# Theory of phase locking of an array of lasers

A. A. Golubentsev, V. V. Likhanskiĭ, and A. P. Napartovich

(Submitted 15 January 1987)
Zh. Eksp. Teor. Fiz. **93**, 1199–1209 (October 1987)

A theoretical analysis is reported of the mode structure of the radiation produced by an array of periodically distributed lasers with effective coupling between the elements. The coupling is based on the reproduction of the periodic field structure. The possible mode types, losses, and eigenfrequencies are determined for infinite and finite sets of lasers assembled in triangular and rectangular arrays. The influence of mirror misalignment, saturation of the active medium, and random phase spread among the individual channels on the modes and their $Q$-factors is examined.

One way of increasing total laser power output is to increase the number $N$ of individual lasers, each with its pumping and cooling system.[1] The power output of an array of lasers is proportional to $N$. On the other hand, it is well-known that, when the radiation from the individual lasers is mutually coherent, and is focused to a common point, the power density at the central spot can be proportional to $N^2$ (the spot size is proportional to $1/N$). It follows that the phase locking of lasers as a means of increasing their power output is a very important question.

A relatively large number of papers, both theoretical and experimental, has been devoted to this problem. It appears that the first theoretical paper was that by Spencer and Lamb[2] on the phase locking of two lasers. As far as we know, experimental work has been confined to the phase locking of semiconductor lasers (see, for example, Refs. 3 and 4) and $CO_2$ lasers (Refs. 5 and 6).[1)] Phase locking is achieved by transferring a fraction of the output of each laser to the neighboring lasers. As a rule, this fraction is small and, in arrays consisitng of a large number of lasers, coupling can be achieved by diffraction "leakage" into neighboring lasers. In this paper, we examine an effective method of phase locking a set of lasers, proposed earlier in Ref. 8 and based on the Talbot effect (self-reproduction of periodic fields).

## FORMULATION OF THE PROBLEM

It is obvious that the parameters of the lasers to be phase locked must be as close to one another as possible. It is technologically convenient to make a compact assembly of almost identical lasers so that it forms a laterally regular array. To simplify the optical system, it is natural to use resonators with mirrors that are common to all the lasers. Coupling between individual lasers is then most simply accomplished by diffractive spreading of the individual beams.[5] The geometry of the system shows that diffractive coupling is accompanied by losses of radiation between the lasers. If the construction of the array of lasers is such that diffractive coupling inside the active volume is not possible (this is the typical situation for gas lasers), one of the common mirrors of the resonator must be moved away from the active medium so that the diffractive coupling becomes sufficiently effective. It is clear that, in the situation described above, it is difficult to attain a large coupling factor and, hence, stable phase locking. Actually, experiment[5] shows that this mode of operation of $CO_2$ laser arrays is frequently interrupted. The situation becomes radically different[8] when one of the plane mirrors is placed at a distance equal to half the self-reproduction distance of the periodic pattern of the ends of the individual channels, i.e., the so-called Talbot distance. The diffractive image of the channel ends, if they radiate in phase, is located precisely at these ends. On the other hand, since a relatively large number of neighbors contributes to the diffraction spot, this situation corresponds to strong coupling between the channels. If the radiation issuing from the channel ends is not phase-locked, the diffractive image is destroyed and radiation losses increase abruptly. The above effect is thus seen to lead to the selection of the phase locked regime. Since the self-reproduction length depends on wavelength, an array of lasers with this type of coupling can be frequency-tuned by displacing the mirror.

To evaluate possible practical applications of the above method of phase locking, we must determine the spectrum and the losses of collective modes of an array of lasers, and then estimate their sensitivity to mirror adjustment (longitudinal displacement and misalignment) and to the spread among the parameters of the individual channels. These questions are examined below.

## FIELD STRUCTURE FOR AN INFINITE ARRAY

The propagation of radiation in an array of lasers with Talbot coupling can be divided into two basic stages. One is the double transit along the channels containing the active medium, and the other is propagation in the atmosphere up to the reflecting mirror $M_T$ and back again. It is clear that a general description is possible for the second stage. As far as propagation along an individual channel is concerned, it is natural to suppose that it can be characterized by a set of lateral eigenmodes and a discrete spectrum of eigenfrequencies. We shall confine our attention to one selected lateral mode (e.g., with Fresnel number evaluated along the radius and the length of the channel $\lesssim 1$) that can be described at the channel end by the field distribution $f(r)$. Without loss of generality, the function $f(r)$ can be taken to be real and normalized, so that

$$\int_S f^2(\mathbf{r})\, d\mathbf{r}=1,$$

where $S$ is the area of the end of the channel. We shall neglect differences in $f(r)$ between different channels.

The field on the plane containing all the channel ends can then be written in the form

$$E(\boldsymbol{\rho})=\sum_{m,n} C(\mathbf{R}_{mn}) f(\boldsymbol{\rho}-\mathbf{R}_{mn}), \tag{1}$$

where $\mathbf{R}_{mn}$ is the coordinate of the center of the channel ($m$,

 0038-5646/87/100676-07$04.00  

$n$), i.e., $\mathbf{R}_{mn} = m\mathbf{a}_1 + n\mathbf{a}_2$, where $\mathbf{a}_1$, $\mathbf{a}_2$ are the translation vectors of the channel ends in the array, $\boldsymbol{\rho}$ is the coordinate in the plane, and $C(\mathbf{R}_{mn})$ is the amplitude of the field envelope. [In writing (1), we assumed that the field was linearly polarized and that the scalar approximation was valid.] The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are listed in the Table for different types of array.

Over a distance $z$, the field transforms so that

$$E(z,\boldsymbol{\rho}) = \frac{ik_0}{2\pi z}\exp(ik_0z)\int E(\boldsymbol{\rho}')\exp\left[\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right]d\boldsymbol{\rho}', \quad (2)$$

where $k_0$ is the wave vector. Assuming that a plane mirror of sufficiently large aperture is placed at a distance $z/2$ from the plane containing the channel ends, we find that the lateral structure of the reflected radiation is described by (2). Next, we assume that the projection of this distribution onto (1) is reproduced to within a constant factor as radiation propagates back and forth along the system of channels. The problem of determining the field in the resonator then reduces to the eigenvalue problem for a set of linear equations:

$$\gamma' C(\mathbf{R}) = A\sum_{\mathbf{R}'} M(\mathbf{R},\mathbf{R}')C(\mathbf{R}'), \quad (3)$$

where

$$M(\mathbf{R},\mathbf{R}') = \frac{ik_0}{2\pi z}\int d\boldsymbol{\rho}\, d\boldsymbol{\rho}' f(\boldsymbol{\rho}-\mathbf{R})f(\boldsymbol{\rho}'-\mathbf{R}')\exp\left[\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right], \quad (4)$$

$\gamma'$ is the eigenvalue whose modulus determines the resonator losses and whose phase determines the eigenfrequencies,[2)] and $A$ is a constant representing the total phase gain and the change in the amplitude in the channels. In our analysis, we shall consider the quantity $\gamma'/A = \gamma$.

When the distance between the perfectly reflecting mirror and the channel ends is $z_T/2$, i.e., half the Talbot distance, there is a known limiting solution ($N\to\infty$) of (3) that corresponds to the complete reproduction of the equal-phase field distribution over the tubes, $C(\mathbf{R}) = \text{const}$. The expressions for $z_T$ for arrays of different types are listed in Ref. 9. It is obvious that $|\gamma| = 1$.

This is most simply verfied by rewriting (2) for the Fourier transforms of the field

$$E(z,\mathbf{q}) = E(0,\mathbf{q})\exp(-iz\mathbf{q}^2/2k_0)\exp(ik_0z) \quad (5)$$

and recalling that $E(\boldsymbol{\rho})$ is a periodic function, so that $\mathbf{q}$ is a multiple of $2\pi\mathbf{b}$, where $\mathbf{b}$ is the reciprocal lattice vector. We then have $\mathbf{b} = m\mathbf{b}_1 + n\mathbf{b}_2$, where $m$, $n$, are integers; the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are defined in the Table.

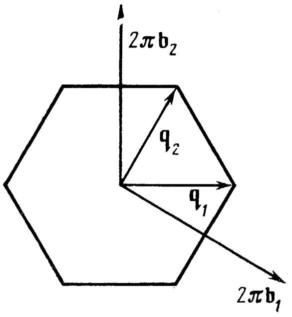


FIG. 1.

Since $2\pi^2 z_T\mathbf{b}^2/k_0$ is a multiple of $2\pi$ for all $\mathbf{b}$, the phase factor in (5) is the same for all the Fourier components, which means that we have the self reproduction effect.

Two questions arise. (1) Does this solution exhaust all the self-reproducing modes and (2) what is the minimum distance for the self-reproduction effect to occur?

The eigenvalues of (3) are readily found in the case of an infinite array of lasers ($N\to\infty$):

$$\gamma = M(\mathbf{q}) = \sum_{\mathbf{R}} M(\mathbf{R})e^{-i\mathbf{q}\mathbf{R}}, \quad (6)$$

since $M$ is a difference matrix, i.e., $M(\mathbf{R},\mathbf{R}') = M(\mathbf{R}-\mathbf{R}')$. The eigenvectors satisfy $C(\mathbf{R}) = \exp(i\mathbf{q}\cdot\mathbf{R})$, i.e., the field structure is the discrete analog of a plane wave. The expression for $M(\mathbf{q})$ can be simplified when $z = z_T$:

$$M(\mathbf{q}) = \sum_{\mathbf{R}}\int d\mathbf{r} f(\mathbf{r})f\left(\mathbf{r}+\mathbf{R}+\mathbf{q}\frac{z_T}{k_0}\right)\times\exp\left\{i\mathbf{q}\cdot\left(\mathbf{R}+\frac{z_T}{k_0}\mathbf{q}\right)-i\frac{z_T}{2k_0}\mathbf{q}^2\right\}. \quad (7)$$

Since the function $f(\mathbf{r})$ is finite and normalized $[\int f^2(\mathbf{r})d\mathbf{r} = 1]$, it is readily seen from (7) that $M(\mathbf{q}=0) = 1$, as expected. It is also readily verified that, when $\mathbf{q} = \mathbf{q}_1 = k_0\mathbf{a}_1/z_T$ and $\mathbf{q}_2 = k_0\mathbf{a}_2/z_T$, we again have $|M(\mathbf{q})| = 1$.

Figure 1 shows the Brillouin zone for a triangular array, together with the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Figure 2 shows the phase distribution for the sources in the array in the case of a self-reproducing mode with $\mathbf{q} = \mathbf{q}_1$ [$\varepsilon = \exp(2\pi i/3)$]. We note that $M(\mathbf{q}_1) = M(\mathbf{q}_2) = \exp(-2\pi i/3)$, i.e., the modes are

TABLE I.

| Type of array (elementary cell) | Translation vectors | Reciprocal lattice vectors | Number $n$ of modes as a function of distance $z_M$ from the mirror |
|---|---|---|---|
| Triangular | $\mathbf{a}_1 = (1,0)\,a$ | $\mathbf{b}_1 = \frac{2}{a\sqrt{3}}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ | If $z_M = m\frac{z_T}{2}$ ($m$—integer), then $n = 3m^2$ |
| | $\mathbf{a}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)a$ | $\mathbf{b}_2 = \frac{2}{a\sqrt{3}}(0,1)$ | |
| Rectangular with integer ratio of periods $P$ | $\mathbf{a}_1 = (Pa, 0)$ | $\mathbf{b}_1 = \left(\frac{1}{Pa}, 0\right)$ | If $z_M = m\frac{z_T}{4}$, |
| | $\mathbf{a}_2 = (0, a)$ | $\mathbf{b}_2 = \left(0, \frac{1}{a}\right)$ | then $n = m^2P^2$ |

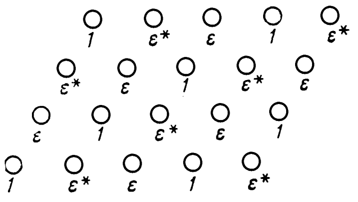


FIG. 2.

degenerate in frequency and differ from the fundamental mode ($\mathbf{q}=0$) by the amount $c/6L_0$, where $c$ is the velocity of light and $L_0$ the separation between the cavity mirrors.

In the case of an infinite rectangular array, the system defined by (3) can be factored. It will therefore be sufficient to consider a uniform periodic structure of radiators. In this case, in addition to the mode $\mathbf{q}=0$, we have the self-reproducing mode with $\mathbf{q}=k_0\mathbf{a}/z_T$, when neighboring channels radiate in antiphase.

It is thus clear that we have a positive answer to the first question. The Table shows the number of self-reproducing modes for distances between the mirror and the ends that are multiples of $z_T/4$.

The answer to the second question depends on the symmetry of the array. For a triangular array, it can be shown that the minimum $z$ for which the self-reproduction effect occurs is $z_T$. For a one-dimensional periodic structure, the minimum distance is $z=z_T/4$. This will reproduce a single structure with phase modulation $(0,\ \pi)$, i.e., $C(n)=\exp(i\pi n)$.

In applications in which phase screens are not used, the most convenient situation is that of a uniform field phase distribution over the channels. Since, for a sufficiently large number $N$ of tubes, there are a number of modes with similar losses, we have to consider the question of mode selection. We shall examine this for a triangular array. Some of the mode-selection methods can be based on the difference between the eigenfrequencies, i.e., on the use of elements with loss dispersion. Here, we shall consider the method based on the difference between the lateral structure of self-reproducing modes in an intermediate plane, e.g., at a distance $z_T/3$ from the ends of the tubes. Figure 3 shows the distribution of spots produced at this distance for different modes ($q=0, q=q_1, q=q_2$). The circles in Fig. 3 represent the channel ends and the points are the coordinates of the spot centers in the $z_T/3$ plane. The numbers 0, 1, 2 represent spot centers formed in the case where, for $z=0$, the field distribution has the eigenvalues $C(\mathbf{R})=1$, $C(\mathbf{R})=\exp(i\mathbf{q}_1\cdot\mathbf{R})$, and $C(R)=\exp(i\mathbf{q}_2\cdot\mathbf{R})$, respectively. An analogous picture is obtained in the $z=2z_T/3$ plane by introducing the replacement $1\to 2$, $2\to 1$. It is clear from these distributions that, by using the corresponding amplitude screen, located at a distance $z_T/3$ from the channel ends (i.e., at a distance $x_T/6$ from the mirror), it is possible to use the losses to select the mode with $C(\mathbf{R})=1$.

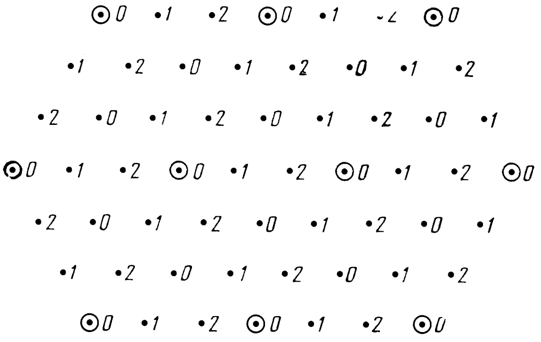


FIG. 3.

## THE INFLUENCE OF FINITE *N* AND OF ADJUSTMENT PRECISION

Having established the field structure in the limit of an infinite array of lasers, we can now proceed to the question of the influence of a finite aperture and of effects associated with mirror adjustment errors. It is clear that, as the number of tubes is reduced, the diffractive image deteriorates, which should lead to an increase in the losses. The characteristic size for which the tubes are efficiently coupled is determined by the angle of divergence of radiation issuing from a single laser, $\theta_f \gtrsim \theta_{\text{diff}} \sim \lambda/\Delta$, where $\Delta$ is the channel radius. We then have $|\delta\mathbf{R}|_M \sim \theta_f z_T \gtrsim a^2/\Delta$, i.e., the number of strongly coupled channels is $\sim a^2/\Delta^2$. Suppose that the size of the array is $L \gg |\delta\mathbf{R}|_M \gtrsim a^2/\Delta$. One would then expect that the structure of the fundamental modes is not greatly distorted. We shall seek solutions of (3) in the form

$$C(\mathbf{R})=\Psi(\mathbf{R})\exp(i\mathbf{q}_0\cdot\mathbf{R}),\ \mathbf{q}_0=0,\ \mathbf{q}_1,\ \mathbf{q}_2,$$

where $\mathbf{q}_0$ corresponds to one of the self-reproducing modes and $\Psi(\mathbf{R})$ is a function that varies appreciably over a characteristic scale $L$. Since $L \gg |\delta\mathbf{R}|_M$, we shall suppose that $\Psi(\mathbf{R})$ is a continuous function of its argument, which enables us to expand $\Psi(\mathbf{R})$ in (3) into a Taylor series in powers of $\mathbf{R}'-\mathbf{R}$. We can then reduce (3) to a differential equation for the envelope of the source amplitudes:

$$(\gamma-\gamma_0)\Psi=\gamma_0 D\Delta_\perp\Psi, \tag{8}$$

where

$$\gamma_0=\exp(-iz_T\mathbf{q}_0^2/2k_0),$$

$$D=\left(\frac{z_T}{2k_0}\right)^2\int(\nabla f(\rho))^2\,d\rho-i\frac{z_T}{2k_0}. \tag{9}$$

In deriving (8), it was assumed that the matrix $M$ had a finite second moment, i.e., $\Sigma\mathbf{R}^2M(\mathbf{R})<\infty$. In other words, the Fourier transform of $M(\mathbf{q})$ [see (6)] should have the following form for $|\mathbf{q}-\mathbf{q}_0|\to 0$:

$$M(\mathbf{q}_0+\delta\mathbf{q})=M(\mathbf{q}_0)[1-D(\delta\mathbf{q})^2+o(\delta\mathbf{q})^2],\ |\delta\mathbf{q}|\ll|\mathbf{q}_0|. \tag{10}$$

It can be shown that this expansion is obtained when $f(\rho)$ has no discontinuities, i.e., it vanishes on the boundary of the radiating region. The precise form of $f(\rho)$ is unknown, but the assumption that $f(\rho)$ has this property seems reasonable.

To estimate the complex "diffusion coefficient," we note that $|\nabla_f|\sim f/\Delta$ if the end of a channel radiates a plane wave. It follows that $\operatorname{Re} D\sim(a^2/\Delta^2)\operatorname{Im} D$. In the weak-coupling limit ($a\gg\Delta$), Eq. (8) reduces to the well-known diffusion eigenvalue problem.

Equation (8) requires the boundary conditions before the amplitude envelope can be determined for the array. It is clear that, near the edge of the region occupied by the lasers, the radiation amplitude should fall because of uncompensated diffractive losses. The shape of $\Psi(\mathbf{R})$ at this edge can be found by solving (3) by the Wiener-Hopf method (see, for example, Ref. 13) for a semi-infinite array of lasers. Using the analogy with the Milne problem in neutron transport theory,[14] one can argue physically that the solution of the problem for a semi-infinite array of lasers will yield the "extrapolated" length $l$, whose order of magnitude is the same as that of $a^2/\Delta$. The boundary condition for (8) is that $\Psi(\mathbf{R})$ must vanish at a distance $l$ from the boundary of the array.

Suppose that the lasers are confined to the band $-(L/2)\leqslant x\leqslant(L/2)$. The solution of (8) for the lowest mode satisfying the conditions $\Psi[\pm(L/2+l)]=0$ (or the equivalent "impedance conditions"[15] in the leading order in $l/L$, i.e., $\partial\Psi/\partial x\pm\Psi/l=0$ for $x=\pm L/2$) has the form

$$\Psi_1(x)=\cos[\pi x/(L+2l)]. \tag{11}$$

We then have

$$\gamma/\gamma_0=1-[\pi^2D/(L+2l)^2].$$

It is important to remember that the length $l$ is, in general, complex. Since Re $D\neq0$, and $|l|\ll L$, it is sufficient to confine our attention to the following expression when the losses, ie., $|\gamma|$, are calculated:[3)]

$$|\gamma|\approx1-(\pi^2/L^2)\,\mathrm{Re}\,D. \tag{12}$$

We have also taken into account the fact that $a^2/L\Delta\ll1$ and $a/L\ll1$. If the array occupies a circle of radius $L$, the lowest mode is described by a Bessel function,

$$\Psi_1(\mathbf{R})=J_0(\mu_0|\mathbf{R}|/(L+l)), \tag{13}$$

where $\gamma/\gamma_0=1-\mu_0^2D/(L+l)^2$ and $\mu_0$ is the first zero of the Bessel function $J_0$.

Thus, the approach formulated above enables us to find the modes for a finite array of sufficiently large size and a given $f(\boldsymbol{\rho})$.

We now turn to the question of the effect of a deviation of the distance to the mirror from the value $z_T/2$. It is clear that, for $z\neq z_T$, the diffractive images of the individual ends will spread. We shall calculate the correction to the eigenmode losses, neglecting the overlap between the image of a given channel and the ends of neighboring channels. It can be shown that, in this case, the eigenvalues given by

$$\frac{\gamma}{\gamma_0}=\int|f(\mathbf{q})|^2\exp\left(-i\frac{\delta z\mathbf{q}^2}{2k_0}\right)\frac{a\mathbf{q}}{(2\pi)^2}, \tag{14}$$

where $f(\mathbf{q})$ is the Fourier transform of $f(\boldsymbol{\rho})$ and $\delta z=z-z_T$. The correction to the frequency that is connected with Im $(\gamma/\gamma_0)$ can be found in an obvious way from the linear term in the expansion in $\delta z$. To find the losses due to the fact that $\delta z\neq0$, we must expand the exponential in (14) up to second order. The result can be reduced to the form

$$\delta|\gamma|\approx\left(\frac{\delta z}{2k_0}\right)^2\left[\left(\int\mathbf{q}^2|f(\mathbf{q})|^2\frac{d\mathbf{q}}{(2\pi)^2}\right)^2-\int\mathbf{q}^4|f(\mathbf{q})|^2\frac{d\mathbf{q}}{(2\pi)^2}\right]$$
$$=\left(\frac{\delta z}{2k_0}\right)^2\left\{\left[\int(\nabla f)^2d\boldsymbol{\rho}\right]^2-\int(\nabla^2f)^2d\boldsymbol{\rho}\right\}. \tag{15}$$

Order-of-magnitude estimates of the integrals yield

$$\delta|\gamma|\approx-\left(\frac{\delta z}{z_T}\right)\left(\frac{a^2}{2\pi\Delta^2}\right)^2.$$

Hence it is clear that, as the relative spacing $\Delta/a$ of the lasers is reduced, losses associated with the misalignment of the mirror rise rapidly. This behavior is consistent with intuitive ideas.

Finally, consider the effect of the angular adjustment of the mirror. Suppose that the inclination of the mirror is characterized by the angle $\boldsymbol{\theta}/2$, $\boldsymbol{\theta}=(\theta_x,\theta_y)$. In this case, the matrix $M(\mathbf{R},\mathbf{R}')$ can be a difference matrix. The inclination of the mirror gives rise to a shift of the diffractive image and to additional losses. The expression for the elements of the matrix $M$ can be conveniently written in the form

$$M(\mathbf{R},\mathbf{R}')=\int\frac{d\mathbf{q}}{(2\pi)^2}\exp\left(-i\frac{z_T\mathbf{q}^2}{2k_0}\right)f\left(\mathbf{q}-k_0\frac{\boldsymbol{\theta}}{2}\right)f^*\left(\mathbf{q}+k_0\frac{\boldsymbol{\theta}}{2}\right)$$
$$\cdot\exp\left[i\mathbf{q}\cdot(\mathbf{R}-\mathbf{R}')+ik_0\frac{\boldsymbol{\theta}}{2}\cdot(\mathbf{R}+\mathbf{R}')\right], \tag{16}$$

where $f(\mathbf{q})$ is the Fourier transform of the function $f(\boldsymbol{\rho})$. It is clear from (16) that the plane wave $\exp(i\mathbf{q}\cdot\mathbf{R})$ rotates during the propagation to the mirror and back, so that $q\to q+k_0\theta$. The matrix $M$ is nearly a difference matrix when $\theta\ll\lambda/a$, since $\exp(ik_0\boldsymbol{\theta}\cdot\mathbf{R})$ varies slowly in comparison with $M(\mathbf{R})$. As before, transforming to the continuous variable $\mathbf{R}$, and expanding $\Psi(\mathbf{R}')$ into a Taylor series, we obtain the large-scale equation

$$\gamma\Psi(\mathbf{R})=\gamma_0\Big[\exp(ik_0\boldsymbol{\theta}\cdot\mathbf{R})$$
$$+D\exp(ik_0\boldsymbol{\theta}\cdot\mathbf{R}/2)\frac{\partial^2}{\partial\mathbf{R}^2}\exp(ik_0\boldsymbol{\theta}\cdot\mathbf{R}/2)\Big]\Psi(\mathbf{R}). \tag{17}$$

This cannot be solved in general. We shall therefore find the change in $\gamma$ due to the inclination of the mirror, using perturbation theory in the parameter $k_0\theta L$ and taking as our starting point the eigenfunctions of (9) that form the orthonormal set $\Psi_{mn}(\mathbf{R})$. The expression for the correction to the fundamental-mode eigenvalue, for which $\int(\mathbf{Q}\cdot\mathbf{R})|\Psi_1|^2d\mathbf{R}=0$ is

$$\frac{\delta\gamma}{\gamma_1}=-\frac{1}{2}k_0^2\int(\boldsymbol{\theta}\cdot\mathbf{R})^2|\Psi_1|^2d\mathbf{R}-k_0^2\sum_{n,m}\frac{\gamma_{nm}|V_{nm}|^2}{\gamma_1-\gamma_{nm}}, \tag{18}$$

where $V_{nm}=\int\Psi_{nm}\boldsymbol{\theta}\cdot\mathbf{R}_1d\mathbf{R}$. Since $\gamma_1-\gamma_{nm}\sim D/L^2$, we find that, in second-order perturbation theory,

$$\delta\gamma\approx\frac{k_0^2\theta^2L^4}{D}\sim\left(\frac{\theta}{\theta_{\text{diff}}}\right)^2\left(\frac{L}{a}\right)^4.$$

Estimates show that, as the size of the array of lasers increases, the precision of adjustment rapidly becomes critical. The formula given by (18) is obviously valid for $|\delta\gamma|\ll1$.

When the mirror inclination angle is not small, the solutions of (3) with $M(\mathbf{R},\mathbf{R}')$ given by (16) exhibit a number of singularities for certain values of $\boldsymbol{\theta}$.

In particular, when $k_0\boldsymbol{\theta}=4\pi\mathbf{b}$, where $\mathbf{b}$ is an arbitrary reciprocal lattice vector, $M(\mathbf{R},\mathbf{R}')$ is again a difference matrix. The mode structures of the infinite array of lasers is then the same as that found for $\boldsymbol{\theta}=0$, but the expression for the eigenvalues $\gamma$ now acquires the factor

$$\mu=\int d\boldsymbol{\rho}f^2(\boldsymbol{\rho})\exp(ik_0\boldsymbol{\theta}\cdot\boldsymbol{\rho}), \tag{19}$$

which governs the losses due to the phase modulation of the field amplitude reflected from the inclined mirror. These losses are small for $\Delta\ll a$:

$$\delta\gamma/\gamma \sim -(\Delta/a)^2(k_0\theta a)^2.$$

For $k_0\boldsymbol{\theta} = 2\pi\mathbf{b} \neq 4\pi\mathbf{b}'$, we again have three high-$Q$ modes, namely, three plane waves with wave vectors $\mathbf{q} = k_0\boldsymbol{\theta}/2$, $k_0\boldsymbol{\theta}/2 + \mathbf{q}_1$, and $k_0\boldsymbol{\theta}/2 + \mathbf{q}_2$.

The foregoing discussion can be illustrated by a simple geometric analysis (Fig. 4). Suppose that the field amplitude of one of the collective modes for $\boldsymbol{\theta} = 0$ is $E(\boldsymbol{\rho})$. After reflection from the inclined mirror, a field of amplitude $E(\boldsymbol{\rho})\exp(-ik_0\boldsymbol{\theta}\cdot\boldsymbol{\rho}/2)$ will again enter the channel ends and acquire an additional phase factor $\exp(ik_0\boldsymbol{\theta}\cdot\boldsymbol{\rho})$.

For $k_0\boldsymbol{\theta} = 2\pi\mathbf{b}$, the distances $l_1$ and $l_2$ from the ends of neighboring channels will again differ by an integral number of half waves, and the function $C(\mathbf{R})\exp(-ik_0\boldsymbol{\theta}\cdot\mathbf{R}/2)$ will be a mode of the resonator with the rotated mirror. On the other hand, when $k_0\boldsymbol{\theta} = 4\pi\mathbf{b}$, the modes become identical with those corresponding to $\boldsymbol{\theta} = 0$. The factor given by (19) describes losses due to the oblique incidence of radiation on the channels.

In the case of a triangular array, there are a number of values of $\boldsymbol{\theta}$ for which high-$Q$ modes are possible (for $\Delta \ll a$).

Let us consider, for example, the case $k_0\boldsymbol{\theta} = \mathbf{q}_1$. The vector $3k_0\boldsymbol{\theta}$ then coincides with one of the reciprocal lattice vectors. Consider three plane waves with wave vectors $\mathbf{q}_a = k_0\boldsymbol{\theta}/2 = \mathbf{q}_1/2$, $\mathbf{q}_b = \mathbf{q}_1/2$, $\mathbf{q}_c = 5\mathbf{q}_1/2$. When the matrix $M(\mathbf{R}, \mathbf{R}')$, given by (16), is applied to one of the three plane waves, it turns it into one of the other two:

$$\sum_{\mathbf{R}'} M(\mathbf{R},\mathbf{R}')\exp(i\mathbf{q}_a\cdot\mathbf{R}') = \mu\exp(i\mathbf{q}_b\cdot\mathbf{R})\exp(-2\pi i/3),$$
$$\sum_{\mathbf{R}'} M(\mathbf{R},\mathbf{R}')\exp(i\mathbf{q}_b\cdot\mathbf{R}') = \mu\exp(i\mathbf{q}_c\cdot\mathbf{R})\exp(-2\pi i/3),$$
$$\sum_{\mathbf{R}'} M(\mathbf{R},\mathbf{R}')\exp(iq_c\cdot\mathbf{R}') = \mu\exp(i\mathbf{q}_a\cdot\mathbf{R}). \qquad (20)$$

It is therefore natural, in this case, to seek high-$Q$ modes (for $\Delta \ll a$) in the form of linear combinations of these three plane waves. The eigenvalues turn out to be

$$\gamma_n = \mu\exp\left(\frac{2\pi i}{3}n - \frac{4\pi i}{9}\right), \qquad n=0, 1, 2.$$

The field amplitude distribution in the channels is shown in Fig. 5 for one of the modes. The fields of the two other modes are obtained by applying translations $\mathbf{a}_1$ and $2\mathbf{a}_1$, and the amplitudes $c_1$, $c_2$, $c_3$ are, respectively, given by $c_1 = 1 + 2\cos(2\pi/9)$, $c_2 = 1 - 2\cos(\pi/9)$, $c_3 = 1 + 2\cos(4\pi/9)$. The particular feature of this case is the considerable difference between the intensities in different channels.

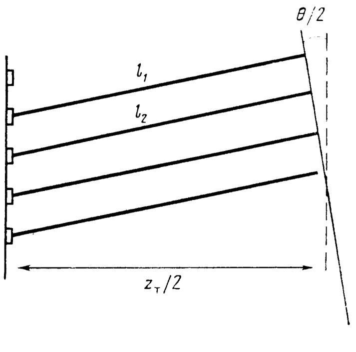


FIG. 4.

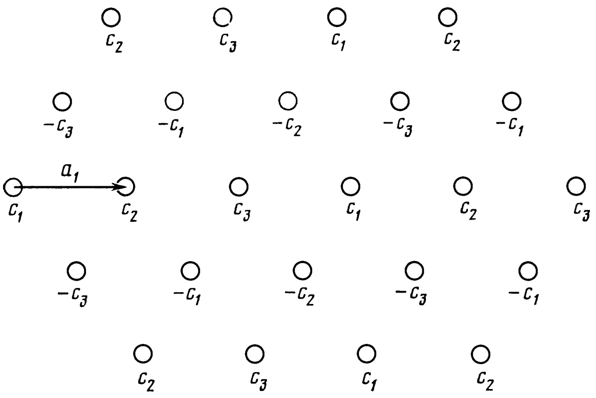


FIG. 5.

It is thus clear that, when $k_0\boldsymbol{\theta} = m\mathbf{q}_1 + n\mathbf{q}_2$, we have virtually dissipation-free modes (for $\Delta \ll a$). If, in addition, we have $k_0\boldsymbol{\theta} = 2\pi\mathbf{b}$, the mode frequencies are equal to those found for $\boldsymbol{\theta} = 0$ and, for $k_0\boldsymbol{\theta} = 4\pi\mathbf{b}$, the modes themselves are the same.

Analysis shows that there are no other values of $\boldsymbol{\theta}$ for which such modes are possible. The dependence of the mode $Q$- values for $\boldsymbol{\theta} \neq 0$ on the dimensions of the array and on a small deviation of the inclination of the mirror from the fixed values defined by $k_0\boldsymbol{\theta} = m\mathbf{q}_1 + n\mathbf{q}_2$ can be found by the method used for $\boldsymbol{\theta} = 0$, and turns out to be similar to that found for the latter case.

## INFLUENCE OF THE ACTIVE MEDIUM

We have assumed, so far, that all the channels were identical. This enabled us to characterize the propagation of radiation along the channels by the single constant $A$ [see (13)], which was the same for all the channels. In practice, when an active medium is present in the channels, the quantity $A$ is a functional of the field amplitude $C(\mathbf{R})$ and of its distribution $f(\boldsymbol{\rho})$ over the channel aperture. Moreover, there are several factors that will ensure that the phase gain in the different channels will be different even in the absence of an active medium. In the situation most often encountered in experiments the phase gain is uncorrelated in different channels and constitutes a random variable. Nevertheless, direct averaging of the system (3) over the phase fluctuations is unjustified in general. Note that in the limit of low relative pulse duration, $\Delta \ll a$, or, more precisely, if $\theta_s \gg \lambda/a$ holds ($\theta_f$ is the divergence angle of radiation leaving a channel), then diffraction couples a large number of channels: $\sim(\theta_f a/\lambda)^2 \gg 1$. This condition corresponds to effective self-averaging of phase fluctuations per transit across the resonator, which enables us to average (e). We take $A$ in the form $A(\mathbf{R}) = A_0\exp\{i\delta\varphi(\mathbf{R})\}$. We assume further that the random quantity $\delta\varphi(\mathbf{R})$ has a Gaussian distribution and that there is no correlation between phases in neighboring channels. We then obtain $\langle A\rangle = A_0\exp(-\langle\delta\varphi^2\rangle/2)$. The spread in the phase gain in different channels thus leads to additional damping of the field and to a higher generation threshold:

$$\gamma' \sim A_0 \gamma \exp(-\langle \delta\varphi^2 \rangle /2).$$

When the active medium is present, we have $A_0 \sim \exp(gL_k)$, where $g$ is the gain and $L_k$ the channel length [it is more correct to speak of the gain corresponding to a mode with amplitude distribution $f(\boldsymbol{\rho})$]. If the array of lasers has a large enough aperture, and if we neglect random spreads, we find that $A(\mathbf{R})$ is a smooth function of the channel coordinates as compared with $M(\mathbf{R}-\mathbf{R}')$. This enables us to transform in (3) to the large-scale description. It can be shown from this that the equation for the field amplitude then becomes

$$[(\gamma/\gamma_0)\exp(-gL_k)-1]\Psi = D\Delta_\perp \Psi, \tag{21}$$

where the eigenvalue is naturally written in the form $\gamma = |\gamma|\exp(i\nu) = \exp(g_n L_k + i\nu)$, $g_n$ is the threshold gain for an infinite array, and $\nu$ is a phase factor determined by the resonator parameters. Since the gain depends on the field intensity in the channel, i.e., $g = g(|\Psi(\mathbf{R})|^2$, Eq. (21) is a nonlinear complex second-order equation that cannot be solved in the general case. To illustrate the application of (21), consider the case of near-threshold generation for which $\exp[(g_n - g)L_k] \simeq 1 + (g_n - g)L_k$ and the radiation intensity in a channel is less than the saturation value $|\Psi_s|^2$, $g = g_0(1 - |\Psi|^2/|\Psi_s|^2)$, where $g_0$ is the unsaturated gain. Using the approximate boundary condition $\Psi(\pm L/2) = 0$ for a band of width $L$ occupied by the channel ends, it can be shown that Eq. (21) for the lowest mode reduces to a real equation for $Y = |\Psi|$:

$$Y'' + \beta(g_0 - g_n - g_0 Y^2/|\Psi_s|^2)Y = 0 \tag{22}$$

where $\beta = L_k \operatorname{Re} D/|D|^2$. This equation has been extensively investigated and the qualitative behavior of its solutions can be understood in terms of the analogy with the classical anharmonic oscillator. The generation threshold is defined by $g_{\text{th}} = g_n + \pi^2/\beta L^2$. The field amplitude can be expressed in terms of the Jacobi elliptic function:[16]

$$Y = p\,\mathrm{sn}\left[\frac{\alpha}{p}\left(x + \frac{L}{2}\right), \varkappa\right], \tag{23}$$

where

$$p = |\Psi_s| \left(\frac{g_0 - g_n}{g_0}\right)^{1/2} \left[1 - \left(1 - \frac{\alpha^2}{\alpha_m^2}\right)^{1/2}\right],$$

$$\alpha_m^2 = \beta|\Psi_s|^2 \frac{(g_0 - g_n)^2}{2g_0}$$

and the parameter $\alpha$ is determined from the given parameter $L$ and the excess above the generation threshold, using the equation

$$L = \frac{2p(\alpha)}{\alpha} \int_0^1 dt\{(1-t^2)[1-\varkappa^2(\alpha)t^2]\}^{-1/2},$$

$$\varkappa^2 = 2\left(\frac{\alpha_m}{\alpha}\right)^2 \left[1 - \left(1 - \frac{\alpha^2}{\alpha_m^2}\right)^{1/2}\right] - 1.$$

When the excess above the generation threshold is small, i.e.,

$$(g_0 - g_n - \pi^2/\beta L^2) \ll \pi^2/\beta L^2$$

the field distribution has the usual form

$$Y \sim \sin\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right].$$

When the laser array is large enough, so that $g_0 - g_n \gg \pi^2/\beta L^2$, the amplitude distribution has the form

$$Y = |\Psi_s| \left(\frac{g_0 - g_n}{g_0}\right)^{1/2} \times \frac{\mathrm{ch}[(2\beta(g_0-g_n))^{1/2}L/2] - \mathrm{ch}[(2\beta(g_0-g_n))^{1/2}x]}{\mathrm{ch}[(2\beta(g_0-g_n))^{1/2}L/2] + \mathrm{ch}[(2\beta(g_0-g_n))^{1/2}x]}.$$

The generated power is then determined by the excess of the gain above the generation threshold, and can be calculated from the Rigrod formula.[17]

In conclusion, let us summarize our main results. The collective modes of a periodic laser array are similar to the modes found for a plane-parallel resonator. Mode discrimination based on losses is, however, found to be appreciably greater. The number of low-loss modes depends on the separation between the array and the plane mirror, and on the symmetry of the laser array.

In addition to the in-phase mode, there are modes with phase modulation of low-loss channels. Selection of a phase-locked mode in the triangular array can be achieved by using an amplitude array inside the resonator. For a quadratic laser array in which the distance to the coupling mirror is $z_T/4$, only the antiphase mode exhibits low losses. Its radiation can be corrected by inserting a periodic phase screen at the exit.

Our analysis has shown that the angular misalignment of the mirror has a significant influence on the modes and the structure of the generated radiation. Random phase gains in the individual channels tend to raise the collective generation threshold.

[1)]A phase-locked array of seven Nd lasers was recently constructed[7] using a phase-conjugate mirror.

[2)]It is important to note that a similar eigenvalue problem for a resonator with periodic boundaries was solved in Ref. 10. The distinctive feature of our approach is that the field distribution over the ends of all the channels is assumed to be given. Resonators with lattice mirrors, and also with a set of retroreflecting mirrors, were investigated in Refs. 11 and 12, but the influence of the finite size of the mirrors and of the misadjustment of the resonator was not examined.

[3)]However, we note that the loss difference between self-reproducing modes with $\mathbf{q}_0 = 0$, $\mathbf{q}_1$, $\mathbf{q}_2$ is due to the difference between their extrapolated lengths, since $D$ is independent of $\mathbf{q}_0$ and additional losses due to a slight misalignment of the mirror and a difference between $z$ and $z_T$ are also the same for all three modes.


[1]G. I. Kozlov, V. A. Kuznetsov, and V. A. Masyukov, Pis'ma Zh. Tekh. Fiz. **4**, 129 (1978) [Sov. Tech. Phys. Lett. **4**, 53 (1978)].
[2]M. B. Spencer and W. E. Lamb Jr., Phys. Rev. A **5**, 893 (1972).
[3]W. Streifer, D. Jevick, R. D. Burnham *et al.*, Appl. Phys. Lett. **45**, 343 (1984).
[4]J. Katz, E. Kapon, S. Margalist, and A. Yariv, IEEE J. Quantum. Electron. **QE-20**, 875 (1984).
[5]A. F. Glova, Yu. A. Dreizin, O. R. Kachurin *et al.*, Pis'ma Zh. Tekh. Fiz. **11**, 249 (1985) [Sov. Tech. Phys. Lett. **11**, 102 (1985)].
[6]G. E. Palma and W. J. Fader, Proc. SPIE **440**, 153 (1984).
[7]A. F. Vasil'ev, A. A. Mak *et al.*, Zh. Tekh. Fiz. **51**, 300 (1986) [Sov. Phys. Tech. Phys. **31**, 191 (1986)].
[8]V. V. Antyukhov, A. F. Glova, O. R. Kachurin *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 63 (1986) [JETP Lett. **44**, 78 (1986)].
[9]V. K. Ablekov, Yu. N. Babaev, M. F. Dmitriev *et al.*, Dokl. Akad. Nauk SSSR **267**, 1114 (1982) [Sov. Phys. Dokl. **267**, 1058 (1982)].
[10]V. M. Marchenko, T. M. Makhvaladze, A. M. Prokhorov *et al.*, Zh. Eksp. Teor. Fiz. **74**, 868 (1978) [Sov. Phys. JETP **47**, 453 (1978)].

[11]V. K. Ablekov, V. S. Belyaev, V. M. Marchenko *et al.*, Dokl. Akad. Nauk SSSR **230**, 1066 (1976) [Sov. Phys. Dokl. **21**, 575 (1976)].
[12]E. E. Bagdasarov, Ya. Z. Virnik, S. P. Vorotilin *et al.*, Kvantovaya Elektron. (Moscow) **8**, 2397 (1981) [Sov. J. Quantum Electron. **8**, 465 (1981)].
[13]B. Noble, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*, Pergamon Press, 1958 [Russ. transl., IIL, M., 1962].
[14]B. Davison, *Neutron Transport Theory*, Oxford University Press, Oxford, 1957 [Russ. transl., Atomizdat, M., 1960, Chap. 6].
[15]L. A. Vaĭnshteĭn, *Open Resonators and Waveguides* [in Russian], Sovetskoe Radio, M., 1966, p. 35.
[16]M. A. Lavrent'ev and B. V. Shabat, *Methods in the Theory of Functions of a Complex Variable* [in Russian], Nauka, Moscow, 1973.
[17]A. Maitland and M. H. Dunn, *Laser Physics*, Am. Elsevier, 1970 [Russ. transl., Nauka, Moscow, 1978].

Translated by S. Chomet