

$$s_p*T\ln\left(1+\exp\frac{\zeta(\Lambda)}{T}\right)\approx\frac{2}{N}\sin\frac{\pi p}{N}\,\mathrm{ch}\frac{2\pi\Lambda}{N}\cdot\int_{-\infty}^{+\infty}d\Lambda\exp\left(\frac{2\pi\Lambda}{N}\right)T\ln\left(1+\exp\frac{\zeta(\Lambda)}{T}\right).\tag{38}$$

The integrand goes over to $\zeta^{(+)}$ in the limit $T\to 0$. Under what conditions can we neglect the temperature dependence in Eq. (38)? We make an estimate of the temperature correction:

$$T\int\ln\left(1+\exp\frac{\zeta(\Lambda)}{T}\right)\exp\frac{2\pi\Lambda}{N}d\Lambda-\Delta_0\cdot$$

$$\approx T\int_{-\infty}^{+\infty}\exp\left\{-\frac{2\pi\Lambda}{N}\right\}\ln\left(1+\exp\left(-\frac{\zeta'|\Lambda-Q|}{T}\right)\right)$$

$$\sim\frac{T^2}{v\rho(Q)}\exp\left(-\frac{2\pi Q}{N}\right)\sim\frac{T^2}{c\varepsilon_F}\exp\left(-\frac{2\pi Q}{N}\right).$$

The temperature correction is small for

$$T^2\ll\Delta_0 c\varepsilon_F\exp(2\pi Q/N)\sim(c\varepsilon_F{}^2).$$

In this case the equations determining the free energy of the system can be written as follows:

$$\frac{F}{L}=-\frac{\pi T^2}{6c^*}\left(1+\frac{N(M^2-1)}{N+M}\right)-\sum_{p=1}^{N-1}\frac{\Delta_0}{c^*}\sin\frac{\pi p}{N}\cdot\int d\Lambda\,\mathrm{ch}\frac{2\pi\Lambda}{N}\ln\left(1+\exp\frac{\varepsilon_p(\Lambda)}{T}\right),\tag{39}$$

$$\Delta_0\sin\frac{\pi p}{N}\,\mathrm{ch}\frac{2\pi\Lambda}{N}=T\ln\left(1+\exp\frac{\varepsilon_p(\Lambda)}{T}\right)$$
$$-\tilde{D}*\mathscr{A}_{pq}^{(N)}*T\ln\left(1+\exp\left(-\frac{\varepsilon_q(\Lambda)}{T}\right)\right)$$
$$-a_n*T\ln\left(1+\exp\left(-\frac{\varepsilon_p^{(n)}(\Lambda)}{T}\right)\right)-s*\mathscr{A}_{11}^{(M)}$$
$$*T\ln\left(1+\exp\left(\frac{\varkappa_l^{(p)}(\Lambda)}{T}\right)\right);\quad p,q=1,\ldots,N-1,\tag{40}$$

$$a_n*T\ln(1+\exp(-\varepsilon_p/T))=T\ln(1+\exp(\varepsilon_p{}^{(n)}/T))$$
$$-C_{pq}*A^{nm}*T\ln(1+\exp(-\varepsilon_q{}^{(k)}/T)),\quad n,m=1,2,\ldots,$$
$$\lim_{n\to\infty}\varepsilon_p^{(n)}/n=H,\tag{41}$$

$$s*\mathscr{A}_{l1}^{(M)}*T\ln(1+\exp(-\varepsilon_p/T))=T\ln(1+\exp(-\varkappa_p^{(l)}/T))$$
$$-C_{pq}*\mathscr{A}_{lk}^{(M)}*T\ln(1+\exp(\varkappa_q/T)),\quad l,k=1,\ldots,M-1.\tag{42}$$

Equations (39)–(42) coincide with the thermodynamic equations of the model of relativistic fermions transforming according to the irreducible representation of the group $SU(N)$ with the highest weight $\Lambda=(M,0,\ldots,0)$ (Ref. 3). In the limit $M\to\infty$ these equations go over into the equation for an $SU(N)$-chiral field.

Equation (6) for the statistics of the gap excitations, mentioned in the Introduction, can be obtained from the low-temperature expansion of the part of the free energy (39) corresponding to these excitations.

We shall calculate the first [in $\exp(-\Delta_1/T)$] iteration of the functions $\varepsilon_p(\lambda)$. For $t\ll\Delta_1$ [we recall that the magnitudes of the gaps Δ_j are given by formula (26)], in Eqs. (41) and (42) we can omit, with exponential accuracy in $1/T$, the left-hand side. The equations are transformed from integral to algebraic equations. Their solutions are

$$1+\exp\left(\frac{\varepsilon_p^{(l)}}{T}\right)=\frac{\mathrm{sh}[H(N-p+l)/2T]\,\mathrm{sh}[(p+l)H/2T]}{\mathrm{sh}[H(N-p)/2T]\,\mathrm{sh}(Hp/2T)}$$
$$p=1,\ldots,N-1;\quad l=1,2,\ldots,$$
$$1+\exp\left(-\frac{\varkappa_l^{(p)}}{T}\right)=\frac{\sin[\pi(N-p+l)/(N+M)]\sin[\pi(p+l)/(N+M)]}{\sin[\pi p/(N+M)]\sin[\pi(N-p)/(N+M)]}\tag{43}$$

Substituting the expressions (43) into Eq. (40), we find

$$\varepsilon_1(\lambda)=\Delta_1\,\mathrm{ch}\frac{2\pi\lambda}{N}-T\ln\frac{\mathrm{sh}[NH/2T]}{\mathrm{sh}[H/2T]}\frac{\sin[\pi N/(N+M)]}{\sin[\pi/(N+M)]}.$$

We substitute this expression into Eq. (39); this yields Eq. (6). Of course, the question arises of the correctness of this derivation, since Eq. (6) contains not only the first exponential but also an entire series in $\exp(-\Delta_j/T)$. However, it can be shown[10] that the next terms of the low-temperature expansion of the functions ε_p will give corrections that are small in $(T/\Delta)^{1/2}$ to the coefficients in this series.

The low-temperature expansion has served us only as a means for the derivation of Eq. (6), which has, of course, a wider meaning.

CONCLUSION

Thus, if the coupling constant is small, there are three energy scales in the theory: ε_F, $\varepsilon_F c^{1/2}$, and

$$\Delta_0=\varepsilon_F c^{M/N}\exp(-2\pi/cN).$$

Above energies $\sim\varepsilon_F c^{1/2}$ the goldstone fields become independent. The energies of the remaining excitations are separated from the ground state by gaps

$$\Delta_j=\Delta_0\sin(\pi j/N),$$

and these excitations have parafermion statistics.

As already stated in the preceding section, the equations describing the gap sector of the model (1) are equivalent to the equations derived for the model of relativistic fermions transforming according to a particular, highest-weight representation of the group $SU(N)$ (Ref. 3). This model is simplest in the case $N=M=2$, which is, moreover, the most interesting from a physical point of view. In this case, the gap sector is described by a three-component Majorana (real) fermion field χ_a. The Lagrangian of the model is

$$L=i\bar{\chi}_a\partial_\mu\chi_a-gJ_a{}^\mu J_{\mu a},\quad J_{\mu a}=\bar{\chi}_b\gamma_\mu\chi_c\varepsilon_{abc}.\tag{44}$$

The model (44) is supersymmetric.[11] Nevertheless, its physical excitations are not bosons and fermions, but parafermions [see formula (6) with $N=M=2$].

Because the degrees of freedom decouple, we can make a number of assertions about the behavior of the correlation functions of the theory. It is logical to assume that the field $\psi_{\alpha m}$ "factors" into a product of three independent fields:

$$\begin{aligned}\psi_{\alpha m}(x,t)&=e^{-ip_Fx}\psi_{R\alpha m}(x,t)+e^{ip_Fx}\psi_{L\alpha m}(x,t),\\ \psi_{R\alpha m}(x,t)&=:e^{i\beta\varphi(z)}:O_{1m}(z)O_{2\dot\alpha}(x,t),\\ \beta^2&=2\pi/NM,\quad z=x+iv_Ft,\end{aligned}\tag{45}$$

where φ is a scalar field and O_{1m} is that field of the Wess-Zumino model with $k=N$ on the group $SU(M)$ that transforms according to the fundamental representation of this group. The dimensions of the fields and their correlation functions in the Wess-Zumino model are known.[5]

We shall assume that the field O_{1m} has the following dimensions from the groups of dimensions of the Wess-Zumino model:

$$\Delta_1=(M^2-1)/2M(M+N),\quad \bar\Delta_1=0.\tag{46}$$

The field $O_{1m}(z)\exp(i\beta\varphi(z))$ forms a field with the conformal dimension

$$\Delta=(MN+1)/2N(M+N),\quad \bar\Delta=0.\tag{47}$$

The correlator of the right-handed components of the field $\psi_{\alpha m}$ is

$$\begin{aligned}&\langle\psi_{R\alpha m}(x,t)\psi^{\dagger}_{R\beta l}(0,0)\rangle\\ &=\delta_{\alpha\beta}\delta_{ml}(\pi T/\operatorname{sh}\pi zT)^{2\Delta}\langle O_{2\alpha}(x,t)O_{2\alpha}^{\dagger}(0,0)\rangle.\end{aligned}\tag{48}$$

Concerning the Green's function $\langle O_{2\alpha}(x,t)O^{*}_{2\alpha}(0,0)\rangle$ we can say the following. For $p_F^{-1}\ll|z|\ll\Delta_0^{-1}$ this correlator depends only on z and its dimension is the same as that of the primary field of the Wess-Zumino model with $k=M$ on the group $SU(N)$, which transforms according to the fundamental representation of this group. This means that its dimension is given by Eq. (46), in which it is necessary to replace M by N. Since the dimension Δ_2 obtained in this way satisfies the relation $2(\Delta+\Delta_2)=1$, the correlator (48) for $|z|\ll\Delta_0^{-1}$ has the usual form of a correlator of free Fermi fields:

$$\langle\psi_{R\alpha m}(x,t)\psi^{\dagger}_{R\beta l}(0,0)\rangle=\delta_{\alpha\beta}\delta_{ml}(\pi T/\operatorname{sh}\pi Tz).\tag{49}$$

Next, for $vt\gg\Delta_0^{-1}$, we can assume that by analogy with the familiar result for massive particles the equation

$$\langle O_\alpha(x,t)O_\alpha^{*}(0,0)\rangle\sim(1/\Delta_0v_Ft)^d\exp(-v_Ft\Delta_0-x^2\Delta_0/2v_Ft),$$

holds, where d is an unknown index.

The strong-coupling region completely determines the behavior of the correlation function of the order parameter: $O_{mn}=\psi^{*}_{Rm\alpha}\psi_{Ln\alpha}$. Because of the presence of the goldstone fields, $\langle O_{mn}\rangle=0$ holds, but the leading correlators of the order parameter are nonzero. The field O_{mn} is a matrix transforming according to the fundamental representation of the group $U(M)$. Therefore, its dimension $\Delta=\bar\Delta$ is given by Eq. (47). Thus,

$$\begin{aligned}&\langle\!\langle\psi^{\dagger}_{R\alpha m}(x,t)\psi_{L\alpha n}(x,t)\psi^{\dagger}_{L\beta l}(0,0)\psi_{R\beta k}(0,0)\rangle\!\rangle\\ &\sim\delta_{mk}\delta_{nl}\Delta_0^2|\pi T/p_F\operatorname{sh}\pi Tz|^{4\Delta}.\end{aligned}\tag{50}$$

Consequently, the Fourier transform of the polarization operator of the electrons has a singularity at $p\sim2p_F$:

$$\Pi(\omega,k+2p_F)\sim\Delta_0^2(\omega^2-v_F^2k^2)^{-1+2\Delta}\varepsilon_F^{-(1+4\Delta)},$$

which should be manifested, e.g., in experiments on the absorption of ultrasound.

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APPENDIX

We shall prove that the specific heat in the theory defined by Eqs. (34) coincides with the specific heat of the Wess-Zumino model (2). The specific heat of a conformally invariant theory is uniquely related to the central charge (36) of the theory, and, for the Wess-Zumino model, is known[5] and is given by Eq. (3).

For an integrable theory with a linear spectrum the relations (35) are always valid. Substituting them into the general formula for the entropy

$$\begin{aligned}S=\sum_n\int d\lambda\{&(\rho_n(\lambda)+\tilde\rho_n(\lambda))\ln(\rho_n(\lambda)+\tilde\rho_n(\lambda))\\ &-\rho_n(\lambda)\ln\rho_n(\lambda)-\tilde\rho_n(\lambda)\ln\tilde\rho_n(\lambda)\},\end{aligned}$$

we obtain

$$S/TL=\sum_n L(\max(n(\varepsilon_n)),\ \min(n(\varepsilon_n))),\tag{A1}$$

where

$$L(x,y)=-\int_y^x dt\left(\frac{\ln t}{1-t}+\frac{\ln(1-t)}{t}\right),\qquad n(\varepsilon)=(e^{\varepsilon/T}+1)^{-1}.$$

The thermodynamic equations for the Wess-Zumino model can be obtained using the results of Ref. 3:

$$\begin{aligned}&\varepsilon_n(\lambda)-s*T\ln(1+\exp(\varepsilon_{n-1}(\lambda)/T))(1+\exp(\varepsilon_{n+1}(\lambda)/T))\\ &=-m_0e^{\pi\lambda}\delta_{n,M}-m_0e^{-\pi\lambda}\delta_{n,0},\quad n=-\infty,\ldots,+\infty.\end{aligned}\tag{A2}$$

For the right-handed particles ($\lambda\gg1$), for $T\ll m_0$ we can set $\varepsilon_M=-\infty$. Then the system (A2) decomposes into two independent parts: $n<M$ and $n>M$. The first part ($n<M$) coincides with Eqs. (34), while the second makes no contribution to the specific heat (A2), since we have $\varepsilon_n^{\max}=\varepsilon_n^{\min}$. For the left-handed particles ($-\lambda\gg1$), for $T\ll m_0$ we can set $\varepsilon_m=-\infty$, and energies with $n>0$ make a contribution to the specific heat. The equations for these again coincide with Eqs. (34).

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