Evolution of inhomogeneities in inflationary models in a theory of gravitation with higher derivatives

L. A. Kofman, V. F. Mukhanov, and D. Yu. Pogosyan

Institute of Astrophysics and Physics of the Atmosphere, Estonian Academy of Sciences and Institute of Nuclear Research, USSR Academy of Sciences
(Submitted 16 February 1987)
Zh. Eksp. Teor. Fiz. **93**, 769–783 (September 1987)

The evolution of a weakly inhomogeneous universe is considered in the framework of theories of gravitation with higher derivatives. It is shown that for a large class of initial conditions the period of inflationary expansion of the background isotropic model is realized here as a quasi-de Sitter intermediate asymptote. The problem of the evolution of scalar perturbations of the metric is solved. In the case of pure R^2 gravitation, the results are obtained in analytic form. Numerical integration of the equations shows that the qualitative behavior of the perturbations is unchanged in models with conformal anomaly with allowance for polarization terms. Bounds on the parameters of the theory that follow from astrophysical observations are obtained.

§1. INTRODUCTION

The appearance of inflationary models of the evolution of the universe has led to a real possibility of solving a number of important cosmological problems. By an inflationary stage of evolution is meant a period of exponential (or quasiexponential) expansion of the universe made possible by the presence of an effective cosmological constant in the model at this period.[1] Two simple ways of introducing an effective cosmological constant are known. The first is by means of a scalar field φ with potential $V(\varphi)$ which either possesses a metastable state[2] or makes it possible to realize a regime of slow rolling.[3] The second possibility is through quantum-gravitational effects,[4] which lead to higher derivatives in the equations for the gravitational field. Both of these mechanisms may operate in superstring or Kaluza-Klein models—either separately or in conjunction.

In the original variants of the inflationary models,[2,4] the exact de Sitter solution was used. Subsequently, difficulties in the original variants of the models were overcome by using the more general quasi-de Sitter solution.[3] The presence of an inflation stage is then no longer an exotic phenomenon, realized for specific initial conditions, but a characteristic regime through which an isotropic metric passes under a large set of initial conditions in models of both types.[5,6]

With the passage of time, the inflationary stage comes to an end, and the field that controls the inflation goes over into an oscillatory regime, the effective cosmological constant disappears, and quasiparticles with dust equation of state arise. The massive quasiparticles then decay into an ultrarelativistic plasma, which is thermalized to temperatures $T \sim 10^{10}$–10^{14} GeV, and the universe goes over to a radiation-dominated Friedmann stage of expansion.

In this overall picture, it is convenient to identify three successive periods: 1) the strictly de Sitter or quasi-de Sitter stage; 2) the decay of the effective cosmological constant due to the rapid change in the value of the field that controls the inflation—the "fast rolling" stage; 3) the stage of dominance of the quasiparticles, which subsequently decay into the hot plasma.

Inherent in inflationary models is the following property, important for cosmological applications: In the inflationary stage it is in principle possible to generate from vacuum quantum fluctuations[7,8] long-wave inhomogeneities with an amplitude sufficient for galaxy formation.[7,9,10] In the simplest case of a single field controlling the inflation, these lead to adiabatic (inflation) perturbations of the metric with an almost flat spectrum.[9–11,12] But if in the inflation stage there is present (but not dominant) another lesser field, χ, that itself (or the products of its decay) makes the dominant gravitational contribution in the late Friedmann stages of the expansion of the universe, its fluctuations lead to appreciable isothermal (isoinflation) perturbations. It is important that the spectrum of these fluctuations may deviate strongly from a flat spectrum.[13]

In this paper, we give a systematic quantitative theory of the generation and evolution of adiabatic (inflation) perturbations of the metric in all three stages in the expansion of the very early universe for quantum-gravitational inflationary models, comparing the results with the case of models with scalar field. Various aspects of this problem were considered earlier in Ref. 14.

The paper is arranged as follows. We first describe the stages of the evolution of the background metric in the model with conformal anomaly, when besides the exact de Sitter solution there are, for a large set of initial data, solutions with quasi-de Sitter intermediate asymptotic behavior.

Then, having written down a self-consistent system of equations for the perturbations of the metric, we trace how the finite fluctuations of the metric needed for the formation of galaxies can be generated from initial vacuum fluctuations. In the physically most interesting case without conformal anomaly, we obtain an analytic solution of the problem (Sec. 3). In Sec. 4, we give the results of numerical solution of the problem in the presence of a conformal anomaly. In the conclusions, we obtain constraints on the parameters of the model from astrophysical observations.

§2. BACKGROUND MODEL

Einstein's equations with allowance for the quantum-gravitational corrections (in the class of conformally flat metrics with small perturbations) have the form

 0038-5646/87/090433-08$04.00

$$R_k^{\ i} - \frac{1}{2}\delta_k^{\ i} R$$
$$= \frac{1}{H_0^2}\left(R_l^{\ i} R_k^{\ l} - \frac{2}{3} R R_k^{\ i} - \frac{1}{2}\delta_k^{\ i} R_m^{\ l} R_l^{\ m} + \frac{1}{4}\delta_k^{\ i} R^2\right)$$
$$- \frac{1}{6M^2}\left(2R^{;i}_{\ ;k} - 2\delta_k^{\ i} R^{;l}_{\ ;l} - 2RR_k^{\ i} + \frac{1}{2}\delta_k^{\ i} R^2\right). \qquad (1)$$

The original Lagrangian contains the combination

$$L = -\frac{1}{16\pi G}\left(R - \frac{R^2}{6M^2}\right) \qquad (2)$$

quadratic in the curvature R and the nonlocal term

$$\frac{1}{16\pi G}\frac{1}{H_0^2} F(g_{ik}),$$

which leads to a gravitational conformal anomaly of the quantized fields,

$$\langle T_l^{\ l}\rangle = -\frac{1}{2880\pi^2}\left[k_1 C_{iklm} C^{iklm} + k_2\left(R_{ik} R^{ik} - \frac{1}{3} R^2\right) + k_3 \Box R\right] \qquad (3)$$

with model-dependent numerical coefficients k_1, k_2, k_3. Comparing (1), (2), and (3), we find

$$\frac{1}{H_0^2} = \frac{k_2}{360\pi}\frac{1}{M_{Pl}^2}, \qquad \frac{1}{M^2} = \frac{1}{M_0^2} + \frac{k_3}{360\pi}\frac{1}{M_{Pl}^2},$$

where $M_{pl} = G^{-1/2}$, $c = \hbar = 1$. In what follows, we shall not distinguish M_0 and the (renormalized) M. In the case $k_2 \neq 0$, Eqs. (1) have the exact de Sitter solution $R^{\ i}_k = (1/4)\delta^i_k R$ with curvature $R = -12H_0^2$. However, it can be shown that in the most characteristic cases of the sets of fields in supergravity either $H_0 \sim M_{Pl}$ (going beyond the framework of the single-loop approximation) or $H_0 \to \infty$, $k_2 = 0$, and there is no anomaly at all. In the physically interesting case without conformal anomaly only the second term obtained from the Lagrangian (2) remains on the right-hand side of Eq. (1). Such terms quadratic in the curvature appear in numerous theories: in quantum gravity with higher derivatives,[15] in Kaluza-Klein models, in superstring models.[16] However, what is important for us is that the theory (2) can have a bearing on the very early universe, admittedly under the rather strong restriction $M \lesssim 10^{14}$ GeV (see below, and also Ref. 17) on the bare mass M, which occurs in (2) and determines the cosmological status of the model.

The time evolution of the background isotropic metric (with flat comoving space) is determined by the 0–0 component of Eq. (1) for the scale factor $a(t)$, various aspects of which have been considered, for example, in Refs. 4 and 18. In addition to these studies, we give here a complete qualitative investigation of the equations for the background metric and describe in detail their inflationary solutions. The equation for $a(t)$ is most readily investigated by going over to the variable $H = \dot{a}/a$:

$$H^2 = \frac{H^4}{H_0^2} - \frac{1}{M^2}(2H\ddot{H} + 6\dot{H}H^2 - \dot{H}^2), \qquad (4)$$

where the dot denotes differentiation with respect to the physical time t. In the dimensionless $y = \dot{H}/M^2$ and $x = H/M$, Eq. (4) becomes

$$\frac{dy}{dx} = -3x + \frac{y}{2x} - \frac{x}{2y} + \beta\frac{x^3}{2y}, \qquad (5)$$

where $\beta = (M/H_0)^2$. The phase portrait of this equation

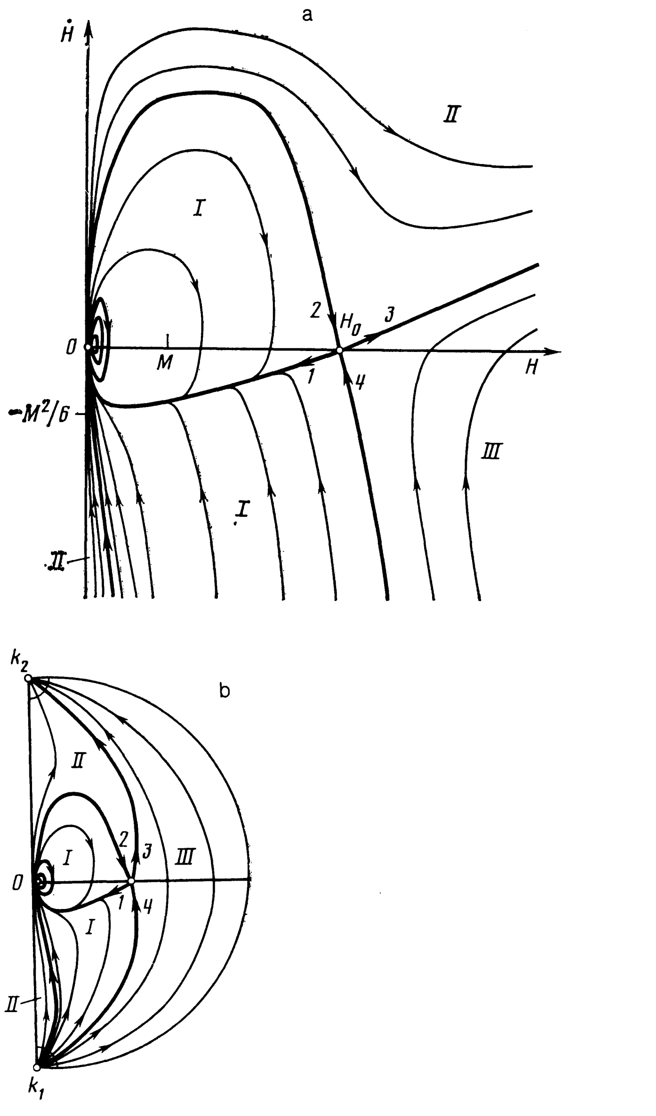

FIG. 1. Phase portrait and projection of the Poincaré sphere for Eq. (4) at finite H_0.

(Fig. 1a) in the coordinates y–x is symmetric about the y axis. The half-plane $x < 0$ corresponds to contraction of the universe, the half-plane $x > 0$ to expansion. Here, there are two singular points: a complicated singularity with the nature of a focus at (0,0) with twisting trajectories (all trajectories pass through the zero point) that corresponds to Friedmann asymptotic behavior, and the saddle $(0, 1/\beta^{1/2})$, which corresponds to the exact de Sitter solution $a(t) = a_0 \times \exp(H_0 t)$ of Eqs. (1) and (4). The complete half-plane is divided into three regions, I–III, by the separatrices 1–4 of the saddle point. The equations of the separatrices near it have the form

$$y \approx \frac{\pm 3 - (9 - 4\beta)^{1/2}}{2}\left(x - \frac{1}{\beta^{1/2}}\right).$$

Separatrix 1 leaves the saddle point and enters the lower quadrant and then winds into the focus. Separatrix 2 leaves the saddle point and enters the upper quadrant and, passing through the singularity at (0, 0), goes away to infinity in the lower quadrant; the point (0,0) is not a stationary point for this separatrix, but it is such a point (a focus) for separatrix

1, the complexity of this singularity being due to these facts. The separatrices 3 and 4 leave the saddle point on opposite sides of the x axis and go away to infinity.

The trajectories in region I begin at infinity for $y<0$, corresponding to a singularity at $t=0$. Then, passing between separatrices 2 and 4, they reach separatrix 1 and continue to move near it to the singularities at (0,0), and, passing through them, enter the interior region between separatrices 1 and 2. It is remarkable that the entire separatrix 1 is the intermediate asymptotic behavior for all trajectories of region I, some of which reach it from below, while others, close to separatrix 2 (for $y<0$), reach it from above from the interior region, passing before this through the point (0,0). All the trajectories of region I terminate in the singularity (0,0) as $t\to\infty$ with regular Friedmann asymptotic behavior. The trajectories of region II begin at infinity for $y<0$ from the singularity at $t=0$ between the y axis and separatrix 2, pass through the point (0,0), which is not singular for them, and go away at infinity into a singularity as $t\to\infty$. The trajectories of region III pass to the right of separatrices 3 and 4 and behave like the trajectories of region II. The qualitative picture of Eq. (4) is completed by an investigation of the singular points at infinity by means of the Poincaré sphere, the projection of which is shown in Fig. 1b. At infinity, there are two complicated singularities of the type of a repulsive node, k_1, and an attractive node, k_2.

In the theory without conformal anomaly with the Lagrangian (2), it is necessary to go to the limit $H_0\to\infty$ in Eqs. (1) and (4). Then in the phase portraits of Figs. 1a and 1b the saddle singularity is displaced to infinity along the x axis, and there remains the single separatrix 1, which begins at infinity and ends at a singularity at (0,0) of focus type. In this case, the behavior of the trajectories is completely analogous to the behavior of the trajectories from region I in the case of finite H_0, k_2 (see Figs. 2a and 2b).

We consider in more detail the trajectories of the physically interesting region I, which have regular Friedmann asymptotic behavior. The typical phase trajectories of this region rapidly (as measured by the time t) reach the separatrix I from below. The most interesting fact is that near this separatrix there is a quasi-de Sitter regime (intermediate asymptotic behavior) for which

$$a(t)\infty\exp\left(\int H(t)\,dt\right),\qquad |\dot H|\ll H^2,$$

and an inflationary stage can be realized. We recall that in the considered model there exists an exact de Sitter solution $(0,1/\beta^{1/2})$, from which, however, one cannot depart without invoking additional considerations (stationary point). A physical reason for deviation from the exact de Sitter solution is provided by quantum fluctuations of the metric.[9] In the original models with conformal anomaly, it was assumed that by virtue of these deviations some of the space will evolve directly along the separatrix 1. In such a scenario, there must be strong spatial inhomogeneities, since quantum fluctuations "throw" some of the geometry into region I and some into region II.

In the model with conformal anomaly, this difficulty can be avoided if one starts with the trajectories of region I [it is merely required that they reach separatrix 2 in order to ensure that the subsequent inflation will be sufficient, $\ln(a(t)/a(t_0))\gtrsim 65$]. Then the trajectories of this region will

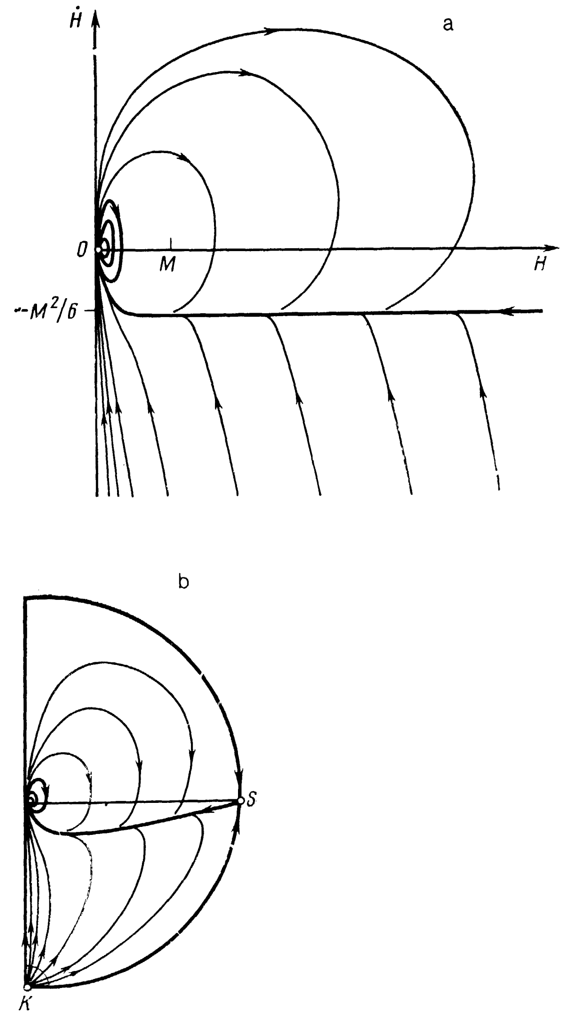

FIG. 2. Phase portrait and projection of the Poincaré sphere for Eq. (4) in the case $H_0\to\infty$.

have an inflationary intermediate asymptotic behavior corresponding to motion along separatrix 1. It is easy to see in the physically most interesting case $\beta\ll 1$ that this separatrix in the section from the saddle to the first intersection point (0, 0) is situated near the zero isocline (on which $\ddot H=0$), which is described by the equation

$$\dot H=3H^2-[(9-\beta)H^4+M^2H^2]^{1/2}. \qquad (6)$$

Therefore, the equation of the separatrix itself on this section is equal up to terms $O(\beta^2)$, to Eq. (6).

The solution of Eq. (6) has the form

$$H(t)\approx 2H_0\frac{\operatorname{sh}(\tau-M^2t/6H_0)\operatorname{sh}[\tau-M^2t/6H_0-\operatorname{arcth}(\beta^{1/2}/3)]}{\operatorname{sh}[2\tau-M^2t/3H_0-\operatorname{arcth}(\beta^{1/2}/3)]}. \qquad (7)$$

From (7) in the limit $\beta\to 0$ we obtain

$$H=H_0\operatorname{th}(\tau-M^2t/6H_0). \qquad (8)$$

In (7) and (8), τ is related to the initial value of H_1 at $t=0$ as follows:

$$\tau=-\frac{1}{2}\ln\left(\frac{H_0-H_1}{H_0+H_1}\right)\approx-\frac{1}{2}\ln\frac{\delta R}{2R_0}. \quad (9)$$

Integrating Eq. (7), we find the scale factor $a(t)$:

$$a(t)=a_1\left[\operatorname{ch}\left(\tau-\frac{M^2t}{6H_0}-\frac{1}{2}\operatorname{arcth}\frac{\beta^{1/2}}{3}\right)\right]^{-\gamma_+}\times\left[\operatorname{sh}\left(\tau-\frac{M^2t}{6H_0}-\frac{1}{2}\operatorname{arcth}\frac{\beta^{1/2}}{3}\right)\right]^{\gamma_-},$$

where

$$\gamma_\pm=\frac{3H_0^2}{M^2}\left(\frac{3}{(9-\beta^2)^{1/2}}\pm1\right),\qquad a_1=a(t=0).$$

In the limit $\beta\to0$ we obtain then

$$a(t)\propto\left[\operatorname{ch}\left(\tau-\frac{M^2t}{6H_0}\right)\right]^{-6H_0^2/M^2}$$

(see also Refs. 17 and 19). It follows from (9) that initially, during a time

$$\Delta t_1\propto-\frac{3H_0}{M^2}\ln\frac{\delta R}{R_0},$$

the universe passes through an exponential stage of expansion, $a\propto\exp(H_0t)$, and then, during a time $\Delta t_2\approx3H_0/M^2$, through a quasiexponential stage with

$$a(t)=a_2\exp\left(-\frac{M^2}{12}(t-t_2)^2\right). \quad (10)$$

In the case without conformal anomaly (as $H_0\to\infty$), the equation of the separatrix on its straight section takes the form

$$\dot H=-M^2/6$$

and the scale factor is described by Eq. (10).

The winding part of the separatrix is described by the scale factor

$$a(t)\approx a_1t^{2/3}\left(1+\frac{2\sin(M(t-t_1))}{3M(t-t_1)}\right), \quad (11)$$

which corresponds to "scalaron" fluctuations.[4]

Thus, all solutions of Eq. (4), for arbitrary values of the parameter $\beta>0$, formally begin from a singularity (except for the single exact de Sitter solution). Some of them, belonging to region I, necessarily pass through the quasi-de Sitter asymptote:

$$R_k^i\approx{}^1/_4\delta_k^iR,\qquad R\approx-12H^2,\qquad|\dot H|\ll H^2.$$

The relative fraction of such solutions depends on the size of region I, i.e., on the parameter $\beta=M^2/H_0^2$. With decreasing β, the size of region I increases, and in the limit $\beta\to0$ all solutions have the intermediate de Sitter behavior. It is remarkable that although the Lagrangian (2) does not explicitly contain a Λ term, the R^2 term effectively acts as one and effects the inflationary expansion. In the case of finite H_0, there is an exact de Sitter solution with $R=-12H_0^2$. However, here too a whole family of quasi-de Sitter solutions, whose curvature may differ by several orders of magnitude from the curvature of the exact de Sitter solution, can also be realized. This circumstance significantly extends the regions of values of the complete set of parameters in a number of theories.[20]

§3. PERTURBATIONS IN R^2 GRAVITATION WITHOUT CONFORMAL ANOMALY

In theories with Lagrangians[1)] of the type (2) that contain only terms local in the curvature, the behavior of the perturbations can be investigated by analytic methods. We consider a homogeneous isotropic universe with small scalar perturbations of the metric. Since throughout almost the entire period of evolution in which we are interested the metric of the background homogeneous universe can be approximated to a good degree of accuracy by the flat model, we shall restrict ourselves in what follows for simplicity to considering a universe with zero spatial curvature ($k=0$). The complete metric of the isotropic universe with small scalar perturbations has in the conformally Newtonian gauge[12,21] the form

$$ds^2=a^2(\eta)[(1+2\Phi)d\eta^2-(1-2\psi)\delta_{\alpha\beta}dx^\alpha dx^\beta], \quad (12)$$

where $a(\eta)$ is the scale factor, η is the conformal time, and Φ and ψ are potentials that characterize the physical modes of the metric perturbations. We note that in this gauge fictitious perturbations are absent.[21] In the employed coordinate system, the potentials Φ and ψ are equal to the gauge-invariant quantities Φ_A and $-\Phi_H$ introduced by Bardeen.[22]

The solutions of the equations for the background model were analyzed in detail in the previous section in the general case with conformal anomaly. We are now interested in the case when there is no anomaly ($H_0\to\infty$). In the limit $H_0\to\infty$, the scale factor together with (4) also satisfies the equation

$$\alpha^2-\alpha'=\frac{F''}{2F}-\alpha\frac{F'}{F}, \quad (13)$$

which can be obtained by subtracting from the 0–0 equation one third of the trace of the α–β equations. Here $\alpha=a'/a$, $F=1-R/3M^2$, and the prime denotes differentiation with respect to the conformal time η. Linearizing Eqs. (1) with respect to Φ and ψ in the limit $H_0\to\infty$, we obtain the following equations for the perturbations:

$$\Delta\psi-3\alpha\psi'-3\alpha^2\Phi=\frac{1}{2M^2F}\left[\alpha\delta R'-\alpha'\delta R-\frac{1}{3}\Delta\delta R-2\alpha R'\Phi-R'\psi'\right], \quad (14)$$

$$\psi'+\alpha\Phi=\frac{1}{6M^2F}[R'\Phi+\alpha\delta R-\delta R'], \quad (15)$$

$$\delta R=3M^2F(\Phi-\psi), \quad (16)$$

where δR is the perturbation of the curvature scalar. Equation (14) is obtained from the 0–0 equation for the gravitational field, Eq. (15) corresponds to the 0–α equation, and (16) to the α–β equation for $\alpha=\beta$.

Substituting (16) in (15) and regrouping the corresponding terms, we express Φ and ψ in terms of $Y=\Phi+\psi$:

$$\Phi=-\frac{F^2}{3F'a}\left(\frac{a}{F}Y\right)',\qquad\psi=\frac{1}{3aFF'}(aF^2Y)'. \quad (17)$$

Using (16), (17), and (13), we obtain from Eq. (14) the following equation for Y:

$$Y''-\Delta Y+\left(3\frac{F'}{F}-2\frac{F''}{F'}+2\alpha\right)Y'+\left[\alpha^2+\alpha'+3\alpha\frac{F'}{F}-2\alpha\frac{F''}{F'}-\frac{F''}{2F}\right]=0. \quad (18)$$

Further, replacing Y by the new variable $u=F^{3/2}aY/F'$ and transforming in the obtained equation for u the term in front of u with allowance for (13), we find after rather lengthy calculations

$$u''-\Delta u-\frac{z''}{z}u=0, \quad z=\frac{(aF'^{1/2})'}{a^2F'}. \tag{19}$$

Solutions of Eq. (19) can be readily found in asymptotic situations. We consider a plane wave with $u\propto e^{ikx}$. For $k^2 \ll z''/z$, we obtain

$$u=\tilde{A}_1 z\int\frac{d\eta}{z^2}+\tilde{A}_2 z=A\left[\frac{aF'^{3/2}}{F'}-\frac{(aF'^{1/2})'}{a^2F'}\int a^2F\,d\eta\right]$$

$$=A\left[\frac{\dot{F}^{3/2}}{\dot{F}}-\frac{(a\dot{F}^{1/2})^{\cdot}}{a^2\dot{F}}\int aF\,dt\right], \tag{20}$$

where the constant $\tilde{A}_2$ corresponds to the constant of integration. In the derivation of the final expression for u, we have used the relation

$$F'^2=-\frac{4}{3}\left(\frac{(aF'^{1/2})'}{a^2F}\right)',$$

whose validity can be readily established by taking into account Eq. (13).

In the short-wave limit $(k^2\gg z''/z)$ we obtain

$$u=\tilde{C}\exp\left(i\int\frac{k}{a}dt\right)+\text{c.c.} \tag{21}$$

Taking into account the connection between u and Y and proceeding from Eqs. (17) and (16), we express Φ, ψ, and δR in terms of u:

$$\Phi=-\frac{1}{3\dot{F}^{1/2}}\left[\left(\frac{\ddot{F}}{\dot{F}}-\frac{5}{2}\frac{\dot{F}}{F}+H\right)u+\dot{u}\right],$$

$$\psi=\frac{1}{3\dot{F}^{1/2}}\left[\left(\frac{\ddot{F}}{\dot{F}}+\frac{1}{2}\frac{\dot{F}}{F}+H\right)u+\dot{u}\right],$$

$$\delta R=-2M^2\dot{F}^{1/2}\left[\left(\frac{\ddot{F}}{\dot{F}}-\frac{\dot{F}}{F}+H\right)u+\dot{u}\right]. \tag{22}$$

Substituting (20) in (22), we find for the long-wave perturbations $(k^2\ll z''/z)$

$$\Phi=A\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot} \qquad \psi=\Phi+A\frac{\dot{F}}{aF^2}\int aF\,dt. \tag{23}$$

In the case $k^2\gg z''/z$,

$$\Phi=-\frac{1}{3\dot{F}^{1/2}}\left\{\left(\frac{\ddot{F}}{\dot{F}}-\frac{5}{2}\frac{\dot{F}}{F}+H+\frac{ik}{a}\right)\right.$$

$$\left.\times C\exp\left(i\int\frac{k}{a}dt\right)+\text{c.c.}\right\}$$

$$\psi=\frac{1}{3\dot{F}^{1/2}}\left\{\left(\frac{\ddot{F}}{\dot{F}}+\frac{1}{2}\frac{\dot{F}}{F}+H+\frac{ik}{a}\right)\right.$$

$$\left.\times C\exp\left(i\int\frac{k}{a}dt\right)+\text{c.c.}\right\}. \tag{24}$$

The solutions (23) and (24) are valid in any stage of the evolution of the universe described by Eq. (4) with $H_0\to\infty$. We now consider the behavior of the perturbations in the quasi de Sitter stage (10). Bearing in mind that in this stage $\dot{H}\approx-(1/6)M^2\ll H^2$ and integrating by parts, we find for long-wave perturbations with $k^2\ll z''/z\approx(2/3)M^2a^2$ from (23)

$$\Phi\simeq A\left[\frac{1}{6}\frac{M^2}{H^2(t)}+O\left(\frac{M^4}{H^4}\right)\right],$$

$$\psi=A\left[-\frac{1}{6}\frac{M^2}{H^2(t)}+O\left(\frac{M^4}{H^4}\right)\right].$$

$$\frac{\delta R}{R}=-A\left[\frac{1}{3}\frac{M^2}{H^2(t)}+O\left(\frac{M^4}{H^4}\right)\right]. \tag{25}$$

In the short-wave limit $(k^2\gg(2/3)M^2a^2)$ the terms with $\ddot{F}/\dot{F}$ and $\dot{F}/F$ can be ignored in the quasi de Sitter stage, and we then obtain from (24)

$$\Phi\simeq-\psi=\left(1+\frac{ik}{Ha}\right)C\exp\left(i\int\frac{k}{a}dt\right)+\text{c.c.}, \qquad \frac{\delta R}{R}\approx-2\Phi. \tag{26}$$

It can be seen first of all from (26) and (25) that in the quasi de Sitter stage the perturbations are conformally flat in the lowest order of the expansion in M^2/H^2. The short-wave perturbations prior to the time they come through the horizon and right up to the time at which the perturbation wavelength becomes comparable with the scale M^{-1} $(Ma\ll k\ll\ll Ha)$, remain conformally planar in all stages of the evolution of the universe, and their amplitude $\Phi\approx-\psi$ remains constant.

Further, the amplitude of the long-wave perturbations $(k\ll Ma)$ increases in proportion to $1/H^2(t)$, since $H(t)$ decreases in the quasi-de Sitter stage.

The "scalaron" stage follows the quasi-de Sitter stage. Substituting $a(t)$ from (11) in Eq. (23) and bearing in mind that in this stage $F\approx 1$, we obtain for the long-wave perturbations

$$\Phi\approx A\,(^3/_5-{}^2/_5\cos(M(t-t_1))),$$

$$\psi=A\,(^3/_5+{}^2/_5\cos(M(t-t_1))). \tag{27}$$

Besides the constant part, for which $\Phi=\psi$, the amplitude of the perturbations in the "scalaron" stage contains an undamped oscillating part with constant amplitude. The amplitude of the oscillations will decrease only when allowance is made for decay of the scalarons.

The oscillating parts of Φ and ψ are shifted by a half-period, and this may be important for the analysis of decay of the "scalarons." Comparing (25) and (27), we find the coefficient of amplification $K(k)$ of the perturbations; this is defined as the ratio of the constant part of the amplitude of the metric perturbations Φ in the "scalaron" stage to their amplitude at the time when the perturbations enter the long-wave regime:

$$K(k)\approx\frac{\Phi(t\gg t_1)}{\Phi(k\sim Ma)}\approx 3.6\frac{H^2_{k\sim Ma}}{M^2}. \tag{28}$$

The following question is associated with the initial values of the perturbation amplitudes. As initial perturbations, it is natural to take the quantum fluctuations of the metric.

As was shown in Refs. 9 and 10, quantization of scalar perturbations of the metric with scales much less than the horizon in the de Sitter stage corresponds to a good degree of accuracy to quantization of an ordinary massless scalar field φ. The perturbations δR of the curvature are related to φ by[9]

$$\delta R=(192\pi G)^{1/2}MH\varphi. \tag{29}$$

The amplitude of the Fourier component of the field corresponding to the vacuum fluctuations at the time of emergence of the perturbation through the horizon is $\varphi_k = H/2^{1/2}k^{3/2}$ (Refs. 23 and 10). Therefore, as initial value for the perturbations it is necessary to choose the Fourier component of the perturbations of the metric, which is equal to

$$\Phi_k=-\frac{1}{2}\frac{\delta R_k}{R}=\left(\frac{\pi G}{6}\right)^{1/2}\frac{M}{k^{3/2}}, \tag{30}$$

at the time the mode with wave vector k comes through the horizon.

Further, bearing in mind that the amplitude of Φ_k is constant until the scale of the perturbation is comparable with the scale $\sim (Ma)^{-1}$, and taking into account the coefficient of the subsequent amplification of the amplitude of the perturbations (28), we obtain the following value of the Fourier amplitude of the perturbations of the metric in the "scalaron" stage:

$$\Phi_k \approx \psi_k \approx \left(\frac{54\pi G}{25}\right)^{1/2} \frac{H^2_{k=Ma}}{M} \frac{1}{k^{3/2}}. \qquad (31)$$

On scales greater than the horizon, the amplitude of the metric perturbations in the subsequent stages remains practically unchanged irrespective of the equation of state of the matter that fills the universe.[21] Bearing this in mind, we arrive at the following estimate for the spectrum of perturbations of greater than galactic scales, converted to the present epoch:

$$\Phi_\lambda \approx (|\Phi_k|^2 k^3)^{1/2} \approx 0.3\left(\frac{8\pi}{3}\right)^{1/2} \frac{M}{M_{Pl}} \ln \frac{\lambda}{\lambda_{Ph}}, \qquad (32)$$

where Φ_λ is the characteristic amplitude of the perturbations in the physical scale λ, and λ_{Ph} is the wavelength of the fossil phonons. The result (32) agrees in order of magnitude with the results obtained earlier by qualitative methods.[9,17] As can be seen from (32), the perturbation spectrum increases in the region of large scales. At $M \sim 10^{13}$ GeV, its amplitude is sufficient for the formation of the large-scale structure of the universe (see below). In conclusion, we emphasize that our results relating to the spectrum are valid only if the considered de Sitter stage is not followed by another analogous stage.

§4. PERTURBATIONS IN THE MODEL WITH CONFORMAL ANOMALY

We now consider the problem of the evolution of perturbations in the more general case when a conformal anomaly is present. A self-consistent system of equations for Φ, ψ, and δR can be obtained in this case by linearizing the complete system of Einstein equations (1). The initial conditions are chosen in the form (30).

In the presence of a conformal anomaly, the system of equations was solved numerically. The corresponding results for the time evolution of Φ_k and ψ_k are given in Fig. 3.

For $k\eta \gg 1$, δR_k oscillates with a damped amplitude. Then, in the inflationary stage at the time when $k\eta \sim 1$, the perturbation with given wave vector k comes through the horizon. If this occurs already in the de Sitter stage $[a(t) \propto \exp(H_0 t)]$, then the amplitude of the curvature perturbations increases in it exponentially[9,10]:

$$\delta R = \delta R_0 e^{\lambda t}, \quad \lambda = [-3/2 + (9/4 + (M/M_0)^2)^{1/2}] H_0.$$

After the transition to the quasi-de Sitter regime (10) the amplitude δR approaches a constant value and at the end of the quasi-de Sitter stage relaxes.

The initial fluctuations Φ_k and ψ_k increase exponentially from the time $(\eta \sim k^{-1})$ they come through the horizon: $\Phi_k \approx -\psi_k \propto \delta R/R$. Then in the quasi-de Sitter stage following the de Sitter stage ($a \propto e^{H_0 t}$), the values of Φ_k and ψ_k, as in the case without conformal anomaly, grow considerably, $\propto 1/H^2(t)$ [see (25)], and in the scalaron stage the expressions (27) are valid.

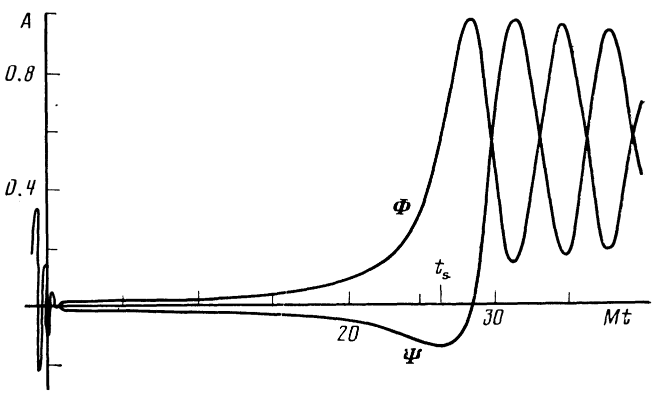

FIG. 3. Dependence of the amplitudes of the metric perturbations Φ and ψ on the time.

To characterize the evolution of the metric perturbations, we use the previously introduced amplification coefficient K, which depends in the general case on the wavelength and the parameter M/H_0, $K \equiv K(M/H_0, k)$. The results of the numerical calculation of K for the physically interesting values $0 < M/H_0 < 0.5$ for wavelengths corresponding to the horizon scales at the present epoch ($\sim 10^{28}$ cm) and different scales of the structure ($\sim 10^{26}$ cm, $\sim 10^{24}$ cm, and $\sim 10^{22}$ cm) are given in Fig. 4. To within 2%, they can be approximated by the analytic formula

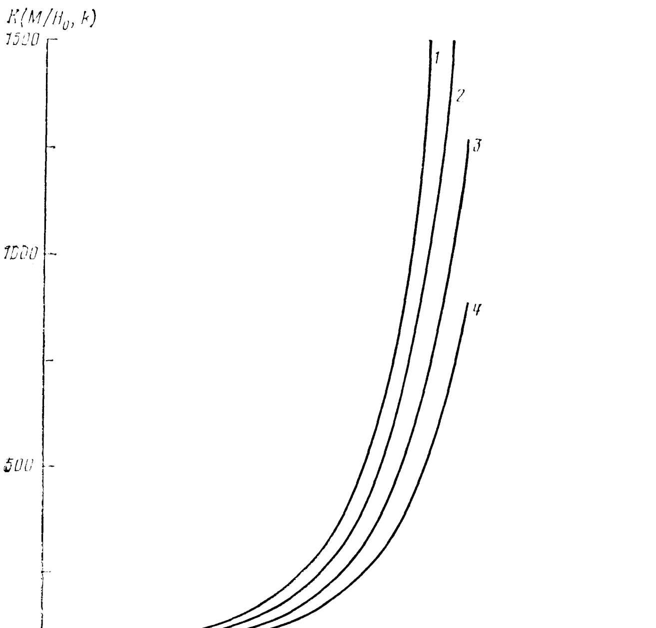

FIG. 4. Coefficient of amplification of the amplitude of the metric perturbations, $K(M/H_0, k)$, as a function of M/H_0 for different wavelengths: curve 1 for 10^{28} cm, curve 2 for 10^{26} cm, curve 3 for 10^{24} cm, and curve 4 for 10^{22} cm.

$$K\left(\frac{M}{H_0},k\right)=3.6\left(\frac{H^2(t_k)}{M^2}\right)\exp\left(\lambda\int_{t_k}^{t_{sc}}H(t)\,dt\right). \qquad (34)$$

Here, $H(t_k)$ is the value of the Hubble parameter at the time t_k at which mode k comes through the horizon, and t_{sc} is the time at which the quasi-de Sitter stage ends and the scalaron stage begins. In the limit $H_0\to\infty$, the expression (34) goes over into (28).

The expression (34) makes it possible to establish the analytic dependence on K (and, hence, the spectrum of perturbations of the metric) on the wave number k,

$$K\infty\left(1-\frac{\dot H(t_k)}{H_0^2}\ln\frac{k}{H}\right)k^{-\lambda}, \qquad (35)$$

and this describes the distortion of the flat perturbation spectrum.

For $H_0/M\gg 1$, $K=3.6(H/M)^2$. With decreasing H_0, the value of K increases, and for $H_0/M\sim 0.5$ the coefficient K is 40 times greater than the analogous coefficient in the case $H_0\to\infty$ at the scales of the contemporary horizon.

We note that the growth of the inhomogeneities Φ_k and ψ_k by the time t_{sc} corresponds to a qualitative explanation of the occurrence of inhomogeneities as the result of the spatial fluctuations at the time inflation ends.

§5. CONCLUSIONS

We conclude by giving the bounds which follow from observational data, on the parameters of the theory.

The first bound is associated with the following condition on the duration of the inflationary stage needed to explain the homogeneity of the universe on scales of the contemporary horizon:

$$-\frac{3H_0^2}{M^2}\ln\left(\frac{\delta R}{R}\right)_0>65. \qquad (36)$$

Here, $(\delta R/R)_0$ is determined by the initial deviation of the model from the exact de Sitter solution. If this initial deviation is due to quantum fluctuations of the curvature, $\delta R/R\sim M/M_{Pl}$, then it follows from (36) that $H_0>2M$. We note that at the end of the de Sitter stage in this case the magnitude of the fluctuations δR on the maximal scales becomes comparable with the background value R and, therefore, it is here necessary to take into account the back reaction of the fluctuations on the background metric (fluctuation phase transition). If $H_0\gg M$, then in the calculation of the perturbations on the scales corresponding to the scales of the currently observed horizon this effect can be ignored, and the theory developed in the previous sections can be used. However, on the maximal scales corresponding to inflation by

$$\frac{3H_0^2}{M^2}\ln\left|\frac{\delta R}{R}\right|\gg 65 \text{ times,}$$

the reaction is important, and the expressions obtained above for the perturbations are invalid. At large H_0 ($H_0\gg M$), the duration of the quasi-de Sitter inflationary stage (10) is determined by the initial value of H. In this inflationary stage, inflation to the scales corresponding to the currently observed horizon is achieved only if

$$3H^2/M^2>65, \qquad (37)$$

and this gives the bound $H/M\gtrsim 5$. Note that the duration of the de Sitter stage is always about $\ln(R/\delta R)_0\gtrsim 10$ times greater than the duration of the quasi-de Sitter stage.

The next most important restriction on the parameters of the theory is associated with the observed bounds on the anisotropy of the microwave background radiation, $\Delta T/T$, due to adiabatic perturbations of the metric. The metric perturbations considered in Secs. 3 and 4 with wavelength corresponding to scales comparable with the scale of the contemporary horizon lead to angular fluctuations in the temperature of the microwave background[24]:

$$\left(\frac{\Delta T}{T}\right)_l=\frac{A}{(l(l+1))^{1/2}}\frac{\mathcal{K}_l}{10\pi^{1/2}}, \qquad (38)$$

where l is the number of the harmonic in the expansion of $\Delta T/T$ in multipoles, and the constant A is related to the perturbations of the metric as follows:

$$\Phi_k=\alpha A/k^{3/2}. \qquad (39)$$

The numerical coefficients α and $\mathcal{K}_l$ depend on the particular cosmological model with hidden mass. In the case when the equation of state in the late stages is the dust equation, $\alpha=3/5$, $\mathcal{K}_l=1$; in less trivial cases, these values are changed slightly.[24] From the bounds on $\Delta T/T$ there follows the allowed range of values of A: $A\sim 10^{-3}$–10^{-4}. Bearing in mind that $\Phi=\Phi_0 K$, where $K(M/H_0,k)$ is the coefficient of amplification of the initial perturbations, and

$$\Phi_0=\left(\frac{\pi}{6}\right)^{1/2}\frac{M}{M_{Pl}}\frac{1}{k^{3/2}},$$

we obtain

$$\frac{M}{M_{Pl}}K\left(\frac{M}{H_0}\right)<\left(\frac{6}{\pi}\right)^{1/2}\alpha A. \qquad (40)$$

Using the results of the calculation of $K(M/H_0)$ from Sec. 4, we can find bounds on M and H_0. The allowed values for these parameters are given in Fig. 5. In the case $H_0\gg M$, we obtain $M\lesssim 10^{13}$–10^{14} GeV. For $H_0=2M$, we have $M<2.5\cdot 10^{11}$–10^{12} GeV (see also Ref. 17). The bounds on M that follow from the bounds on the amplitude of the gravitational waves generated in the inflationary stage are much weaker.

The temperature of the heating after the decay of the scalarons (with decay probability $\Gamma\sim\beta M^3/M_{Pl}^2$) is

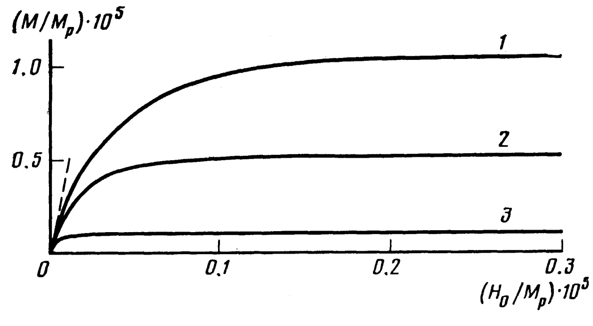

FIG. 5. Bounds on the parameters M and H_0 for different allowed values of A: curve 1 for $A=10^{-3}$, curve 2 for $A=0.5\cdot 10^{-3}$, and curve 3 for $A=10^{-4}$.

$$T_R \approx \left(\frac{5\beta^2}{4\pi^3 N}\right)^{1/4} \left(\frac{M^3}{M_{Pl}}\right)^{1/2}, \tag{41}$$

where N is the number of degrees of freedom of the created particles, and β is a numerical coefficient that depends on the nature of the decays. For $M \sim 10^{13}$ GeV, $T_R \sim 10^8$–10^{10} GeV. Thus, this model is at the limit of compatibility with the standard picture of baryosynthesis, which requires $T_R > 10^{10}$ GeV (as regards this criterion, the model without anomaly or with $H_0 \gg M$ is preferable). However, there exist models of baryosynthesis in which the baryon asymmetry can also arise at energies much lower than 10^{10} GeV.[25] Therefore, this criterion does not rule out scenarios with $M < 10^{13}$ GeV.

In the case $M > 10^{14}$ GeV, the inflationary stages of the types considered in this paper can occur only if they are followed by another inflationary stage at lower energies, brought about, for example, by a scalar field.

We note finally that the models discussed here can be realized both in the framework of quantum cosmology[19,26] and in the framework of chaotic scenarios.[3]

[1)]In an isotropic universe, the terms quadratic in the Weyl tensor in the Lagrangian do not lead to corrections in the linearized equations for the perturbations by virtue of the conformal flatness ($C_{iklm} \approx 0$) of the background model.

[1]A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[2]A. D. Linde, Phys. Lett. **108B**, 384 (1982); **114B**, 431 (1982); **116B**, 335 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[3]A. D. Linde, Phys. Lett. **129B**, 177 (1983).

[4]A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).

[5]L. A. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Lett. **157B**, 361 (1985).

[6]V. A. Belinskiĭ, L. P. Grishchuk, I. M. Khalatnikov, and Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **89**, 346 (1985) [Sov. Phys. JETP **62**, 195 (1985)].

[7]G. V. Chibisov and V. F. Mukhanov, "Galaxy formation and phonons," Preprint No. 162 [in English], P. N. Lebedev Physics Institute (1980); Mon. Not. R. Astron. Soc. **200**, 535 (1982).

[8]V. N. Lukash, Zh. Eksp. Teor. Fiz. **79**, 1601 (1980) [Sov. Phys. JETP **52**, 807 (1980)].

[9]V. F. Mukhanov and G. V. Chibisov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 54 (1981) [JETP Lett. **33**, 48 (1981)].

[10]V. F. Mukhanov and G. V. Chibisov, Zh. Eksp. Teor. Fiz. **83**, 473 (1982) [Sov. Phys. JETP **56**, 258 (1982)].

[11]A. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982); S. W. Hawking, Phys. Lett. **115B**, 295 (1982); J. M. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **26**, 679 (1983).

[12]V. F. Mukhanov, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 1333 (1984) [JETP Lett. **40**, 496 (1984)].

[13]A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 1333 (1984) [JETP Lett. **40**, 496 (1984)]; L. A. Kofman and A. D. Linde, Nucl. Phys. **282B**, 555 (1987); L. A. Kofman, Phys. Lett. **173B**, 400 (1986).

[14]L. A. Kofman and V. F. Mukhanov, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 481 (1986) [JETP Lett. **44**, 619 (1986)].

[15]K. S. Stelle, Phys. Rev. D **16**, 953 (1977); I. Antoniadis and E. T. Tomboulis, "Gauge invariance and unitarity in higher derivatives quantum gravity," Preprint UCLA/84/TEP/10.

[16]B. Zwiebach, Phys. Lett. **156B**, 315 (1985); A. A. Tseytlin, Phys. Lett. **176B**, 92 (1986).

[17]A. A. Starobinskiĭ, Pis'ma Astron. Zh. **9**, 579 (1983) [Sov. Astron. Lett. **9**, 302 (1983)].

[18]T. V. Ruzmaĭkina and A. A. Ruzmaĭkin, Zh. Eksp. Teor. Fiz. **57**, 680 (1969) [Sov. Phys. JETP **30**, 372 (1970)]; V. Ts. Gurovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **77**, 1683 (1979) [Sov. Phys. JETP **50**, 844 (1979)]; M. B. Fishetti, J. B. Hartle, and B. L. Hu, Phys. Rev. D **20**, 1757 (1979); P. Anderson, Phys. Rev. D **29**, 615 (1984).

[19]A. Vilenkin, Phys. Rev. D **32**, 2511 (1985).

[20]Q. Shafi and C. Wetterich, Phys. Lett. **152B**, 51 (1985).

[21]G. V. Chibisov and V. F. Mukhanov, "Theory of Relativistic potential: cosmological perturbations," Preprint No. 154 [in English], P. N. Lebedev Physics Institute (1983).

[22]J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[23]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. **A360**, 117 (1978).

[24]L. A. Kofman and A. A. Starobinskiĭ, Pis'ma Astron. Zh. **11**, 643 (1985) [Sov. Astron. Lett. **11**, 271 (1985)]; L. A. Kofman, D. Yu. Pogosyan, and A. A. Starobinskiĭ, Pis'ma Astron. Zh. **12**, 419 (1986) [Sov. Astron. Lett. **12**, 175 (1986)].

[25]I. Affleck and M. Dine, Nucl. Phys. **B249**, 361 (1985); A. D. Linde, Phys. Lett. **160B**, 243 (1985).

[26]Ya. B. Zel'dovich, Pis'ma Astron. Zh. **7**, 579 (1981) [Sov. Astron. Lett. **7**, 322 (1981)]; J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); A. D. Linde, Zh. Eksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)].

Translated by Julian B. Barbour