

# Dynamics of point vortex dipoles and spontaneous singularities in three-dimensional turbulent flows

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Equations describing the dynamics of  $\delta$ -singular vortices (vortex dipoles) and corresponding to the three-dimensional hydrodynamic equations of an ideal incompressible liquid are obtained. On the basis of an exact solution of the dynamical problem it is shown that explosive growth of localized vorticity is possible in a system consisting of two vortex dipoles.

Progress in our understanding of the nature of the strong vortex interactions in advanced three-dimensional turbulent flows can be achieved, as in the two-dimensional case also, on the basis of distinguishing localized elementary vortex objects such that a finite set of them is fully able to give rise to (and adequately simulate) complex dynamical processes in such nonlinear infinite-dimensional systems.<sup>1,2</sup>

In this connection, in the present paper we consider three-dimensional  $\delta$ -singularities of the vortex field—point vortex dipoles. For these we have obtained dynamical equations possessing (in contrast to the dynamical system of vortons of Ref. 2) the same invariants of motion as the original three-dimensional Euler hydrodynamic equations. It is shown that for two vortex dipoles irreversible catastrophic growth of the quadratic vorticity in a finite time is possible; this phenomenon is investigated, e.g., in Ref. 3 and 4, and is connected, in the final analysis, with the well-known phenomenon of the stretching of vortex tubes in three-dimensional turbulent flows.<sup>5</sup>

1. An elementary example of a localized vortex object having point support is a vortex dipole, for which the vortex field  $\omega$  is solenoidal ( $\text{div } \omega = 0$ ) and has the form

$$\omega_i(\mathbf{x}) = \varepsilon_{ijl} \gamma_l \frac{\partial \delta(\mathbf{x})}{\partial x_j}, \quad (1)$$

where  $\delta$  is a delta function in three-dimensional space,  $\varepsilon_{ijl}$  is the completely antisymmetric unit pseudotensor of rank three, and  $\gamma_i$  is the intensity vector of the vortex dipole situated at the coordinate origin (here and below, repeated indices imply summation from 1 to 3).

In the more general case we can consider solenoidal vortex objects possessing like (1), point support, but having the form of, e.g., a superposition of vortex multipoles:

$$\omega_i = \varepsilon_{ijl} \gamma_l \frac{\partial F}{\partial x_j}, \quad F = \sum_{n=1}^{\infty} a_n \bar{L}^n \delta(\mathbf{x}),$$

$$a_n = \text{const}, \quad \bar{L} \equiv \gamma_l \frac{\partial}{\partial x_l}.$$

In the present paper, however, we shall confine ourselves to studying only vortex dipoles (1).

In unbounded space a vortex dipole corresponds to a solenoidal velocity field  $\mathbf{v}$  ( $\text{div } \mathbf{v} = 0$ ), which is established from (1) in the standard manner on the basis of the well-known relation between  $\omega$  and  $\mathbf{v}$ , analogous to the Biot-Savart law in the theory of electromagnetism.<sup>6</sup> Outside the vortex dipole this velocity field is described by the potential

$$(\mathbf{v} = \text{grad } \Phi)$$

$$\Phi = \gamma_i x_i / 4\pi |\mathbf{x}|^3, \quad (2)$$

which coincides with the potential of a doublet, of the source-sink type, of intensity  $\gamma$  (Ref. 6). In this case the flow of the liquid outside the vortex dipole occurs as if it were formed by motion of a solid sphere of radius  $a$  with velocity  $\mathbf{U}$  in an unbounded liquid at rest at infinity (if  $\gamma = 2\pi a^3 \mathbf{U}$ ). We note also that the potential (2) coincides with the potential created by a closed vortex filament at large distances from it, if  $\gamma = \kappa \mathbf{S}$ , where  $\mathbf{S}$  is the total area vector of any open surface bounded by the vortex filament and  $\kappa$  is the vortex intensity.<sup>6</sup> An electromagnetic analog of the vortex dipole (1) was considered in Ref. 7 in connection with the problem of the generation of a magnetic field. Thus, the vortex object (1) has a clear physical meaning, corresponding to taking the limit

$$\gamma = \lim_{\substack{\kappa \rightarrow \infty \\ |\mathbf{S}| \rightarrow 0}} \kappa \mathbf{S}$$

for a vortex ring (or to taking the analogous limit for a spherical Hill vortex<sup>6</sup>).

2. For a system consisting of  $N$  vortex dipoles, the vortex field  $\omega$  has the form

$$\omega_i = \sum_{\alpha=1}^N \varepsilon_{ijl} \gamma_l^\alpha(t) \frac{\partial}{\partial x_j} \delta[\mathbf{x} - \mathbf{x}^\alpha(t)], \quad (3)$$

here  $\mathbf{x}^\alpha$  and  $\gamma^\alpha$  determine the position and intensity of the vortex dipole labeled  $\alpha$  ( $\alpha = 1, 2, \dots, N$ ) as functions of time, since in a system consisting of several vortex dipoles there is hydrodynamic interaction between them.

The energy  $T$ , Lamb momentum  $P_i$ , and angular momentum  $M_i$  are invariants of the initial Euler hydrodynamic equations and are expressed in terms of the vortex field (3) as follows<sup>6</sup>:

$$T' = \frac{\rho}{8\pi} \sum_{\alpha=1}^N \sum_{\beta=1}^N \gamma_i^\alpha \gamma_j^\beta \left[ -\frac{\delta_{ij}}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^3} + \frac{3(x_i^\beta - x_i^\alpha)(x_j^\beta - x_j^\alpha)}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^5} \right], \quad (4)$$

$$P_i = \rho \sum_{\alpha=1}^N \gamma_i^\alpha, \quad M_i = \rho \sum_{\alpha=1}^N \varepsilon_{ijl} x_j^\alpha \gamma_l^\alpha, \quad (5)$$

where  $T'$  determines the invariant, regular (finite) part of the total kinetic energy  $T$  of the liquid (naturally, the quantity  $T$  itself is infinite for vortex dipoles, as it is for the field of a flow induced by a vortex filament of arbitrary shape<sup>6</sup>) and

corresponds to the interaction of the vortex dipoles. We note that the helicity invariant

$$\mathcal{H} = \int d^3x (\mathbf{v} \cdot \text{rot } \mathbf{v})$$

for the system of vortex dipoles (3) is identically equal to zero.

The dynamical system describing the evolution of  $\gamma^\alpha$  and  $\mathbf{x}^\alpha$  should leave the quantities (4) and (5) invariant. It is not difficult to convince oneself that this system is determined by the Hamiltonian  $H = T'/\rho$  (where  $\rho$  is the density of the incompressible liquid), and has the form

$$\begin{aligned} \frac{dx_i^m}{dt} &= -\frac{\partial H}{\partial \gamma_i^m} \\ &= \frac{1}{4\pi} \sum_{\substack{\alpha=1 \\ (\alpha \neq m)}}^N \gamma_i^\alpha \left( \frac{\delta_{ij}}{|\mathbf{x}^m - \mathbf{x}^\alpha|^3} - \frac{3(x_i^m - x_i^\alpha)(x_j^m - x_j^\alpha)}{|\mathbf{x}^m - \mathbf{x}^\alpha|^5} \right) \\ &\equiv \sum_{\substack{\alpha=1 \\ (\alpha \neq m)}}^N \frac{\partial \Phi(\mathbf{x}^m - \mathbf{x}^\alpha)}{\partial x_i^m}, \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{d\gamma_i^m}{dt} &= \frac{\partial H}{\partial x_i^m} \\ &= \frac{3\gamma_i^m}{4\pi} \sum_{\substack{\alpha=1 \\ (\alpha \neq m)}}^N \gamma_j^\alpha \left[ \delta_{ij}(x_i^m - x_i^\alpha) + \delta_{li}(x_j^m - x_j^\alpha) + \delta_{ji}(x_l^m - x_l^\alpha) \right. \\ &\quad \left. - \frac{5(x_i^m - x_i^\alpha)(x_l^m - x_l^\alpha)(x_j^m - x_j^\alpha)}{|\mathbf{x}^m - \mathbf{x}^\alpha|^2} \right] |\mathbf{x}^m - \mathbf{x}^\alpha|^{-5}, \end{aligned}$$

where the potential  $\Phi$  is defined in (2) and the superscripts correspond to the labeling of the vortex dipoles ( $m = 1, 2, \dots, N$ ).

In Refs. 8–10 systems of the form (6) were introduced on the basis of a different approach that does not exclude the effects of self-action of the vortex dipoles. In our analysis these undesirable effects, which lead to uncertainties (see Ref. 10) in the magnitude of the self-induced velocity, are excluded automatically in the derivation of (6) as a result of the renormalization (regularization) of the total energy  $T$  of the liquid. Then the system (6) can indeed serve as a basis for the extension of gridless algorithms for calculating the flows (see Ref. 11) to the three-dimensional case, since none of the restrictions on the mutual dynamics of point vortex dipoles that usually apply (see Ref. 10) in the analysis of the dynamics of small but finite vortex rings are present in this case.

**3.** We shall consider the dynamics of the relative motion of two vortex dipoles in the case when the total momentum  $\mathbf{P} = 0$  and the angular momentum  $\mathbf{M}$  is oriented along the  $z$  axis.

Let the vector  $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^1 = \boldsymbol{\gamma}^2$  and  $\mathbf{l} = \mathbf{x}^1 - \mathbf{x}^2$ , lie in the  $(x, y)$  plane at the initial time  $t = 0$ . Since the quantity  $\mathbf{M} = \rho \mathbf{l} \times \boldsymbol{\gamma}$  is an invariant of the motion [see (5)], for all values of the vectors  $\mathbf{l}$  and  $\boldsymbol{\gamma}$  remain as before in the  $(x, y)$  plane and are entirely determined by the values of the moduli  $l = |\mathbf{l}|$  and  $\gamma = |\boldsymbol{\gamma}|$  and by the values of the corresponding polar angles  $\varphi$  (for  $\mathbf{l}$ ) and  $\varphi_1$  (for  $\boldsymbol{\gamma}$ ). For  $l, \gamma, \varphi$ , and  $\varphi_1$ , from (6) we obtain the following system of equations:

$$\begin{aligned} \frac{dl}{dt} &= -\frac{\gamma \cos(\varphi - \varphi_1)}{\pi l^3}, \\ \frac{d\gamma}{dt} &= \frac{3\gamma^2 \cos(\varphi - \varphi_1) (3 - 5 \cos^2(\varphi - \varphi_1))}{4\pi l^4}, \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d\varphi}{dt} &= -\frac{\gamma \sin(\varphi - \varphi_1)}{2\pi l^4} = -\frac{M}{2\pi l^3 \rho}, \\ \frac{d\varphi_1}{dt} &= \frac{3\gamma \sin(\varphi - \varphi_1) (1 - 5 \cos^2(\varphi - \varphi_1))}{4\pi l^4} \\ &= \frac{3M(1 - 5 \cos^2(\varphi - \varphi_1))}{4\pi l^3 \rho}, \end{aligned} \quad (8)$$

$$M = |\mathbf{M}| = \rho \gamma l \sin(\varphi - \varphi_1).$$

From (7) and (8) it follows that

$$\frac{d(\gamma l)}{dt} = \frac{5\gamma^2}{4\pi l^3} \left( 1 - \frac{3(\gamma l)^2}{\gamma^2 l^2} \right) \equiv 5H = \text{const},$$

where  $(\gamma l) \equiv \gamma l \cos(\varphi - \varphi_1)$ . Therefore, from (7) we obtain as a result of elementary integration the exact solution

$$\gamma^2 = 4\pi H l^2 + \frac{3}{l^2} (5Hl + (\gamma_0 l_0)^2)^2, \quad (9)$$

$$\varphi = \varphi(t=0) + \frac{M}{25\varphi_0 H (t^{(+)} - t^{(-)})} \ln \left| \frac{(t^{(+)} - t) t^{(-)}}{(t - t^{(-)}) t^{(+)}} \right|, \quad (10')$$

$$\begin{aligned} l &= \left[ l_0^2 - \frac{5}{\pi} \left( t(\gamma_0 l_0) + \frac{5Ht^2}{2} \right) \right]^{1/2} \\ &= \left( \frac{25H}{2\pi} \right)^{1/2} (t^{(+)} - t)^{1/2} (t - t^{(-)})^{1/2}, \end{aligned} \quad (10)$$

where

$$l_0 = l(t=0), \quad \gamma_0 = \gamma(t=0),$$

$$\begin{aligned} t^{(\pm)} &= \frac{-(\gamma_0 l_0) \pm [2\pi H l_0^2 + (\gamma_0 l_0)^2]}{5H} \\ &= \frac{4\pi l_0^3 (-\cos \Psi_0 \pm 2^{-5/2} |\sin \Psi_0|)}{5\gamma_0 (1 - 3 \cos^2 \Psi_0)}, \quad \Psi_0 \equiv (\varphi - \varphi_1)|_{t=0}. \end{aligned}$$

From (9) and (10) it can be seen the the angle  $\Psi_0$  between the vectors  $\boldsymbol{\gamma}_0$  and  $\mathbf{l}_0$  is the fundamental parameter determining the qualitatively different types of dynamical regimes for the vortex dipoles.

In fact, for  $|\cos \Psi_0| < 1/\sqrt{3}$  (i.e., for  $H > 0$ ), in a finite time  $t^{(+)}$  a  $t$ -irreversible collapse (coalescence to a point) of two, initially isolated vortex dipoles occurs. Changing the sign of  $t$  here does not cause the vortex dipoles to move apart. We note also that the system (6) is invariant to the replacement  $t \rightarrow -t$ ,  $\gamma \rightarrow -\gamma$ . For  $0 < \cos \Psi_0 < 1/\sqrt{3}$  the distance  $l$  between the vortex dipoles at  $t > 0$  decreases monotonically with increase of  $t$ , and  $l \rightarrow 0$  as  $t \rightarrow t^{(+)}$ . In the case  $-1/\sqrt{3} < \cos \Psi_0 < 0$  the vortex dipoles first move apart [ $l = l_{\text{max}} = l_0 \sin^{2/5} \Psi_0 (1 - 3 \cos^2 \Psi_0)^{1/5}$ ] at  $t = t_0 = -4\pi l_0^4 \cos \Psi_0 / 5\gamma_0 (1 - 3 \cos^2 \Psi_0)$ , and then, for  $t > t_0$ , begin to collapse to a point. Here, for  $\cos \Psi_0 > 0$  the collapse time  $t^{(+)}$  is shorter than in the case when  $\cos \Psi_0 < 0$ . In the limit  $t \rightarrow t^{(+)}$  this mutual approach of the vortex dipoles occurs along trajectories corresponding to logarithmic spirals, since from (8) and (10') it follows that the angular velocity  $\omega \equiv d\varphi/dt \rightarrow -\sqrt{2}/10(t^{(+)} - t)$ , and the frequen-

cy ratio  $\omega/\omega_1 \rightarrow 1$  in this limit (here  $\omega_1 \equiv d\varphi_1/dt$ ). The collapse of the vortex dipoles occurs at the point of their "center of gravity"  $\mathbf{B} = (\mathbf{x}^1 + \mathbf{x}^2)/2$ , which is immobile in the laboratory coordinate frame, since from (6) it follows that  $d(\mathbf{x}^1 + \mathbf{x}^2)/dt = 0$ , and so absolute motion is absent at the point  $\mathbf{B}$ . In this dynamical regime (i.e., for  $H > 0$ ) the solution (9), (10) becomes indeterminate for  $t \geq t^{(+)}$ , since for  $t > t^{(+)}$  the quantity  $l$  becomes negative, which is inadmissible from a physical point of view.

At the same time, for  $|\cos\Psi_0| > 1/\sqrt{3}$  (i.e. for  $H < 0$ ) and  $\cos\Psi_0 < 0$  there is a monotonic increase of  $l$  for  $t > 0$ , and in the limit  $t \rightarrow \infty$  the moving apart of the vortex dipoles is determined by the asymptotes  $l \cong O(t^{2/5})$  and  $\gamma^2 \cong O(t^{6/5})$ . However, for  $|\cos\Psi_0| > 1/\sqrt{3}$  and  $\cos\Psi_0 > 0$  the vortex dipoles come together in a finite time

$$t^{(+)} = \frac{4\pi l_0^4 (\cos\Psi_0 - 2^{-3/2} |\sin\Psi_0|)}{5\gamma_0 (3 \cos^2\Psi_0 - 1)}.$$

Then, in the time interval

$$\Delta t = t^{(-)} - t^{(+)} = \frac{4\pi \cdot 2^{-3/2} l_0^4 |\sin\Psi_0|}{5\gamma_0 (3 \cos^2\Psi_0 - 1)}$$

the solution (10) is physically not determined, since for  $t^{(+)} < t < t^{(-)}$  the quantity  $l$  takes negative values (from physical considerations we can assume that for  $t^{(+)} < t < t^{(-)}$  the vortex dipoles are together at the same point  $\mathbf{B}$ ). Then, for

$$l > t^{(-)} = \frac{4\pi l_0^4 (\cos\Psi_0 + 2^{-3/2} |\sin\Psi_0|)}{5\gamma_0 (3 \cos^2\Psi_0 - 1)}$$

the vortex dipoles move apart, so that again  $l \cong O(t^{2/5})$  as  $t \rightarrow \infty$ . In particular, for  $\Psi_0 = 0$  we have  $t^{(+)} = t^{(-)} = \tilde{t}_0 = 2\pi l_0^4 / 5\gamma_0$ , and the regime of collapse at  $0 \leq t \leq \tilde{t}_0$  is immediately replaced at  $t > \tilde{t}_0$  by a process in which the vortex dipoles move apart.

Also special is the case when  $|\cos\Psi_0| = 3^{-1/2}$  (i.e.,  $H = 0$ ). For  $\cos\Psi_0 = -3^{-1/2}$  the vortex dipoles move apart, in such a way that  $l \cong O(t^{1/5})$  and  $\gamma^2 \cong O(t^{8/5})$  in the limit  $t \rightarrow \infty$ . At the same time, in the case when  $\cos\Psi_0 = 3^{-1/2}$ , in a finite time  $t^{(+)} = \pi \cdot 3^{1/2} l_0^4 / 5\gamma_0$  the vortex dipoles collapse to the point  $\mathbf{B}$ , where, evidently, they should remain for an unlimited time, since for  $t > t^{(+)}$  the quantity  $l$  determined from the solution (10) becomes negative.

In addition, it follows from the solution (9) that in the dynamical regimes with  $H > 0$  and  $H \leq 0$  (in the case  $\cos\Psi_0 > 0$ ), the quantity  $\gamma^2$  grows without limit as the vortex dipoles come together in the finite time  $t^{(+)}$ . For  $H > 0$  this  $t$ -irreversible process (like the collapse of the vortex dipoles itself for such initial conditions) is qualitatively insensitive to a change of sign of  $t$ , in contrast to the case  $H \leq 0$ , for which the operation  $t \rightarrow -t$  leads to a qualitative change of the dynamical regime, since the collapse of the vortex dipoles in this case is replaced by their moving apart (and vice versa). In the case when  $H < 0$  and  $\cos\Psi_0 > 0$  the spontaneous singularity of  $\gamma^2$  as  $t \rightarrow t^{(+)}$  is replaced at  $t > t^{(-)} \geq t^{(+)}$  by a monotonic increase ( $\gamma^2 = O(t^{6/5})$ ) as  $t \rightarrow \infty$ ; see above).

We note also that for coaxial vortex dipoles (i.e., for  $\Psi_0 = 0$  and  $M = 0$ ) their coming together at the time  $t = \tilde{t} = 2\pi l_0^4 / 5\gamma_0$  (see above) occurs along the straight line

joining them, and not along a logarithmic spiral as in the case of noncoaxial vortex dipoles (i.e., for  $\Psi_0 \neq 0$  and  $M \neq 0$ ). In this special case, when the angular momentum  $M = 0$ , as  $t \rightarrow \tilde{t}$  we have  $\gamma^2 \rightarrow 0$  (since  $\gamma^2 \cong O[(\tilde{t} - t)^{6/5}]$ ), while for  $t > \tilde{t}$ , while the vortex dipoles move apart, their intensity increases monotonically ( $\gamma^2 \cong O(t^{6/5})$ ) as  $t \rightarrow \infty$ ), just as in other cases with  $H < 0$ .

Thus, the dynamical regime of the mutual approach of coaxial vortex dipoles is in qualitative agreement with the well-known phenomenon of the attraction of coaxial vortex rings with opposite momenta.<sup>6</sup> At the same time, in classical hydrodynamics, solutions describing the dynamics of two noncoaxial vortex rings are not known.<sup>6</sup> Thus, the exact solution obtained above for noncoaxial vortex dipoles with  $M \neq 0$  makes it possible, in particular, to analyze characteristic tendencies in the dynamics of two finite noncoaxial vortex rings, if at the initial time  $t = 0$  they are separated by a distance much greater than their own sizes.

4. The above exact solution (9), (10) of the three-dimensional hydrodynamic equations of an ideal incompressible liquid can be used to construct a model turbulent regime on the basis of the method, developed in Refs. 12 and 13, of randomization of integrable problems of hydrodynamics. For this, in accordance with Refs. 12 and 13, it is necessary to introduce a probability measure on the ensemble of realizations of the turbulent velocity field corresponding to the solution (9), (10) with random initial data. We note that in such modeling the magnitude of the mean-square vorticity (the enstrophy) of the turbulent flow can be determined:  $\tilde{\Omega} = \langle \Omega \rangle$ , where  $\Omega = (\partial v_i / \partial x_j)^2$ ,  $\mathbf{v}$  is the velocity field created by the pair of vortex dipoles (the dynamics of which is described by the exact solution (9), (10) for random initial data), and the angular brackets denote statistical averaging over the ensemble of realizations of the random (turbulent) velocity field  $\mathbf{v}$ .

From the solution (9), (10) there follows then the possibility of explosive growth of the quantity  $\Omega$  in spatially localized regions, even for each realization (of this statistical ensemble) satisfying the conditions  $|\cos\Psi_0| < 3^{-1/2}$  or  $1 > \cos\Psi_0 \geq 3^{-1/2}$ . Therefore, explosive growth of  $\tilde{\Omega}$  certainly also occurs irrespective of the specific form of the probability measure, for which it is sufficient to assume that it tends to zero for realizations with  $\Psi_0 = 0$  and  $-1 \leq \cos\Psi_0 \leq -3^{-1/2}$ . In fact, from (2), (6), and (9), (10) it follows that  $\Omega \cong O(\gamma^2 l^2 / |\mathbf{x} - \mathbf{x}_1|^{10})$ , i.e., in the neighborhood  $|\mathbf{x} - \mathbf{B}| \leq l(t)$  of the point  $\mathbf{x} = \mathbf{B} = (\mathbf{x}_1 + \mathbf{x}_2)/2$  (to which the vortex dipoles collapse) we have the estimate  $\Omega \cong O(\gamma^2 / l^2)$  or  $\Omega \cong O[1/(t^{(+)} - t)^q]$  as  $t \rightarrow t^{(+)}$ , where the explosive-growth exponent  $q = 2$ . Here, at the other points for which  $|\mathbf{x} - \mathbf{x}_1| > l(t)$ , the quantity  $\Omega$  no longer has a singularity as  $t \rightarrow t^{(+)}$ . In particular, as a result of averaging  $\Omega$  over the probability measure introduced in Ref. 13 (which is a finite discrete set of  $\delta$ -functions), for a statistical ensemble consisting of a finite number of terms we obtain the estimate  $\tilde{\Omega} \cong O[1/(t_{\min}^{(+)} - t)^2]$  as  $t \rightarrow t_{\min}^{(+)}$ , where  $t_{\min}^{(+)}$  is the smallest value of the positive quantity  $t^{(+)}$  among all the possible discrete values  $\{t^{(+)}\}$  corresponding to different realizations of the random initial data. Of course, the estimate  $\tilde{\Omega}$  obtained in this way corresponds to an extremely simplified model of turbulence. We note, however, that in Ref. 4, from considerations of scaling and dimensionality,

exactly the same exponent of the explosive growth of the enstrophy was obtained:

$$\tilde{\Omega} = \int_0^{\infty} dk k^2 E(k, t) \approx O\left(\frac{1}{(t^* - t)^2}\right)$$

as  $t \rightarrow t^*$  for the case when the spectrum  $E(k, t)$  of the energies of the turbulence is determined by the flow of energy through the spectrum and corresponds to the Kolmogorov–Obukhov “two-thirds law” in the inertial interval of scales  $k^{-1}$  for  $t \rightarrow t^*$ , i.e.,

$$E(k, t) \approx \varepsilon^{2/3} k^{-5/3} g(k\varepsilon^{1/3}(t^* - t)^{-1})$$

(where  $g \approx O(1)$  for  $t \rightarrow t^*$ , and  $\varepsilon$  is the average specific rate of dissipation of turbulent energy). At the same time, for an energy spectrum determined by the flow of helicity (or gyro-tropy)  $\mathcal{H}$  through the spectrum, an exponent  $q \neq 2$  for the explosive growth of the enstrophy was obtained in Ref. 4. Here, the fact that our result accords only with the first case considered in Ref. 4 is entirely natural, since it has already been noted above that for the system of vortex dipoles (3) the helicity invariant  $\mathcal{H} \equiv 0$ , and, therefore, for such a system flow of helicity through the spectrum cannot be realized at all. The comparison of our estimate of the explosive growth of the enstrophy  $\tilde{\Omega}$  with the corresponding result of Ref. 4 can be used, e.g., to determine the arbitrary empirical constant  $t^*$  introduced in Ref. 4, since it is obvious that  $t^* \approx t_{\min}^+$ . In addition, from the qualitative agreement between the above estimate of  $\tilde{\Omega}$  and the results of Refs. 3 and 4 it follows that a system of two nonlinearly interacting vortex dipoles is already able to model reasonably well the elementary act of strong interaction in developed turbulent flow. In fact, analysis of this relatively simple dynamical system may turn out to be useful for modeling the phenomenon of a sink of turbulent energy of an ideal incompressible liquid at spatially localized (point) vortex singularities, although such modeling, of course, is not exhaustive or obligatory, inasmuch as it certainly does not ensure completeness of the description of the observable vortex interactions that lead to the energy sink in real three-dimensional turbulent flows. The possibility of the existence of such an energy-sink phenomenon was first pointed out in a paper of Onsager.<sup>14</sup> The results of the numerical experiments of Ref. and the conclusions of Ref. 4 are in accordance with the assumption made in Ref. 14. In fact, the unlimited growth of the enstrophy  $\tilde{\Omega}$  in a finite time, as obtained in Refs. 3 and 4 and in our work, implies the admissibility of the existence, for the average specific rate of dissipation of turbulent energy, of a finite limit  $\varepsilon = \lim_{\nu \rightarrow 0} \nu \tilde{\Omega}$  as the kinematic viscosity  $\nu$  tends to zero.<sup>2-4</sup> This, in its turn, as noted in Ref. 2, makes it possible to extend the Kolmogorov–Obukhov “two-thirds law” to the region of arbitrarily small scales in the limit  $\nu \rightarrow 0$  (for finite  $\nu$ , the “two-thirds law” certainly does not hold for scales smaller than the internal scale  $\eta = (\nu^3/\varepsilon)^{1/4}$  of the turbulence<sup>5</sup>).

The experimentally observed sharply expressed intermittency of turbulent flows confirms the possibility of the onset of spatiotemporal vortex-field singularities, modeled in the present paper on the basis of an analysis of the dynamics of vortex dipoles.

In conclusion, we note that analysis of the dynamical

system (6), even for a small number of vortex dipoles, can also be of interest, e.g., in connection with the problem of predictability (integrability of the hydrodynamic equations<sup>15,16</sup> and for the solution of problems concerning the transport of impurities in the atmosphere and in a plasma on the basis of the use of the method of randomization of integrable problems.<sup>12,13</sup> In addition, three-dimensional  $\delta$ -singular vortex dipoles are the most adequate objects for modeling turbulent flows in three-dimensional problems of applied aerodynamics. In fact, for the modeling of two-dimensional turbulent flows around obstacles wide use is currently made of the method of point vortices—rectilinear vortex filaments.<sup>1</sup> However, for the calculation of, e.g., turbulent flow around moving elastic profiles such two-dimensional modeling is no longer appropriate because of the presence of important three-dimensional effects.<sup>17</sup> In three-dimensional turbulent flow the vortex filaments are deformed in the interaction, and, accordingly, strong self-interaction effects, hindering the use of these objects, arise. At the same time, in principle, point vortex dipoles do not change their structure upon interaction, and therefore their application for the modeling of three-dimensional turbulence has obvious advantages.

We must also stress the fundamental importance of the fact that in three-dimensional hydrodynamics the correct introduction of point vortex dipoles of the form (3) is, in general, possible. For example, in a recent paper,<sup>18</sup> Saffman and Meiron make the opposite statement, based on the fact that for the system (3) the quantity  $\mathbf{A} = (-1/2) \int d\mathbf{x}(\mathbf{x})^2 \boldsymbol{\omega}$  does not remain invariant. However, in the three-dimensional case  $\mathbf{A}$  is only a part of the total invariant angular momentum of the liquid:

$$\mathbf{M} = \rho \int d^3x [\mathbf{x} \times \boldsymbol{\omega}] / 3$$

(see Ref. 6), although in the two-dimensional case  $\mathbf{M} = \rho \mathbf{A}$  (Ref. 6). The invariance of  $\mathbf{M}$  for the system of vortex dipoles (3) has already been noted above (see also Ref. 9). In addition, the self-interaction effect for an isolated point vortex dipole (describable by the term  $\gamma_i \delta(\mathbf{x})$  in the expression for the liquid velocity  $\tilde{v}_i = \gamma_i \delta(\mathbf{x}) + \partial\Phi/\partial x_i$  induced by a vortex dipole situated at the point  $\mathbf{x} = 0$ ) is excluded automatically when the term  $\partial\Phi/\partial x_i$  is subjected to the usual regularization (see Ref. 19) needed by virtue of the fact that the function  $\partial\Phi/\partial x_i$  for  $\Phi$  from (2) has a locally nonintegrable singularity at the point  $\mathbf{x} = 0$ . In fact, since this regularization for  $\partial\Phi/\partial x_i$  for  $\Phi$  from (2) is always defined only to within a term  $c_i \delta(\mathbf{x})$  ( $c_i$  is arbitrary),<sup>19</sup> for  $c_i = -\gamma_i$  the self-interaction effect for a point vortex dipole is actually eliminated. This circumstance was also not taken into account in Ref. 18.

Thus, point vortex objects can have meaning not only in two-dimensional but also in three-dimensional hydrodynamics.

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