Dissipationless shock waves in media with positive dispersion

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We consider nonstationary one-dimensional flows in dissipationless hydrodynamics with positive dispersion which is described by the nonlinear Schrödinger equation. We determine the conditions for the occurrence of dissipationless shock waves (DSW). We find analytical solutions describing the structure of the DSW. We point out the phenomenon of the overturning of DSW in a medium with positive dispersion, caused by cutoff of the flow due to the vanishing of the density near the trailing edge of the DSW. We obtain an asymptotically exact solution of the problem of the decay of an arbitrary initial discontinuity.

§1. INTRODUCTION

The studies of nonlinear nonstationary flows of dispersive hydrodynamics are not only important for many problems in the dynamics of a rarefied plasma, the hydrodynamics of waves on water, nonlinear waves in dielectrics, electro-acoustic waves, and so on, but are also of theoretical interest in their own right. Indeed, if we neglect the small dispersive terms, the flow is, as in ordinary hydrodynamics, described by the Euler equations. The most important feature of nonlinear Euler dynamics is the overturning of a front leading to the appearance of a shock wave (Ref. 1, § 101). The most important role in ordinary hydrodynamics is in this case played by dissipative processes which lead to the establishment of a shock-wave front which has a finite width.

In dispersive hydrodynamics the motion following the onset of a singularity is, when there is no dissipation at all, of a nature which is different in principle. Here there appears a region which expands continuously with time and which is filled with undamped small-scale oscillations. This is a dissipationless shock wave (DSW).[2] A consistent description of a DSW is rather complicated. It has only been possible to study small amplitude waves which are described by the Korteweg–de Vries (KdV) equation.[3] Moreover, a numerical and analytical study of the decay of initial discontinuities in dispersive hydrodynamics made it possible to distinguish a DSW as an expanding region where the oscillations have a quasi-stationary nature and to find a relation which determines the change in the mean values when one passes through a DSW, similar to the Rankine-Hugoniot adiabat in ordinary hydrodynamics.[4]

It is convenient to use for the description of an expanding region filled with quasi-stationary oscillations Whitham's method[5] which is based upon averaging the integrals of the initial equation over the oscillations. The resultant system of averaged hydrodynamic equations is rather complicated. These equations can be appreciably simplified if there exist Riemann invariants for them. Earlier, Riemann invariants have been found only for systems of averaged equations corresponding to an initial KdV equation.[5] Dubrovin and Novikov[6] posed the general problem of a connection between the existence of Riemann invariants of averaged equations and the complete integrability of the initial equation (see also Ref. 7 and the comprehensive survey in Ref. 8). On the basis of these studies Pavlov[9] recently determined Riemann invariants for a set of averaged equations corresponding to the nonlinear Schrödinger equation.[10]

We shall consider a nonlinear Schrödinger equation with repulsion (or defocusing), which describes dispersive hydrodynamics with an adiabatic index $\gamma = 2$ and with positive dispersion (see Ref. 11, §§ 15, 16). The latter means that the wave velocity in the medium increases with increasing wave vector k. A study of the dynamics of nonlinear flows in such a medium is just the aim of the present paper. The use here of the Riemann invariants for the averaged equations simplifies their solution appreciably. We formulate in § 2 the averaged Whitham equations in Riemannian form and consider their limiting transitions into a KdV system and into Eulerian hydrodynamics. We give in § 3 the theory of DSW. We distinguish the concepts of quasi-simple and simple DSW. We obtain analytical solutions which completely describe a simple DSW. We show that the structure of a DSW in a medium with positive dispersion is qualitatively different from the previously considered DSW for negative dispersion: the wave starts, for instance, from very weak small-scale oscillations and not from solitons, and has on the fast leading edge a weak rather than a singular discontinuity (cf. Refs. 3,4). On the contrary, close to the trailing edge the density and the wave velocity undergo a very sudden change. The region occupied by the oscillations expands rapidly with time.

In § 3 we also introduce the phenomenon of the overturning of a DSW with positive dispersion. It comes about because the density in the negative solitons near the trailing edge of the DSW vanishes, so that a singularity connected with the cutoff of the flow appears. The limiting value of the jump in the density at the DSW for which overturning occurs is $(\rho_2/\rho_1)_{cr} = 4$. We consider in § 4 the decay of initial discontinuities. We give a complete classification and obtain analytical solutions describing the decay process for any initial discontinuity in dispersive hydrodynamics with positive dispersion.

An important application of the theory considered here refers to nonlinear optics and the nonlinear theory of the propagation of radiowaves in a plasma (in particular, in the ionosphere). The Euler equations correspond then to the geometric-optics approximation: the density corresponds to the wave intensity, and the velocity to the direction of their propagation in a plane at right angles to the beam axis. It is well known that nonlinear geometric optics is valid only up

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to the moment when singularities appear on the profile of the beam intensity (see, e.g., Ref. 11, § 13). Nonlinear oscillations develop beyond the singular point and diffraction effects become important. The method used in the present paper allows us to describe the structure of the beam both in the smooth and in the oscillatory region. The DSW demonstrates, for instance, how the oscillatory structure develops after the overturning of the profile of the beam intensity.

§2. WHITHAM'S EQUATIONS AND RIEMANN INVARIANTS

The nonlinear Schrödinger equation with repulsion, which describes the wave propagation in a nonlinear defocusing medium, has the form[10]

$$2iu_t+u_{xx}-2|u|^2u=0. \tag{1}$$

The subscripts t and x here and henceforth indicate derivatives with respect to t and x. A change of variables $u=\rho^{1/2}e^{i\varphi}$, $\varphi_x=v$ reduces Eq. (1) to hydrodynamic form (Ref. 11, § 12):

$$\rho_t+(\rho v)_x=0, \tag{2a}$$

$$v_t+vv_x+\rho-\frac{1}{4}\left(\frac{\rho_{xx}}{\rho}-\frac{\rho_x^2}{2\rho_2}\right)_x=0. \tag{2b}$$

This set of equations describes one-dimensional hydrodynamic flows when there is no dissipation and when the dispersion is positive. Indeed, the dispersion equation for the set (2) has the form

$$\omega-kv_0=\pm k\rho_0^{1/2}(1+k^2/4\rho_0)^{1/2}, \tag{3}$$

whence it follows that there is no dissipation and that the dispersion is positive, since the phase and the group velocities in the medium increase with increasing wave vector k. We emphasize that we use in (2) dimensionless functions and variables: the density ρ is normalized by some characteristic density value ρ_1, the velocity v by the sound velocity c_s (at $\rho=\rho_1$), the coordinate x by a characteristic length D of the oscillations which is determined by the small dispersion parameter δ, and the time t by a characteristic time of the oscillations D/c_s (see Ref. 11). The dispersion parameter in (2) is therefore of order unity, and the characteristic length and time of the oscillations are of the same order. If we neglect the last (dispersive) term in Eq. (2b), which is justified for sufficiently smooth large-scale motions,

$$\Delta x\sim\rho/\rho_x\gg1,\quad \Delta t\sim\rho/\rho_t\gg1, \tag{4}$$

Eqs. (2) go over into the equations of ideal Eulerian hydrodynamics with adiabatic index $\gamma=2$ (i.e., with pressure $p=\rho^2/2$).

Equations (2) have stationary solutions in the form of a periodic traveling wave $f(x-Ut)$:

$$\rho=b_1-(b_1-b_3)\mathrm{dn}^2[(b_1-b_3)^{1/2}(x-Ut)],\quad v=U-(b_1b_2b_3)^{1/2}/\rho. \tag{5}$$

Here U and b_i are arbitrary constants ($b_1>b_2>b_3$); dn ξ is a Jacobi elliptic function. The parameters of the elliptic function dn ξ are connected with the b_i through the usual relations:[3,12]

$$s^2=(b_2-b_3)/(b_1-b_3),$$

$$a=(b_2-b_3)/2,\ b_1-b_3=2a/s^2,\ b_1>b_2>b_3>0. \tag{6}$$

The wave vector k of the oscillations is then given by the formula

$$k=(\pi/K(s))(2a/s^2)^{1/2}. \tag{7}$$

($K(s)$ and $E(s)$ are complete elliptic integrals of the first and second kind.)

The values of the hydrodynamic quantities ρ and v averaged over the period of the stationary wave (5) can be expressed in terms of the parameters U and b_i through the formulae (see Ref. 13)

$$\begin{aligned}\bar\rho&=b_1-(b_1-b_3)E(s)/K(s),\\ \bar v&=U-(b_2b_3/b_1)^{1/2}\\ &\quad-(b_1-b_3)^{1/2}[E(\varphi,s_1)-F(\varphi,s_1)(1-E(s)/K(s))].\end{aligned} \tag{8}$$

Here $F(\varphi,s)$ and $E(\varphi,s)$ are incomplete elliptic integrals of the first and second kind,

$$\varphi=\arcsin[(b_1-b_3)^{1/2}/b_1^{1/2}],\quad s_1^2=1-s^2.$$

One can look for quasi-stationary solutions of the set (2) in the same form (5) assuming that the parameters b_i and U are not constant, but vary slowly as functions of x and t. Their evolution is described by Whitham's equations, which in the case considered here can be written in Riemannian form. The Riemann invariants r_i are connected with the parameters b_i, U of the stationary wave through the following relations:

$$\begin{aligned}U&={}^1/_4(r_1+r_2+r_3+r_4),\\ b_1&={}^1/_{16}(r_1+r_2-r_3-r_4)^2={}^1/_{16}(\Delta_{13}+\Delta_{24})^2,\\ b_2&={}^1/_{16}(r_1+r_3-r_2-r_4)^2={}^1/_{16}(\Delta_{12}+\Delta_{34})^2,\\ b_3&={}^1/_{16}(r_1+r_4-r_2-r_3)^2={}^1/_{16}(\Delta_{12}-\Delta_{34})^2.\end{aligned} \tag{9}$$

Here

$$\Delta_{ik}=r_i-r_k,\quad r_1>r_2>r_3>r_4. \tag{10}$$

Whitham's equations which describe the evolution of the invariants r_i have the form (Ref. 9)[1)]

$$\partial r_k/\partial t+V_k(r)\partial r_k/\partial x=0,\ k=1,\ 2,\ 3,\ 4, \tag{11a}$$

where

$$\begin{aligned}V_k(r)&=U(r)+W_k(r),\\ W_1&=\Delta_{12}[2(1-\Delta_{24}E(s)/\Delta_{14}K(s))]^{-1},\\ W_2&=-\Delta_{12}[2(1-\Delta_{13}E(s)/\Delta_{23}K(s))]^{-1},\\ W_3&=\Delta_{34}[2(1-\Delta_{24}E(s)/\Delta_{23}K(s))]^{-1},\\ W_4&=-\Delta_{34}[2(1-\Delta_{13}E(s)/\Delta_{14}K(s))]^{-1}.\end{aligned} \tag{11b}$$

Here

$$s^2=\Delta_{12}\Delta_{34}/\Delta_{13}\Delta_{24},\quad s_1^2=1-s^2=\Delta_{14}\Delta_{23}/\Delta_{13}\Delta_{24}. \tag{12}$$

The inverse formulae expressing the Riemann invariants in terms of the parameters of the quasi-stationary wave have the form

$$r_k=U+R_k(b_1,\ b_2,\ b_3).$$

The expressions for the R_k depend here on the sign of ($\Delta_{34}-\Delta_{12}$). If ($\Delta_{34}-\Delta_{12})>0$, we have

$$R_1=b_1^{1/2}+b_2^{1/2}-b_3^{1/2},\ R_2=b_1^{1/2}-b_2^{1/2}+b_3^{1/2},\ R_3 =-b_1^{1/2}+b_2^{1/2}+b_3^{1/2},\ R_4=-b_1^{1/2}-b_2^{1/2}-b_3^{1/2}.$$

If, however, $(\Delta_{34}-\Delta_{12})<0$, we have

$$R_1=b_1^{1/2}+b_2^{1/2}+b_3^{1/2},\ R_2=b_1^{1/2}-b_2^{1/2}-b_3^{1/2},\ R_3 =-b_1^{1/2}+b_2^{1/2}-b_3^{1/2},\ R_4=-b_1^{1/2}-b_2^{1/2}+b_3^{1/2}.$$

We consider limiting cases. We assume that the invariants r_1, r_2, r_3 are close in magnitude and differ considerably from r_4:

$$\Delta\approx\Delta_{\alpha 4}\gg\Delta_{\beta\gamma},\ \alpha,\ \beta,\ \gamma=1,\ 2,\ 3. \tag{13}$$

It then follows from (11) that

$$W_1\to W_1^0\approx{}^1/_2 s_0^2\Delta_{13}(1-E(s_0)/K(s_0))^{-1},$$

$$W_2\to W_2^0\approx-{}^1/_2 s_0^2 s_{10}^2\Delta_{13}K(s_0)/(s_{10}^2K(s_0)-E(s_0)),$$

$$W_3\to W_3^0\approx{}^1/_2 s_{10}^2\Delta_{13}K(s_0)/E(s_0),\ W_4\to W_4^0\approx-{}^1/_2\Delta,$$

$$s_0^2=\Delta_{12}/\Delta_{13},\ s_{10}^2=\Delta_{23}/\Delta_{13}.$$

Neglecting the change in the invariant r_4, i.e., putting $r_4=r_{40}$ we rewrite Eqs. (11a) in the form

$$\frac{\partial r_\alpha}{\partial t}+\left[\frac{1}{4}(r_{40}+r_1+r_2+r_3)+W_\alpha^0\right]\frac{\partial r_\alpha}{\partial x}=0,\quad \alpha=1,2,3. \tag{14}$$

We replace now the r_α by new values $r_\alpha^*=\frac{3}{4}r_\alpha+\frac{1}{4}r_{40}$ and go over to the equations for the r_α^* which are exactly the same as the Whitham equations for KdV.[5,3] We have thus made the transition to the KdV system (14) by assuming the invariant r_4 to be constant: $r_4=r_{40}$ (the last of Eqs. (11a) is then identically satisfied) and to differ appreciably from the invariants r_1, r_2, r_3 which lie close to one another [Eq. (13)].

We now consider the singular points where the invariants intersect (or merge). Let, for instance, $r_1=r_2=r_0$. Then

$$\Delta_{12}=0,\ \Delta_{13}=\Delta_{23},\ \Delta_{14}=\Delta_{24},\ \Delta_{13}-\Delta_{24}=\Delta_{23}-\Delta_{14}=-\Delta_{34},$$

and, hence,

$$s^2=0,\ E/K=1,\ U={}^1/_4(r_3+r_4)+{}^1/_2 r_0,\ b_2=b_3={}^1/_{16}(\Delta_{34})^2. \tag{15}$$

In this case

$$W_1=W_2=\frac{\Delta_{13}\Delta_{14}}{\Delta_{13}+\Delta_{14}},\quad V_1=V_2=U+\frac{\Delta_{13}\Delta_{14}}{\Delta_{13}+\Delta_{14}},$$

so that the characteristics of the invariants r_1 and r_2 are joined together. We then get for the invariants r_3 and r_4

$$W_3=-{}^1/_2\Delta_{23},\ W_4=-{}^1/_2\Delta_{24},\ V_3={}^3/_4 r_3+{}^1/_4 r_4,\ V_4={}^3/_4 r_4+{}^1/_4 r_3$$

and Eqs. (11) take the form

$$\frac{\partial r_3}{\partial t}+({}^1/_4 r_4+{}^3/_4 r_3)\frac{\partial r_3}{\partial x}=0,\quad \frac{\partial r_4}{\partial t}+({}^1/_4 r_3+{}^3/_4 r_4)\frac{\partial r_4}{\partial x}=0. \tag{16}$$

One checks easily that these equations are identically the same as the Riemannian form of the hydrodynamical Eqs. (2) in the Euler limit, i.e., when one neglects the dispersion terms. Indeed, the Euler equations

$$\rho_t+(\rho v)_x=0,\quad v_t+vv_x+\rho_x=0 \tag{17}$$

are reduced to the Riemannian form using the invariants (Ref. 1, § 104)

$$r_+=v+2\rho^{1/2},\quad r_-=v-2\rho^{1/2},$$

$$v={}^1/_2(r_++r_-),\quad \rho={}^1/_{16}(r_+-r_-)^2. \tag{18}$$

One easily sees that the equations for the Riemann invariants r_+ and r_- are exactly the same as (16). From the continuity condition for the hydrodynamic quantities ρ and v we find that also the values of the invariants r_3 and r_4 in that case are the same as r_+ and r_-:

$$r_3=r_+,\ r_4=r_-. \tag{19}$$

Therefore, when r_1 and r_2 merge the invariants r_3 and r_4 in fact become invariants of the Euler Eqs. (17). Therefore, as should be the case, when $r_1=r_2$ we have by virtue of (15) $b_1=b_2$ and the amplitude of the oscillations in (5) vanishes. Hence, the oscillations vanish and Whitham's Eqs. (11) must then be identical with the Euler Eqs. (17).

The same transition also occurs when $r_2=r_3$. In this case

$$\Delta_{23}=0,\ \Delta_{12}=\Delta_{13},\ \Delta_{24}=\Delta_{34},\ \Delta_{12}+\Delta_{34}=\Delta_{13}+\Delta_{34}=\Delta_{14}$$

and hence

$$s^2=1,\ E/K=0,\ k=0,\ W_2=W_3=0,$$

$$V_1={}^3/_4 r_1+{}^1/_4 r_4,\ V_4={}^3/_4 r_4+{}^1/_4 r_1, \tag{20}$$

so that now the invariants r_1 and r_4 turn out to be identical with the invariants of the Euler equations:

$$r_1=r_+,\ r_4=r_-. \tag{21}$$

The oscillations vanish in this case as follows: they turn into separate negative solitons (see (5)) and the distance between them tends to infinity, as $k\to 0$, when $r_2\to r_3$ [Eq. (20)].

§3. DISSIPATIONLESS SHOCK WAVE

Broad discontinuity

In Euler hydrodynamics the evolution of a finite-amplitude perturbation in a compression wave always leads to the appearance of a singular point on the wave front (Ref. 1, §§ 101, 103). Beyond the singular point in dissipationless dispersive hydrodynamics there appears and gradually expands in time a region filled with small-scale oscillations (Fig. 1). This is the DSW region.

The hydrodynamic flow and the oscillations in the DSW region are, when the conditions (4) for the average quantities $\bar{\rho}$ and $\bar{v}$ of (8) are satisfied, described by Whitham's equations. Outside the DSW there are in the same limit as $t\to\infty$ no oscillations, and the flow has here a purely hydrodynamical character and is described by the Euler equations.

We formulate the conditions for the matching of the Whitham and Euler equations at the boundaries of the DSW. To do this it is convenient to use the Riemann invariants. In the DSW region there are the four invariants (r_1,r_2,r_3,r_4) of (10) and in the Euler region the two (r_+,r_-) of (18). Their matching is accomplished in the general case in the way shown in Fig. 1. The boundaries of the DSW are defined as the lines where the invariants reclose (or merge): on the one boundary r_1 merges with r_2 (i.e., $s=0$) and on the other one r_2 merges with r_3 (i.e., $s=1$). In that case on the first boundary according to (19) r_3 (i.e., $s=1$). In that case on the first boundary according to (19) r_3 links up with r_+ and r_4 with r_-; on the second boundary according to (21) r_1 links up with r_+ and r_4 again with r_-. Thus

$$r_1|^1=r_2|^1,\ r_3|^1=r_+|^1,\quad r_4|^1=r_-|^1;\ s=0, \tag{22a}$$
$$r_2|^2=r_3|^2,\ r_4|^2=r_-|^2,\quad r_1|^2=r_+|^2;\ s=1. \tag{22b}$$

The indexes $|^1$ indicate here the leading boundary of the DSW and the index $|^2$ the trailing one. It is clear from (22) that in the positive-dispersion case considered here always at the leading edge of the DSW, small amplitude oscillations ($s^2 \to 0, a \to 0$) arise which qualitatively distinguishes it from the negative-dispersion DSW that starts with a steep soliton front (see Refs. 3, 4).

We show that the boundary conditions (22) are sufficient for an unambiguous determination of the solution. The Euler invariants r_+ and r_- ahead of the DSW are determined at any point by Eqs. (17) and (18) and also by the initial and boundary conditions of the problem, and we must therefore consider them in (22) as being given. The four conditions [the first two from (22a) and the first two from (22b)] constitute a complete set of boundary conditions for the parameters r_i. Equations (11) determine them next in the whole DSW region. The last two conditions in (22a) and (22b) serve as boundary conditions for the invariants r_+ and r_- in the Euler region beyond the DSW.

We emphasize that the DSW boundaries $s=0$ and $s=1$ are not the location of a singularity for the external, Euler part of the solution. At the same time they are singular lines where the characteristics for the interior Whitham region merge. For an effective use of the boundary conditions (22) in that region it is necessary to elucidate the form of the solution of Eqs. (11) near the singular points. An analysis similar to the one given in Ref. 3 leads to the following results. Near the leading edge of the wave ($s^2=0$) we have, when $r_1^1(t) = r_2^1(t) = r^1(t)$,

$$s^2=4\left[2\frac{dr^1}{dt}(x^1(t)-x)\right]^{1/2}|2r^1-r_3^1-r_4^1|\,[3(r_3^1-r_4^1)^2$$
$$+8(r^1-r_3^1)(r^1-r_4^1)]^{-1/2}(r_3^1-r_4^1)(r^1-r_3^1)^{-1}(r^1-r_4^1)^{-1}, \tag{23}$$

where according to (22a) $r_3^1 = r^1_+(t)$, $r_4^1 = r^1_-(t)$. Here $x^1(t)$ is a point on the leading edge. It is clear from (23) that the amplitude a of the oscillations which is proportional to s^2 increases near the leading edge $x^1(t)$ as $(x^1(t)-x)^{1/2}$.

Similarly, near the trailing edge of the wave ($s=1$) we have, when $r_2^2 = r_3^2 = r^2(t)$,

$$s_1^4\left(\ln\frac{16}{s_1^2}+\frac{1}{2}\right)=-\frac{16(r_1^2-r_4^2)^2}{(r_1^2-r^2)^2(r^2-r_4^2)^2}\frac{dr^2}{dt}(x-x^2(t)). \tag{24}$$

Here, according to (22b)

$$r_1^2=r_+^2(t),\ r_4^2=r_-^2(t),$$

where $r_i^2 = r_i|^2$. It is clear that near the trailing edge $x^2(t)$ the wave vector k vanishes in proportion to $[\ln(x-x^2(t))]^{-1}$. It follows from (24) and (8) that as $x \to x^2(t)$ the average values $\bar{\rho}$ and $\bar{v}$ also tend to their limiting values according to the law $[\ln(x-x^2(t))]^{-1}$. On the trailing edge $\bar{\rho}$ and $\bar{v}$ have therefore in the general case a singularity with an infinite derivative.

Conditions (22)–(24) show that a DSW occupying a spatial region which is finite and continuously expanding with time can be inserted in the solution of the Euler equations. In that sense one can say that the Euler equations allow strong discontinuities which expand with time and the DSW is such a discontinuity. Following Ref. 4 one may call this discontinuity an expanding or even a broad discontinuity, since one cannot make it infinitesimally narrow through any choice of variables whatever. Such discontinuities arise only in dispersive hydrodynamics.

We now dwell on a few particular cases.

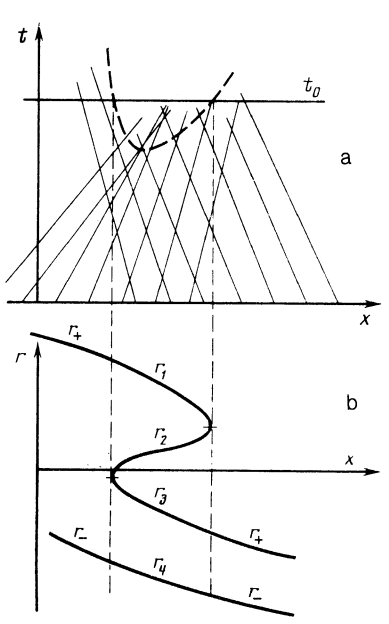

FIG. 1. Occurrence of a dissipationless shock wave. a: intersection of the characteristics of the Euler equations in the x, t plane (occurrence of a singularity). The dashed lines are the boundaries of the DSW region (leading edge for the larger x). b: Riemann invariants as functions of x for $t = t_0$.

Quasi-simple wave

We consider a simple Riemann wave in Euler hydrodynamics (Ref. 1, §§ 101, 104). This a particular solution of the Euler Eqs. (17) or (16), for which one of the invariants, for instance, r_-, is constant:

$$r_-=\text{const}=r_0. \tag{25}$$

It follows from (16) that the evolution of the invariant r_+ is then described by the equation of a simple wave:

$$\partial r_+/\partial t+(^3/_4 r_+ + {}^1/_4 r_0)\partial r_+/\partial x=0.$$

The hydrodynamic velocity v and the density ρ are in the simple wave connected through the relation (18), (25):

$$v-2\rho^{1/2}=r_0. \tag{26}$$

After the overturning of the simple wave a DSW is formed in dispersive hydrodynamics. The equation (11) for the invariant r_4 with the boundary conditions (22), (25) is then identically satisfied if we put

$$r_4=\text{const}=r_0. \tag{27}$$

It is thus natural to call the DSW quasi-simple. It arises from a simple Riemann wave and its evolution is completely described by the change in only three invariants: r_1, r_2, r_3 according to Eqs. (11) with the boundary conditions (22). The fourth invariant then stays constant: $r_4 = r_- = r_0$. We show in Fig. 2 the behavior of the invariants in a quasi-simple wave.

The quasi-simple waves are a broad class of DSW. They are, in particular, all DSW described by the KdV equation—

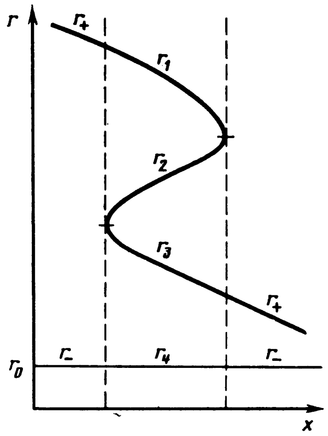

FIG. 2. Riemann invariants for a quasi-simple DSW.

the quasi-simple waves go over into them as $r_0 \to -\infty$. It is also important that quasi-simple waves appear when a simple wave overturns in any Eulerian hydrodynamics. Moreover, a quasi-simple wave always describes the process of the appearance of a DSW. Indeed, a DSW emerges after singularities arise in Eulerian hydrodynamics. It is clear from Fig. 1 (see Ref. 1, § 103) that the singular point then corresponds to the intersection of characteristics only of one of the invariants—we call it r_+. The change in the second invariant, r_-, in the region where the DSW arises can then always be neglected, i.e., we may assume it to be constant. And this means that the DSW in the region where it arises is a quasi-simple wave. Moreover, in the vicinity of the point of overturning one can split off the main, non-changing part also of the second invariant: $r_+ = r_{+0} + \Delta r_+$, $|\Delta r_+| \ll r_{+0}$. Equations (11) then reduce to the equations for the Riemann invariants for the KdV system (14) and we are led to an already solved problem of the appearance of DSW in the KdV system.[3]

The boundary condition (22) for quasi-simple waves can be formulated in a compact form not only in terms of Riemann invariants but also as a condition on the hydrodynamic variables. Indeed, using (22), (25), (26), (27), and eliminating the invariant r_0 we find

$$v_2 - v_1 = 2(\rho_2^{1/2} - \rho_1^{1/2}). \tag{28}$$

Here v_1, ρ_1 and v_2, ρ_2 are the values of the hydrodynamic velocity and the density, respectively, at the leading and trailing edges of the DSW. The jump in the velocity when one passes through a quasi-simple DSW is thus uniquely connected with the jump in the density.[2)] The condition (28) in the actual case $\gamma = 2$ considered here is exactly the same as the general condition for passing through a DSW formulated in Ref. 4. This condition replaces the Rankine-Hugoniot adiabat in ordinary hydrodynamics.

We emphasize that condition (28) is valid only for quasi-simple DSW. In the general case conditions (22) hold.

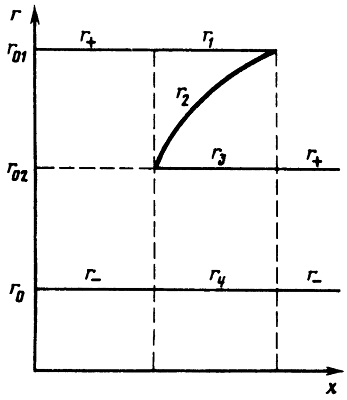

FIG. 3. Riemann invariants for a simple DSW.

Simple DSW

We now consider the case of uniform flow in Euler hydrodynamics when not only the invariant r_- but also the invariant r_+ is constant; only the constant is different at the left and the right boundaries of the DSW:

$$r_- = r_0, \quad r_+|^1 = r_{01}, \quad r_+|^2 = r_{02}. \tag{29}$$

It is clear from (18) that in this case the hydrodynamic quantities, the density ρ and the velocity v, undergo a constant jump on passing through the DSW. This corresponds to the classical statement of the shock wave problem in ordinary hydrodynamics.

The solution of Eqs. (11) with the boundary conditions (22), (29) is clearly:

$$\begin{gathered} r_4 = r_0, \quad r_3 = r_{01}, \quad r_1 = r_{02}, \\ \partial r_2/\partial t + V_2(r_{02};\, r_2;\, r_{01},\, r_0)\, \partial r_2/\partial x = 0, \end{gathered} \tag{30}$$

where r_0, r_{01}, r_{02} are given constants and the velocity V_2 is given by Eq. (11b). It is clear that in the DSW considered only one Riemann invariant, r_2, varies. It is thus a simple Riemann wave in dispersive hydrodynamics. It is natural therefore to call it a simple DSW.

It is well known (Ref. 1, § 101) that one can write the general solution of Eq. (30) in the following form: r_2 depends on only one parameter, τ, and this dependence is given implicitly by the relation

$$V_2(r_{02};\, r_2;\, r_{01};\, r_0) = \tau, \tag{31}$$

while the connection between τ and x and t is given by the equation

$$x = \tau t + P(\tau). \tag{32}$$

Here $P(\tau)$ is an arbitrary function. We show the behavior of the invariants in a simple DSW in Fig. 3.

We analyze in detail the solution obtained. Let the density and the velocity behind the DSW be $\rho = \rho_0$ and $v = v_0$, and in front of it $\rho = \rho_1$ and $v = v_1$. Without losing generality we may assume that $\rho_1 = 1$ (we measure ρ in units ρ_1), $v_1 = 0$ (we go over to the appropriate system of coordinates) and $\rho_0 > 1$ (the DSW is a compression wave). The velocity v_0 is then determined from Eq. (28):

$$v_0 = 2(\rho_0^{1/2} - 1), \tag{33}$$

and, thus, the density jump ρ_0 is the only parameter in the problem. Hence, the dissipationless shock wave is completely determined by specifying the density jump ρ_0.

Using (30), (29), (18) we find the values of the constant invariants:

$$r_1 = 2(2\rho_0^{1/2} - 1), \quad r_3 = 2, \quad r_4 = -2. \tag{34}$$

Moreover, it is convenient to express all quantities in terms of s^2. We have

$$b_1 = \frac{[2\rho_0^{1/2} - 1 + (\rho_0^{1/2} - 1)^2 s^2]^2}{(1 + (\rho_0^{1/2} - 1) s^2)^2},$$

$$b_2 = \frac{[1 + (\rho_0 - 1) s^2]^2}{[1 + (\rho_0^{1/2} - 1) s^2]^2}, \quad b_3 = \frac{[1 - (\rho_0^{1/2} - 1)^2 s^2]^2}{[1 + (\rho_0^{1/2} - 1) s^2]^2}$$

$$r_2 = \frac{2[2\rho_0^{1/2}-1-(\rho_0^{1/2}-1)s^2]}{1+(\rho_0^{1/2}-1)s^2}, \qquad a = \frac{2\rho_0^{1/2}(\rho_0^{1/2}-1)s^2}{1+(\rho_0^{1/2}-1)s^2},$$

$$U = \frac{2\rho_0^{1/2}-1+(\rho_0^{1/2}-1)^2 s^2}{1+(\rho_0^{1/2}-1)s^2}, \qquad k = \frac{2\pi\rho_0^{1/4}(\rho_0^{1/2}-1)^{1/2}}{K(s)[1+(\rho_0^{1/2}-1)s^2]^{1/2}} \tag{35}$$

$$\bar\rho = \{[2\rho_0^{1/2}-1+(\rho_0^{1/2}-1)^2 s^2]^2 - 4\rho_0^{1/2}(\rho_0^{1/2}-1) \times [1+(\rho_0^{1/2}-1)s^2]E/K\} \times [1+(\rho_0^{1/2}-1)s^2]^{-2},$$

$$v = U-(b_2b_3/b_1)^{1/2}-(b_1-b_3)^{1/2}[E(\varphi, s_1)-F(\varphi, s_1)(1-E/K)],$$

$$\varphi = \arcsin[(b_1-b_3)/b_1]^{1/2}.$$

Equation (31) then takes the form

$$\tau = \left\{2\rho_0^{1/2}-1+(\rho_0^{1/2}-1)^2 s^2 + \frac{2\rho_0^{1/2}(\rho_0^{1/2}-1)s^2(1-s^2)}{[1+(\rho_0^{1/2}-1)s^2]E/K+s^2-1}\right\} \times [1+(\rho_0^{1/2}-1)s^2]^{-1}. \tag{36}$$

Equations (35), (36), (32), are the final ones. They allow us to find both all average values (7) and (8) in which we are interested and the oscillatory structure of the DSW at any point x and t for any density jump ρ_0.

It follows from (33), in particular, that the point $\tau = \tau^+$ determined by the position of the leading edge of the wave is

$$s^2=0, \quad \tau^+ = (8\rho_0 - 8\rho_0^{1/2}+1)/(2\rho_0^{1/2}-1), \tag{37}$$

and the point τ^- corresponding to the trailing edge

$$s^2=1, \quad \tau^- = \rho_0^{1/2}. \tag{38}$$

Hence it is clear that the width of the DSW increases rapidly with increasing density jump:

$$\Delta\tau = \tau^+ - \tau^- = [(6\rho_0^{1/2}-1)/(2\rho_0^{1/2}-1)](\rho_0^{1/2}-1). \tag{39}$$

We show in Fig. 4 the behavior of the parameter s^2, the wave vector k, the amplitude a of the oscillations, the average density $\bar\rho$, and the average velocity $\bar v$ in the DSW as functions of τ for various values of the density jump ρ_0. It is clear that both on the leading and on the trailing edge of the wave the average values of the hydrodynamic quantities $\bar\rho$ and $\bar v$ undergo a weak discontinuity. However, on the trailing edge this discontinuity is singular: the derivatives $d\bar\rho/d\tau$ and $d\bar v/d\tau$ tend to infinity (cf. Ref. 3). The velocity of the motion of the trailing edge (29) then turns out to be equal to the sound velocity as $k \to 0$, (3), i.e., the nonlinearity does practically not affect the motion of the solitons. This also distinguishes substantially the wave considered from the negative-dispersion case where the soliton velocity is always determined by the nonlinearity.[2-4]

We study the behavior of the wave in the vicinity of the leading and the trailing edges. Expanding all quantities in powers of s^2 we find near the leading edge τ^+:

$$r_2 = 2(2\rho_0^{1/2}-1) - s^2\cdot 4\rho_0^{1/2}(\rho_0^{1/2}-1),$$
$$a = s^2\cdot 2\rho_0^{1/2}(\rho_0^{1/2}-1),$$
$$k = 4\rho_0^{1/4}(\rho_0^{1/2}-1)^{1/2} - s^2\rho_0^{1/4}(\rho_0^{1/2}-1)^{1/2}(2\rho_0^{1/2}-1), \tag{40}$$

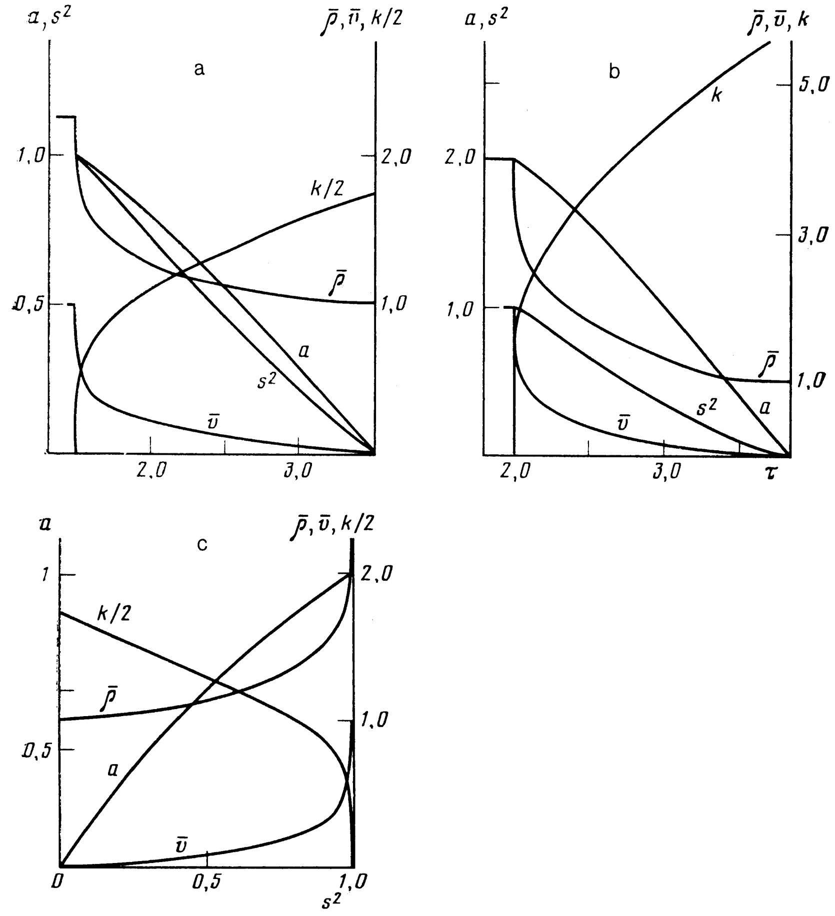

FIG. 4. Average quantities as functions of $\tau = x/t$ in a DSW: a: $\rho_0 = 2.25$; b: $\rho_0 = 4.0$; c: as function of s^2, $\rho_0 = 2.25$.

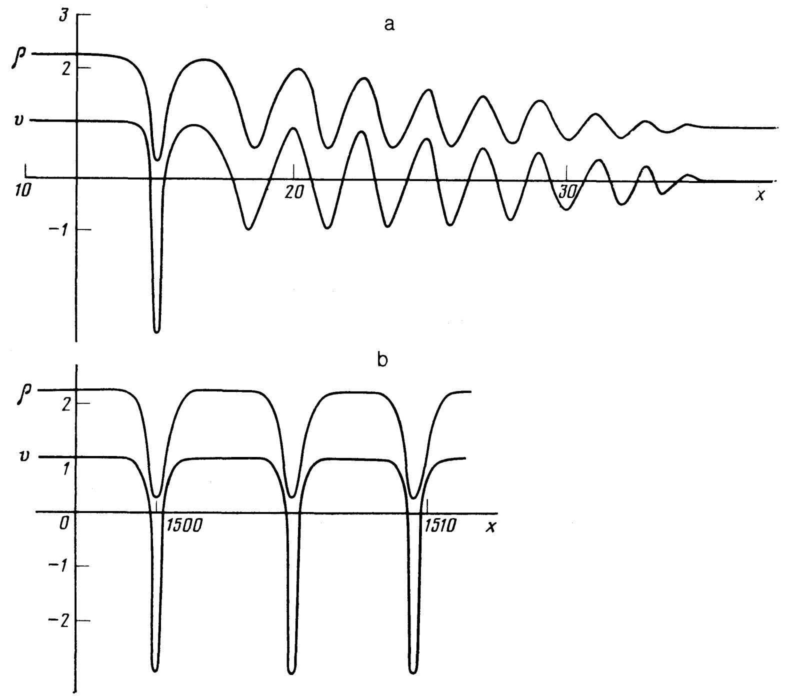

FIG. 5. Oscillatory structure of a DSW for $\rho_0 = 2.25$: a: $t = 10$; b: $t = 1000$ (trailing edge).

$$\bar{\rho}=1+s^4\rho_0^{1/2}(\rho_0^{1/2}-1)(2\rho_0^{1/2}-1)^2/4,$$
$$\bar{v}=s^4\rho_0^{1/2}(\rho_0^{1/2}-1)(2\rho_0^{1/2}-1)/4,$$
$$U=(2\rho_0^{1/2}-1)-s^2\rho_0^{1/2}(\rho_0^{1/2}-1),$$

where

$$s^2=(\tau^+-\tau)(2\rho_0^{1/2}-1)^2\{3\rho_0^{1/2}(\rho_0^{1/2}-1)[(2\rho_0^{1/2}-1)^2+1/2]\}^{-1}. \tag{41}$$

It is clear that when the distance from the leading edge of the wave $x^+ - x$ increases the amplitude of the oscillations increases linearly according to (40) and (41), and the average density and velocity increase only quadratically. This case differs thus from the usual picture for the increase of a perturbation near the leading edge, (23), of a DSW.

Close to the trailing edge we find by expanding in $s_1^2 = 1 - s^2$

$$r_2=2+s_1^2\cdot 4(\rho_0^{1/2}-1)/\rho_0^{1/2},\ a=2(\rho_0^{1/2}-1)-s_1^2(2\rho_0^{1/2}-1)/\rho_0,$$
$$k=4\pi(\rho_0^{1/2}-1)^{1/2}\Lambda^{-1},\ \Lambda=\ln(16/s_1^2),\ \bar{\rho}=\rho_0-8(\rho_0^{1/2}-1)\Lambda^{-1},$$
$$\bar{v}=v_0-4(\rho_0^{1/2}-1)^{1/2}\arcsin[2(\rho_0^{1/2}-1)^{1/2}/\rho_0^{1/2}]\Lambda^{-1}, \tag{42}$$
$$v_0=2(\rho_0^{1/2}-1),$$

where

$$s_1^2\Lambda=[\rho_0^{1/2}/(\rho_0^{1/2}-1)](\tau-\tau^-). \tag{43}$$

It is clear that near the trailing edge the wavelength tends logarithmically to infinity while the averages $\bar{\rho}$ and $\bar{v}$ have a singularity.

We show in Fig. 5 the behavior of the oscillations $\rho(x)$ and $v(x)$ in the DSW, constructed according to (5), (35), (36) (the function $P(\tau)$ is here taken to be zero, which corresponds to a self-similar motion $\tau = x/t$). The minimum of the first soliton is fixed at the point of the leading edge $x^-/t = \rho_0^{\frac{1}{2}}$ (for a more rigorous determination of the phase of the oscillations see Ref. 14). From this figure it is clear that the region occupied by oscillations shifts to the right when t increases and expands rapidly. Near the trailing edge there are then split off negative solitons and the distance between them increases logarithmically, (42), (43). This is clear from Fig. 5b where we show the region near the trailing edge for the large value $t = 1000$ (the whole wave in that case contains already more than 800 oscillations). Attention is called to the steep increase of the magnitude of the velocity near the peaks of the negative solitons. This increase is the consequence of the appreciable lowering of the density: in the peak the density $\rho_m = 0.25$, i.e., almost an order of magnitude less than the average ($\rho \approx 2.2$) which leads through the conservation of flux to a steep increase in the magnitude of the velocity.

We assumed above that $\rho_1 = 1$, $v_1 = 0$, $\rho_0 > 0$. We now turn to a jump with arbitrary values of ρ_2, ρ_1, and v_1. To do this we put $\rho_0 = \rho_2/\rho_1$, and for the reference solution constructed above $\tilde{\rho}(\rho_0,\tilde{x},\tilde{t})$ and $\tilde{v}(\rho_0,\tilde{x},\tilde{t})$. The solution ρ, v can then be expressed for all values of ρ_2, ρ_1, v_1 in terms of the reference solution $\tilde{\rho},\tilde{v}$ through the following formulae:

$$\rho=\rho_1\tilde{\rho},\quad v=\rho_1^{1/2}\tilde{v}+v_1, \tag{44}$$

where

$$\tilde{t}=\rho_1 t,\ \tilde{x}=\rho_1^{1/2}(x-v_1 t),\ \tilde{\tau}=\rho_1^{-1/2}(\tau-v_1). \tag{45}$$

Turning over of the DSW

We determine the value of the density ρ at the peak of the first soliton. It follows from (35) and (5) that $\rho_m = b_3|_{s=1}$, i.e.,

$$\rho=\rho_m=(2-\rho_0^{1/2})^2/\rho_0. \tag{46}$$

Hence it is clear that with increasing ρ_0 (when $\rho_0 < 4$) the density decreases. The flow velocity at the same point is

$$v_m=\rho_0^{1/2}(2-\rho_0^{1/2}-\rho_0)(2-\rho_0^{1/2})^{-1}. \tag{47}$$

Lowering the density in the soliton peaks is the cause of the appearance of singularities. Indeed, it is clear from (46) and (47) that as $\rho_0 \to 4$ the density $\rho_m \to 0$, and the velocity $v_m \to -\infty$. Condition (6) is then violated ($b_3 \to 0$).

The appearance of singularities is connected with the cutoff (or blocking) of the flow in the solitons near the trailing edge. It is interesting that a singularity arises then when the velocity of the outflow of gas from the trailing edge of the DSW vanishes: $v_- = \tau^- - v_0$. Indeed, it is clear from (38) and (33) that

$$v_- = \rho_0^{1/2} - 2(\rho_0^{1/2} - 1) = 2 - \rho_0^{1/2}.$$

When $\rho_0 > 4$ there is no singularity in the region of the first solitons, but in compensation it shifts into the interior of the wave: ρ and b_3 tend to zero and v to infinity when $s = s_k$, where $s_k^2 = (\rho_0^{\frac{1}{2}} - 1)^{-2}$.

The singularity considered here is similar to the singularity arising due to the overturning of solitons in DSW with negative dispersion.[2,4] It is therefore natural to call it the overturning of DSW in media with positive dispersion.

§4. DECAY OF INITIAL DISCONTINUITIES

The general solution of the problem of the decay of an initial discontinuity in dispersive hydrodynamics was constructed in Ref. 4. However, it did not allow one to determine the location and structure of the dissipationless shock waves which arose. We shall obtain here a complete asymptotically exact solution of this problem.

Let at the initial time $t = 0$ the density ρ and the velocity v undergo a discontinuity: in the left-hand half-space $\rho = \rho_2$, $v = v_2$, and in the right-hand one $\rho = \rho_1$, $v = v_1$. Without loss of generality we can (by measuring the density in units of ρ, in a system moving with velocity v_1) put ρ_1, $v_1 = 0$, $\rho_2 > 1$. Thus, for $t = 0$

$$\rho|_{t=0} = \begin{cases} \rho_2 & \text{if} \quad x<0,\ \rho_2>1, \\ 1 & \text{if} \quad x>0 \end{cases}$$
$$v|_{t=0} = \begin{cases} v_2 & \text{if} \quad x<0 \\ 0 & \text{if} \quad x>0 \end{cases}. \tag{48}$$

According to Ref. 4 (cf. Ref. 1, § 100) on decay of an initial discontinuity either a DSW or a rarefaction wave moves on both sides of it. Between these waves there appears a plateau region: $\rho = \text{const} = \rho_0$, $v = \text{const} = v_0$ using Eqs. (28), (33) when passing through the DSW or through the rarefaction wave (see footnote 2). Bearing in mind that the waves move in opposite directions, we have

$$v_0 = 2(\rho_0^{1/2} - 1), \quad v_2 - v_0 = -2(\rho_2^{1/2} - \rho_0^{1/2}),$$

whence

$$\rho_0 = \tfrac{1}{4}(\rho_2^{1/2} + 1 + \tfrac{1}{2}v_2)^2, \quad v_0 = \rho_2^{1/2} - 1 + \tfrac{1}{2}v_2. \tag{49}$$

We have assumed here that the velocity v has a positive sign in the direction of the gradient of ρ, i.e., in the direction of the x-axis. One arrives easily at the same result by using the conservation of the invariants r_+ and r_- of (18) when passing through the DSW or the rarefaction wave.

First of all, starting from the overturning condition $\rho_0 = 4$ (see (46), (47)), we determine from (49) the range of values of the parameters for which the solution of the initial problem (48) exists without the appearance of the overturning singularity: $\rho_2 \leqslant 9(1 - \frac{1}{6}v_2)^2$. Moreover, depending on the relation between ρ_2 and v_2 or ρ_0 and ρ_2 the following four cases are possible (see Figs. 6, 7).

1. $\rho_0 > \rho_2$, i.e., $v_2 > 2(\rho_2^{\frac{1}{2}} - 1)$.

In that case (Fig. 6a) compression waves (DSW) propagate on both sides of the initial discontinuity (cf. Ref. 4). Since the density and the velocity in the DSW change, then from the constant values ρ_0, v_0 on the plateau to other constant values ρ_2 and v_2 as $x \to -\infty$, or $\rho_1 = 1$, $v_1 = 0$ as $x \to \infty$, these are in both cases simple DSW. Using for them the formulae given in the preceding section, we can construct the following solution:

$$\begin{aligned}
&\rho = \rho_2 \quad \text{if} \quad \tau \leqslant (\tilde\tau^+ + v_2)\rho_2^{1/2},\\
&\rho = \rho_2\tilde\rho(\rho_{02}, |\tilde\tau|) \quad \text{if} \quad (\tilde\tau^+ + v_2)\rho_2^{1/2} \leqslant \tau \leqslant (\tilde\tau^- + v_2)\rho_2^{1/2},\\
&\rho = \rho_0 \quad \text{if} \quad \rho_2^{1/2}(\tilde\tau^- + v_2) \leqslant \tau \leqslant \tau^-,\\
&\rho = \tilde\rho(\rho_0, \tau) \quad \text{if} \quad \tau^- \leqslant \tau \leqslant \tau^+,\\
&\rho = 1 \quad \text{if} \quad \tau \geqslant \tau^+,
\end{aligned} \tag{50}$$

$$\begin{aligned}
&v = v_2\rho_2^{1/2} \quad \text{if} \quad \tilde\tau \leqslant (\tau^+ + v_2)\rho_2^{1/2},\\
&v = v_2 + \tilde v(\rho_{02}, |\tilde\tau|) \quad \text{if} \quad \rho_2^{1/2}(\tilde\tau^+ + v_2) \leqslant \tau \leqslant (\tilde\tau^- + v_2)\rho_2^{1/2},\\
&v = v_0 \quad \text{if} \quad \rho_2^{1/2}(\tilde\tau^- + v_2) \leqslant \tau \leqslant \tau^-,\\
&v = \tilde v_0(\rho_0, \tau) \quad \text{if} \quad \tau^- < \tau < \tau^+,\\
&v = 0 \quad \text{if} \quad \tau > \tau^+,\\
&\tau = x/tc_s(\rho_1), \quad \tilde\tau = (\tau - v_2)/\rho_2^{1/2}.
\end{aligned}$$

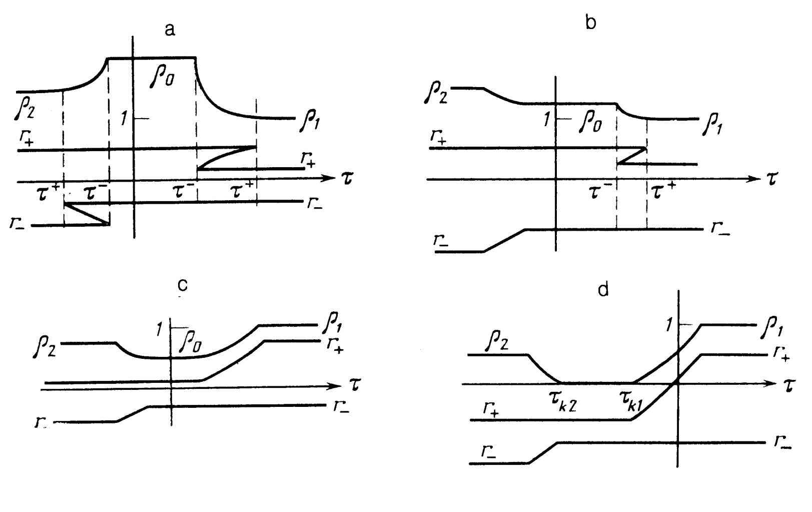

FIG. 6. Decay of initial discontinuities: a: $\rho_0 > \rho_2$; b: $\rho_2 > \rho_0 > 1$; c: $1 > \rho_0 > 0$; d: $\rho_0 = 0$.

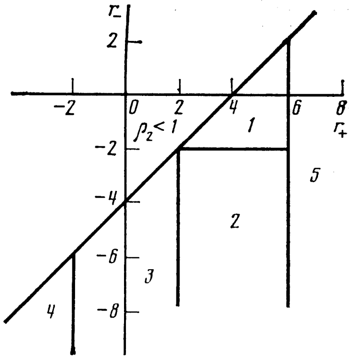

FIG. 7. Classification of the decay of initial discontinuities in the plane of the adiabatic invariants r_+, r_-: 1: $\rho_0 > \rho_2$; 2: $\rho_2 > \rho_0 > 1$; 3: $1 > \rho_0 > 0$; 4: $\rho_0 = 0$; 5: $\rho_0 > 4$—region of overturning of the DSW.

We have given here the asymptotic solution of the Euler-Whitham equations, averaged over the oscillations, with the boundary conditions (48). The functions $\tilde{\rho}(\rho_0, \tau)$, $\tilde{v}(\rho_0, \tau)$ are given by Eqs. (35), (36). Here $\rho_{02} = \rho_0/\rho_2$. The points τ^- and τ^+ are the given functions (37), (38), and the points $\tilde{\tau}^-, \tilde{\tau}^+$ are the same functions of ρ_{02} but taken with the opposite sign ($\tilde{\tau}^- < 0$, $\tilde{\tau}^+ < 0$). Moreover, Eqs. (44), (45) have been used. We give in Eqs. (50) only the averaged quantities.

The oscillations are, generally speaking, excited both in the plateau region $\rho_2^{\frac{1}{2}}(\tilde{\tau}^- + v_2) \leqslant \tau \leqslant \tau^-$ and in the regions of the dissipationless shock waves $\rho_2^{\frac{1}{2}}(\tilde{\tau}^+ + v_2) \leqslant \tau \leqslant \rho_2^{\frac{1}{2}}(\tilde{\tau}^- + v_2), \tau^- \leqslant \tau \leqslant \tau^+$. However, the amplitude of the oscillations on the plateau decreases with time according to a power law $a \propto t^{-1/2}$ (see Ref. 4). In the solution which is asymptotic in t only the quasi-stationary oscillations which are excited in the DSW regions are important. Their structure is completely described by Eqs. (35), (36), and (5), taking (44), (45), and (50) into account. We thus obtain an asymptotically exact solution of the initial problem (48), which gives both the average quantities and the behavior of the oscillations. One determines easily in the case considered also the change in all the adiabatic invariants; it is shown schematically in Fig. 6a.

2. $1 < \rho_0 < \rho_2$, i.e., $2(1 - \rho_2^{\frac{1}{2}}) < v_2 < 2(\rho_2^{\frac{1}{2}} - 1)$.

In this case (see Fig. 6b) there propagates to the left of the initial discontinuity a rarefaction wave ($\rho_2 > \rho_0$), and to the right a compression wave, DSW ($\rho_0 > 1$). The asymptotic solution then has the form

$$\begin{aligned}
&\rho = \rho_2 \quad \text{if} \quad \tau \leqslant v_2 - \rho_2^{1/2},\\
&\rho = {}^1/_9 (2\rho_2^{1/2} + v_2 - \tau)^2 \quad \text{if} \quad v_2 - \rho_2^{1/2} \leqslant \tau \leqslant {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3),\\
&\rho = \rho_0 \quad \text{if} \quad {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3) \leqslant \tau \leqslant \tau^+ = {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 + 1),\\
&\rho = \tilde{\rho}(\rho_0, \tau) \quad \text{if} \quad \tau^+ \leqslant \tau \leqslant \tau^-,\\
&\rho = 1 \quad \text{if} \quad \tau > \tau^-,\\
&v = v_2 \quad \text{if} \quad \tau < v_2 - \rho_2^{1/2},\\
&v = {}^1/_3(v_2 + 2\rho_2^{1/2} + 2\tau) \quad \text{if} \quad v_2 - \rho_2^{1/2} \leqslant \tau \leqslant {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3),\\
&v = v_0 \quad \text{if} \quad {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3) \leqslant \tau \leqslant {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 + 1) = \tau^+,\\
&v = \tilde{v}(\rho_0, \tau) \quad \text{if} \quad \tau^+(\rho_0) \leqslant \tau \leqslant \tau^-(\rho_0),\\
&v = 0 \quad \text{if} \quad \tau > \tau^-(\rho_0).
\end{aligned} \tag{51}$$

The rarefaction wave is determined by the solution of the Euler Eqs. (17) and (16). In the vicinity of weak discontinuities there arise, when the rarefaction wave is matched to the plateau region, oscillations with an amplitude which decreases with time $\propto t^{-1/3}$ (see Ref. 4). In the DSW region the structure of the oscillations is given by Eqs. (35), (36), (5). The variation of the adiabatic invariants is shown in Fig. 6b.

3. $0 < \rho_0 < 1$, i.e., $-2(\rho_2^{\frac{1}{2}} + 1) < v_2 < 2(1 - \rho_2^{\frac{1}{2}})$.

In this case $\rho_0 < 1$ (Fig. 6c) so that on both sides of the initial discontinuity rarefaction waves propagate. In that case no quasi-stationary oscillations arise in the flow and the asymptotic solution has the simple form

$$\begin{aligned}
&\rho = \rho_2 \quad \text{if} \quad \tau < v_2 - \rho_2^{1/2},\\
&\rho = {}^1/_9(2\rho_2^{1/2} + v_2 - \tau)^2 \quad \text{if} \quad v_2 - \rho_2^{1/2} \leqslant \tau \leqslant {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3),\\
&\rho = \rho_0 \quad \text{if} \quad {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3) \leqslant \tau \leqslant -{}^1/_2 + {}^3/_4 v_2 + {}^3/_2 \rho_2^{1/2},\\
&\rho = {}^1/_9(2 + \tau)^2 \quad \text{if} \quad -{}^1/_2 + {}^3/_4 v_2 + {}^3/_2 \rho_2^{1/2} \leqslant \tau \leqslant 1,\\
&\rho = 1 \quad \text{if} \quad \tau \geqslant 1,\\
&v = v_2 \quad \text{if} \quad \tau \leqslant v_2 - \rho_2^{1/2},\\
&v = {}^1/_3(v_2 + 2\rho_2^{1/2} + 2\tau) \quad \text{if} \quad v_2 - \rho_2^{1/2} \leqslant \tau \leqslant {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3),\\
&v = v_0 \quad \text{if} \quad {}^1/_2(\rho_2^{1/2} + {}^1/_2 v_2 - 3) \leqslant \tau \leqslant -{}^1/_2 + {}^3/_4 v_2 + {}^3/_2 \rho_2^{1/2},\\
&v = {}^2/_3(\tau - 1) \quad \text{if} \quad -{}^1/_2 + {}^3/_4 v_2 + {}^3/_2 \rho_2^{1/2} \leqslant \tau \leqslant 1,\\
&v = 0 \quad \text{if} \quad \tau \geqslant 1.
\end{aligned} \tag{52}$$

The behavior of the adiabatic invariants is shown in Fig. 6c.

4. $v_2 < -2(\rho_2^{\frac{1}{2}} + 1)$.

In this case (Fig. 6d) the density ρ_0 on the plateau vanishes, the medium breaks up and there appear a vacuum region and two rarefaction waves in the vacuum. In the Euler approximation the solution is given by Eqs. (52), but $\rho_0 \equiv 0$ and the critical points where the density vanishes are, respectively,

$$\tau_{cr1} = -2, \quad \tau_{cr2} = 2\rho_2^{1/2} + v_2. \tag{53}$$

The width of the vacuum region is $\Delta\tau = -2(\rho_2^{\frac{1}{2}} + 1) - v_2$. In the vicinity of the singular points τ_{cr1} and τ_{cr2} one needs a special study of the solution of the set of Eqs. (2) describing the transition to the vacuum (see Ref. 10, Ch. IV).

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[1)] These equations were also obtained by V. N. Bogaevskiĭ. The authors are grateful to him for acquainting them with his work before its publication.

[2)] We note that the same condition is satisfied when one passes through any simple Riemann wave in Eulerian hydrodynamics, in particular, through a rarefaction wave (only the sign may change).

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