Alfven vortices in a plasma with finite ion temperature

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Nonlinear Alfven waves are considered in a homogeneous plasma with finite ion temperature. Account is taken of linear and nonlinear dispersion effects due to the finite Larmor radius of the ions. It is shown that these effects can be described by hydrodynamic equations similar to those of Grad. Hydrodynamic equations are obtained for nonlinear weakly dispersive Alfven waves in a plasma with arbitrary T_i/T_e ratio. It is shown that Alfven vortices are present in such a plasma.

1. INTRODUCTION

Alfven vortices, initially discussed in Refs. 1 and 2, have recently attracted the attention of many workers.[3–6] The reasons are, on the one hand, that Alfven vortices are among the most striking examples of nonlinear regular magnetoplasma structures and are in this sense of general physical interest, and on the other, the important role played by Alfven waves in the physics of outer-space and laboratory plasma.

The starting point for the development of the theory of Alfven vortices was the idea of the analogy between the nonlinear properties of Alfven waves and those of Rossby waves in a rotating shallow liquid. Taking into account the results of Larichev and Reznik,[7] who obtained an analytic solution for the nonlinear equation of Rossby waves in the form of a solitary dipole vortex, the authors of Refs. 1 and 2 obtained an analogous solution for Alfven waves, and named the corresponding nonlinear object an Alfven vortex.

Common to Refs. 1 and 2, and also to 5, is the fact that, first, they deal with a plasma having $\beta > m_e/m_i$ (β) is the ratio of the plasma pressure to the magnetic field pressure, while m_e and m_i are the masses of the electron and ion; second, they neglect effects due to the finite ion temperature, $T_i \to 0$. This neglect is justified if $T_e \ll T_i$ (T_e is the electron temperature), hereafter called the case of a finite-pressure plasma with cold ions. In contrast to Refs. 1, 2, and 5, Ref. 4 deals with vortices in a plasma with $\beta < m_e/m_i$. In this case one can neglect effects connected both with finite T_i and with finite T_e, meaning a cold plasma. Both types of plasma were discussed in Ref. 3, while the plasma considered in Ref. 6 had $\beta > m_e/m_i$ and $T_i \gtrsim T_e$.

References 1 and 2 were deficient in two respects. First, no account was taken in the initial equations of the nonlinear term connected with the wave component of the magnetic field. Second, the conditions used in Refs. 1 and 2 to match the solutions on different sides of the singular line of the vortex were excessively simplified. The first of these errors was corrected in Ref. 8, where new solution-matching conditions were formulated for a rather large class of magnetoplasma vortices, including Alfven vortices. A corrected variant of Refs. 1 and 2 can be taken to be Ref. 3, in which the errors of the former were clarified and correct prescriptions were obtained for dipole vortices corresponding to the case $\beta > m_e/m_i$ and $T_e \gg T_i$.

We note also that the first of the above deficiencies of Ref. 1 was criticized in Ref. 5, and the second in Ref. 6. The criticism should be regarded as justified. It must be noted at the same time that the same matching conditions as in Refs. 1 and 2 were used both in Ref. 5 and in a paper by the same group of authors.[4] This shortcoming of Refs. 4 and 5 was pointed out in Ref. 3, where correct solutions were obtained for dipole Alfven vortices in the case $\beta < m_e/m_i$, thereby correcting Ref. 4.

A correct answer to the question of what dipole Alfven vortices are at $\beta > m_e/m_i$, $T_e \gg T_i$ and also at $\beta < m_e/m_i$ can be found in Ref. 3. One of the purposes of the present paper is to cast light on Alfven vortices in a plasma with $\beta > m_e/m_i$, $T_i \gtrsim T_e$. This question was the subject of Ref. 6, where an attempt was made to treat it by using a traditional hydrodynamic description of the plasma, based on equations comprising a simplified variant of Braginskiĭ's equations.[9] We shall make it clear, however, that equations of the type given in Ref. 9 are insufficient for the analysis of Alfven vortices under the indicated conditions.

An answer to the question of which hydrodynamic equations are applicable to the problem of nonlinear Alfven waves in a plasma with $\beta < m_e/m_i$ and $T_i \gtrsim T_e$ is one more purpose of the present paper.

Section 2 deals with a decription of nonlinear Alfven waves in a plasma with $\beta > m_e/m_i$ and $T_i \gtrsim T_e$. Account must be taken in this problem of several abstruse phenomena of magnetized-plasma physics. To understand the gist of the present paper we deem it useful to offer the following few explanations.

We note first that the nonlinear objects of interest to us are connected with weakly dispersing Alfven waves, which are described in the linear approximation by a dispersion equation of the form

$$\omega^2 = k_z^2 c_A^2 [1 + k_\perp^2 (\rho_0^2 + 3\rho_i^2/4)]. \tag{1.1}$$

Here ω is the oscillation frequency, k_z and $k_\perp$ are respectively the components of the wave vector along the equilibrium magnetic field $B_0 \| z$ and across $\mathbf{B}_0$, $c_A^2 = B_0^2/4\pi n_0 m_i$ the square of the Alfven velocity, n_0 the equilibrium density of the plasma $\rho_0^2 = T_e/m_i\omega_{Bi}^2$ the square of the ion Larmor radius, $\omega_{Bi} = eB_0/m_i c$ the ion cyclotron frequency, e the ion charge, and c the speed of light. The terms with $k_\perp^2$ in the right-hand side of (1.1) represent the dispersion increment to the oscillation frequency and are therefore called dispersive. The term with ρ_0^2 describes the electron contribution to the dispersion, and the term with ρ_i^2 the ion contribution. Equation (1.1) can be derived with the aid of the kinetic equation for the dielectric tensor of the plasma, by expand-

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ing in powers of the small parameter $k_\perp^2 \rho_i^2$, therefore waves of type (1.1) are called kinetic Alfven waves (see, e.g., Ref. 10). This term reflects also the fact that Eq. (1.1) cannot be derived from the standard magnetohydrodynamics equations, such as of the Braginskiĭ type.[9] This is evidence that hydrodynamics of the type of Ref. 9 is not appropriate to the problem of linear weakly dispersive Alfven wave. It must be noted, however, that formal application of this hydrodynamics leads to a dispersion equation of type (1.1), in which the coefficient of ρ_i^2, while incorrect, still differs from zero and turns out to be of the correct sign. In this sense equations of the type of Ref. 9 describe approximately the ionic contribution to the dispersion of linear Alfven waves. As to the equations used in Ref. 6, they yield an equation of type (1.1) with a zero coefficient of ρ_i^2. In other words, the linear-approximation equation of Ref. 6 (see the equation preceding Eq. (10) of Ref. 6) does not take into account the ion contribution to the dispersion. According to (1.1) this contribution is of the order of T_i/T_e compared with the electron contribution of Ref. 6 to the dispersion. The linear dispersion equation of Ref. 6 is therefore incorrect at $T_i \gtrsim T_e$, i.e., precisely in the case that the authors of Ref. 6 claimed to analyze.

The ion contribution to the dispersion of low-frequency magnetosonic waves was previously considered in Ref. 11. It was shown there that, as in our case of Alfven waves, Braginskiĭ's hydrodynamic approach[9] leads to an incorrect expression for this contribution (both the magnitude and the sign of the coefficient of the term of type $k_\perp^2 \rho_i^2$ turn out to be incorrect). On the other hand, according to Ref. 11, a correct expression for this contribution is obtained by using Grad's hydrodynamic approach,[12-14] which unlike Braginskiĭ's[9] yields more complete expressions for the viscosity tensor and for the heat flux. This circumstance suggested to us the idea of testing the suitability of Grad's hydrodynamics for the calculation of the ion contribution to the dispersion of Alfven waves. It was found that this hydrodynamics yields exactly the same linear dispersion equation as the kinetic approach, i.e., Eq. (1.1). This attests to the compatibility of Grad's hydrodynamics with the linear problem of the kinetic Alfven waves.

The low-frequency magnetosonic solitons discussed in Ref. 11 are due to weakly linear weakly dispersing waves. A feature of the equations that describe such solitons is additivity of the nonlinearity and of the dispersion. In the Alfven-vortex problem, on the contrary, there is no such additivity: the nonlinear dispersion equations play just as an important role in the equations for these vortices as the usual linear ones. This is indicated by the fact that Alfven vortices are strongly nonlinear waves, as is clear from an analysis of Alfven vortices in a plasma with cold ions, $T_i \to T_0$ (Ref. 3). In this case, both the linear and the nonlinear dispersion equations are determined by the electron temperature. All the terms are then accounted for in standard hydrodynamics.[3] If, however, $T_i \gtrsim T_e$, the linear dispersion terms are correctly taken into account, as noted above, in Grad's hydrodynamics. The question is then whether one can expect this hydrodynamics to account adequately also for the nonlinear dispersion term. This question can be answered by recalling that the nonlinear effects due to the existence of Alfven vortices constitute none other than a vector nonlinearity.[8] In a hydrodynamic description of the ions, the effects of the vector nonlinearity are accounted for by terms of the type $\mathbf{V}\nabla$ ($\mathbf{V}$ is the hydrodynamic velocity of the ions), whereas the linear effects are terms of $\partial/\partial t$ type. It is clear therefore that the problem of allowance for nonlinear dispersion terms due to the finite character of T_i can be solved if all the terms of type $\mathbf{V}$, together with the $\partial/\partial t$ terms, are retained in the ion hydrodynamic equations that take adequate account of the linear description. Grad's hydrodynamics satisfies precisely all these requirements. It is clear therefore that this hydrodynamics is suitable for the description of the problem of interest to us.

As to the hydrodynamics used in Ref. 6, it does, just as Grad's hydrodynamics, allow for or neglect terms of type $\mathbf{V}\nabla$ to the same extent as terms of type $\partial/\partial t$. Since, however, as noted above, the hydrodynamics of Ref. 6 does not take into account the ion contribution to the linear dispersion terms, it is clear that it does not allow for the corresponding contribution to the nonlinear dispersion terms.

In Sec. 2 we present nonlinear equations for Alfven waves and simplify them as applied to the problem of Alfven vortices. The procedure for deriving these equations from general hydrodynamic equations of Grad's type is described in the Appendix. The specific features of dipole Alfven waves are investigated in Sec. 3. The results are discussed in Ref. 4.

2. NONLINEAR EQUATIONS FOR ALFVEN WAVES, AND THEIR SIMPLIFICATION IN THE CASE OF STATIONARY WAVES

According to the Appendix, in the case of a plasma with finite ion temperature, the closed-current equation $\operatorname{div} \mathbf{j} = 0$ is of the form

$$\frac{d_0}{dt}\left(\Delta_\perp\varphi+\frac{3}{4}\rho_i^2\Delta_\perp^2\varphi\right)+\frac{c_A^2}{c}\hat{\mathscr{D}}_\parallel\Delta_\perp A=0. \tag{2.1}$$

Here φ and A are the electrostatic and vector potentials given by Eqs. (A.10) of the Appendix,

$$\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2,$$

and the operators d_0/dt and $\hat{\mathscr{D}}_\parallel$ are defined as follows:

$$\frac{d_0}{dt}=\frac{\partial}{\partial t}+\frac{c}{B_0}[\nabla\varphi,\nabla]_z,\qquad \hat{\mathscr{D}}_\parallel=\frac{\partial}{\partial z}-\frac{1}{B_0}[\nabla A,\nabla]_z. \tag{2.2}$$

Equation (2.1) is supplemented by standard electronic continuity and longitudinal-motion equations

$$\frac{d_0}{dt}\tilde{n}+\frac{c}{4\pi e}\hat{\mathscr{D}}_\parallel\Delta_\perp A=0, \tag{2.3}$$

$$\frac{1}{c}\frac{\partial A}{\partial t}+\hat{\mathscr{D}}_\parallel\left(\varphi-\tilde{n}\frac{T_{0e}}{en_0}\right)=0, \tag{2.4}$$

where $\tilde{n}$ is the perturbed electron density. The condition that the electronic thermal conductivity be infinite along the magnetic field, $\mathbf{B}\cdot\nabla T_e = 0$, is assumed satisfied.

Equations (2.1), (2.3), and (2.4) are the initial ones in the problem of interest to us, that of Alfven vortices in a uniform plasma with $m_e/m_i < \beta < 1$ and with an arbitrary ratio of T_i and T_e.

Note that it is convenient also to use in lieu of (2.1) a relation that follows from (2.1) and (2.3)

$$\frac{d_0}{dt}\left[\tilde{n}-\frac{en_0}{m_i\omega_{Bi}^2}\left(\Delta_\perp\varphi+\frac{3}{4}\rho_i^2\Delta_\perp^2\varphi\right)\right]=0, \tag{2.5}$$

and is none other than the ion continuity equation.

It follows from (2.1), (2.3), and (2.4) that $\partial W/\partial t = 0$, where W is the energy of the waves, given by

$$W=\frac{1}{2}\int\left\{\frac{n_0m_ic^2}{B_0^2}\left[(\nabla_\perp\varphi)^2-\frac{3}{4}\rho_i^2(\Delta_\perp\varphi)^2\right]+\frac{(\nabla_\perp A)^2}{4\pi}+\frac{T_{0e}}{n_0}\tilde{n}^2\right\}d\mathbf{r}$$

$$=\frac{1}{2}\int\left\{n_0m_i\left[\mathbf{V}_E^2-\frac{3}{4}\rho_i^2(\mathrm{rot}_z\mathbf{V}_E)^2\right]+\frac{B_\perp^2}{4\pi}+\frac{T_{0e}}{n_0}\tilde{n}^2\right\}d\mathbf{r}. \tag{2.6}$$

With the aid of (2.5) we obtain one more conservation law $\partial K/\partial t = 0$, where

$$K=\int\left(\Delta_\perp\varphi+\frac{3}{4}\rho_i^2\Delta_\perp^2\varphi-\frac{m_i\omega_{Bi}^2}{en_0}\tilde{n}\right)^2d\mathbf{r}. \tag{2.7}$$

By analogy with hydrodynamics,[15] we can call K the generalized entrophy.

A plasma with $T_e = 0$ is subject also to the conservation law $\partial M/\partial t = 0$, where

$$M=\int\mathbf{B}_\perp\left(\mathbf{V}_E+\frac{3}{4}\rho_i^2\Delta_\perp\mathbf{V}_E\right)d\mathbf{r}. \tag{2.8}$$

Note that we have obtained the integrals of motion (2.6)–(2.8) under the assumption that the quantities φ, A, $\Delta_\perp\varphi$, $\Delta_\perp A$, and $\Delta_\perp^2\varphi$ are continuous. We shall show presently that this assumption is valid for our Alfven-vortex problem.

Let φ, A, and $\tilde{n}$ depend only on x and $\eta = y + \alpha z - ut$, where α and u are certain constants (cf. Ref. 8). This corresponds to the case of traveling stationary waves. For such waves it follows from (2.3)–(2.5) that

$$\hat{\mathscr{D}}\tilde{n}-\frac{\alpha c}{4\pi eu}\hat{\mathscr{D}}_1\Delta_\perp A=0. \tag{2.9}$$

$$\hat{\mathscr{D}}_1F_1=0, \tag{2.10}$$

$$\hat{\mathscr{D}}F=0. \tag{2.11}$$

Here

$$F=\Delta_\perp\varphi+\frac{3}{4}\rho_i^2\Delta_\perp^2\varphi-\frac{m_i\omega_{Bi}^2}{en_0}\tilde{n}, \tag{2.12}$$

$$F_1=A-\frac{\alpha c}{u}\left(\varphi-\frac{T_{0e}}{en_0}\tilde{n}\right), \tag{2.13}$$

and the operators $\hat{\mathscr{D}}$ and $\hat{\mathscr{D}}_1$ mean

$$\hat{\mathscr{D}}\equiv\frac{\partial}{\partial\eta}-\frac{c}{uB_0}[\nabla\varphi,\nabla]_z, \tag{2.14}$$

$$\hat{\mathscr{D}}_1\equiv\frac{\partial}{\partial\eta}-\frac{1}{\alpha B_0}[\nabla A,\nabla]_z. \tag{2.15}$$

An important property of Eqs. (2.9)–(2.11) is their vector integrability, the meaning of which is explained in Ref. 8. Using the vector-integration procedure, these equations can be reduced to linear ones. By integration, we get from (2.10) and (2.11)

$$F_1=C_1(A-\alpha B_0x), \tag{2.16}$$

$$F=C(\varphi-uB_0x/c), \tag{2.17}$$

where C and C_1 are certain constants. The curves on which the equalities

$$\varphi(\mathbf{r})=uB_0x/c, \tag{2.18}$$

$$A(\mathbf{r})=\alpha B_0x \tag{2.19}$$

are satisfied are singular lines of the system (2.9)–(2.11). The constant C can have different values on different sides of the line (2.18). We confine ourselves below for simplicity to an analysis of localized solutions for the case $C_1\equiv 0$. This means that in our problem $F_1 = 0$, i.e., according to (2.13)

$$\hat{\mathscr{D}}\tilde{n}=\frac{en_0}{T_{0e}}\hat{\mathscr{D}}_1\left(\varphi-\frac{u}{\alpha c}A\right). \tag{2.20}$$

Taking into account the explicit forms of the operators $\hat{\mathscr{D}}$ and $\hat{\mathscr{D}}_1$ [see (2.14) and (2.15)], we get

$$\varphi=\frac{u}{\alpha c}\left(A+\frac{\alpha^2c_A^2}{u^2}\rho_0^2\Delta_\perp A\right). \tag{2.21}$$

Substituting (2.21) in (2.9), carrying out the vector integration, and assuming the corresponding integration constant to be zero, we arrive at the following relation between φ and A:

$$\tilde{n}=\frac{en_0}{T_{0e}}\left(\varphi-\frac{u}{\alpha c}A\right). \tag{2.22}$$

In addition, from (2.16) and (2.20) we obtain the connection between $\tilde{n}$ and A:

$$\tilde{n}=\frac{\alpha c}{4\pi eu}\Delta_\perp A. \tag{2.23}$$

Substituting (2.22) and (2.23) in (2.12) and taking into account the assumption made above that the wave dispersion is weak, $(\rho_0^2, \rho_i^2)\Delta_\perp \ll 1$, and also the relation $u^2 \approx \alpha^2 c_A^2$ ensuing from this assumption [see (1.1)], we get

$$F=\frac{u}{\alpha c}\left[\left(\rho_0^2+\frac{3}{4}\rho_i^2\right)\Delta_\perp^2A+\left(1-\frac{u^2}{\alpha^2c_A^2}\right)\Delta_\perp A\right]. \tag{2.24}$$

In this approximation we can assume that $\varphi \approx uA/\alpha c$ in the right-hand side of (2.17) [see (2.22)]. It follows then from (2.17) and (2.24) that

$$\left(\rho_0^2+\frac{3}{4}\rho_i^2\right)\Delta_\perp^2A+\left(1-\frac{u^2}{\alpha^2c_A^2}\right)\Delta_\perp A=C(A-\alpha B_0x). \tag{2.25}$$

Note that in the weak-dispersion approximation the difference between (2.18) and (2.19) is inessential.

We have thus reduced the nonlinear Alfven-vortex problem to a linear one characterized by Eq. (2.25) with different values of the constant C on opposite sides of the singular line $A - \alpha B_0x$. This substantial simplification is the result of the integrability of our system of equations (2.9)–(2.11).

3. DIPOLE ALFVEN VORTICES

We consider dipole Alfven vortices, putting

$$(\varphi, A, \tilde{n})=(\Phi, Z, N)\cos\theta. \tag{3.1}$$

Here Φ, Z, and N depend only on the radial coordinate $r\equiv(x^2+\eta^2)^{1/2}$. The angle variable θ is defined by the relation $\theta = \arctan(\eta/x)$. We assume that $C = 0$ at $r > a$ and $C \neq 0$ at $r < a$, where $r = a$ is the singular point of the vortex. We seek the solutions of (2.25) on different sides of the singular point and match them at $r = a$, assuming that at this point the quantities φ, $\partial\varphi/\partial r$, $\Delta_\perp\varphi$, A, $\partial A/\partial r$ are continuous. Taking (2.22) and (2.23) into account, we conclude that this matching makes $\Delta_\perp A$ and $\tilde{n}$ also continuous.

In addition, applying the operator $\Delta_\perp$ on both sides of (2.22) and recognizing the continuity of $\Delta_\perp \varphi$ and $\Delta_\perp A$, we conclude that the quantities $\Delta_\perp^2 A$ and $\Delta_\perp \tilde{n}$ are also continuous. Finally, applying the operator $\Delta_\perp^2$ to both sides of (2.23) we find that if terms of order $\Delta_\perp^3$ are neglected then $\Delta_\perp^2 \varphi$ is also continuous. This justifies, in particular, the procedure used to derive the integrals of motion (2.6)–(2.8).

We assume that r in (2.25) is large enough, and the potential A is proportional to $\exp(-\varkappa r)$. We find then that the argument $\varkappa$ of the decreasing exponential can take on values $\varkappa_1$ and $\varkappa_2$, where $\varkappa_{1,2}$ is defined by the relations

$$\varkappa_1^2=(\rho_0^2+3\rho_i^2/4)^{-1}(1-u^2/\alpha^2c_A^2), \quad \varkappa_2^2=0. \tag{3.2}$$

The presence of zero roots, $\varkappa_2^2=0$, reflects the fact that the asymptote of the Alfven vortices contains, besides the exponentially decreasing part (at $\varkappa_1^2>0$), also a part that decreases as a power law. In this sense the Alfven vortices can be called degenerate two-potential vortices, alternatives to the standard two-potential vortices for which $(\varkappa_1^2,\varkappa_2^2)>0$. Examples of standard two-potential vortices were considered in Ref. 16.

At $r>a$ the radial part of the potential A [see (3.1)] is described by the relation

$$Z(r)=E_1K_1(\varkappa_1 r)+E_2/r, \tag{3.3}$$

where I_1 is a modified Bessel function of the second kind, and E_1 and E_2 are certain constants. In the interior region, however, i.e., at $r<a$, the function $Z(r)$ is given by

$$Z(r)=\alpha B_0 r+D_1J_1(\gamma r)+D_2I_1(\lambda r), \tag{3.4}$$

where J_1 and I_1 are a Bessel function and a modified Bessel function, D_1, D_2, γ, and λ are constants which are determined, just as E_1 and E_2 from the matching conditions. Solutions of the type (3.3) and (3.4) were initially obtained in Ref. 3 for the case $T_1=0$.

It is clear from (3.2) that the role of finite T_i reduces only to a renormalization of the rate at which the potentials fall off in the outer region, a scale characterized by the parameter $\varkappa_1$, and of the analogous scales in the internal region, characterized by the parameters γ and λ.

An important feature of any type of vortex having a singular point is the condition that the vortex parameters be matched. In the case of electrostatic vortices such a condition was cited, e.g., in Ref. 8, while in the case of standard two-potential vortices it was given in Ref. 16. In our case of degenerate two-potential vortices the condition that the vortex parameters be matched means that[3]

$$[\xi(\beta^2+\gamma^2)+(1-\xi)\lambda^2]L+(\lambda^2+\gamma^2)B + [\xi(\beta^2-\gamma^2)+\gamma^2(\xi-1)]G=0. \tag{3.5}$$

Here

$$B=\beta K_2(\beta)/K_1(\beta), \quad L=\lambda I_2(\lambda)/I_1(\lambda),$$
$$G=\gamma J_2(\gamma)/J_1(\gamma), \quad \beta=a\varkappa_1, \quad \xi=1+3T_i/4T_e. \tag{3.6}$$

Plots of the function $\Phi(r)$ and $Z(r)$, obtained with allowance for relations (3.3)–(3.5), are given in Ref. 3.

It is known (see, e.g., Ref. 8) that another important feature of electrostatic vortices is the modified dispersion equation (MDE) of the vortex, which relates the vortex propagation velocity u (and also the propagation angle α) with the fall-off argument $\varkappa$ of the potential of the vortex in its outer region. In all the previously investigated electrostatic-vortex examples (see Ref. 8) the MDE were linear in $\varkappa^2$ and had, in accordance with the asymptote of the potential as $r\to\infty$, a single decreasing exponential. In this case it is possible to solve the MDE for u, i.e., find the function $u=u(\alpha,\varkappa^2)$, and regarding $\varkappa^2$ as a free parameter (limited mainly by the requirement $\varkappa^2>0$) one can determine the possible range of variation of the velocity u. An example of such a procedure can be found in Ref. 8. In the case of standard two-potential vortices, when we deal simultaneously with two values of $\varkappa^2$, the interval of the possible values of the velocity u should be defined differently, namely by the requirement that both $\varkappa_1^2$ and $\varkappa_2^2$ be positive. In the case of degenerate two-potential vortices, however, when $\varkappa_2^2=0$, the quantity $\varkappa_1^2$ plays the same role as $\varkappa^2$ does in the electrostatic-vortex problem. The procedure of finding the possible interval of the velocities u is then similar to that in the case of electrostatic vortices.

From the foregoing and from Eqs. (3.1) it is clear that the Alfven vortices considered by us in a plasma with $\beta>m_e/m_i$ have an Alfven propagation velocity $u<\alpha c_A$. This fact was first noted for $T_i=0$ in Ref. 3.

4. DISCUSSION OF RESULTS

Using Grad's general hydrodynamic equations, we have obtained a set of equations (2.1), (2.3), (2.4) that describes nonlinear Alfven waves in a plasma with allowance for the finite Larmor radius of the ions. These equations agree with the heuristic equations of Ref. 17. We have shown that the conserved quantities in nonlinear Alfven waves are the energy and the generalized entrophy (2.6) and (2.7). At $T_0=0$ there is one more integral of the motion, defined by Eq. (2.8). An additional important properties of the derived nonlinear equations is that in the case of traveling stationary waves they reduce to linear on different sides of the singular lines. When account is taken of the weak dispersion of the waves, the entire system of equations reduces then to the single equation (2.25). We have analyzed this equation for the case of dipole Alfven vortices and have shown that it has a solution of the form (3.1), (3.3), (3.4), which is analogous to that of Ref. 3. A classification of the possible types of two-potential vortices is presented, according to which Alfven vortices are degenerate two-potential vortices. In addition, earlier studies of the theory of Alfven vortices are critically analyzed. The present paper completes therefore, on the one hand, the initial stage of the theory of Alfven vortices, aimed at establishing their possible existence, and on the other it lays the groundwork for the development of a more complete theory of nonlinear Alfven waves. The starting point of this theory should be the equations (2.1), (2.3), and (2.4) obtained above.

APPENDIX

Hydrodynamic description of nonlinear Alfven waves in a hot-ion plasma

The aim of this Appendix is to derive Eq. (2.1) within the framework of Grad's hydrodynamics. We recall that by Grad's hydrodynamics is usually meant a system of hydrodynamic equations that contains, besides the continuity, motion, and heat-balance equation also equations for the viscosity tensor $\hat{\pi}$ and for the heat flux **q**. Using this hydrodynamics, we recognize that in the Alfven-wave case of interest to us only the transverse motion of the ions is significant. In addition, it can be verified that effects connected with the heat flux are inessential in our problem. We therefore put $\mathbf{q} = 0$ and take into account only the viscosity-tensor components $\pi_{xx} = -\pi_{yy}$, $\pi_{xy} = \pi_{yx}$. According to Ref. 13, in the case of a collisionless plasma these quantities are defined by the equations

$$\frac{d\pi_{xx}}{dt} + 2\pi_{xx}\operatorname{div}\mathbf{V} + \pi_{xy}\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right) + p_i\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right) - 2\omega_{Bi}\pi_{xy} = 0,$$
$$\frac{d\pi_{xy}}{dt} + 2\pi_{xy}\operatorname{div}\mathbf{V} + \pi_{xx}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + p_i\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + 2\omega_{Bi}\pi_{xx} = 0. \quad (A.1)$$

Here $d/dt = \partial/\partial t + \mathbf{V}\nabla$ and p_i is the ion pressure.

We solve Eqs. (A.1) by expanding in powers of $1/\omega_{Bi}$. At our required accuracy we obtain then

$$\pi_{ik} = \pi_{ik}^{(0)} + \pi_{ik}^{(1)}, \quad (i, k) = (x, y). \quad (A.2)$$

Here $\pi_{ik}^{(0)}$ is that part of the tensor which corresponds to the Braginskiĭ approximation,[9]

$$\pi_{xx}^{(0)} = -\frac{p_i}{2\omega_{Bi}}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right), \quad \pi_{xy}^{(0)} = \frac{p_i}{2\omega_{Bi}}\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right), \quad (A.3)$$

$\pi_{ik}^{(1)}$ is the Grad correction[12] to the viscosity tensor,

$$\pi_{xx}^{(1)} = -\frac{1}{2\omega_{Bi}}\left[\frac{d\pi_{xy}^{(0)}}{dt} + 2\pi_{xy}^{(0)}\operatorname{div}\mathbf{V} + \pi_{xx}^{(0)}\operatorname{rot}_z\mathbf{V}\right],$$
$$\pi_{xy}^{(1)} = \frac{1}{2\omega_{Bi}}\left[\frac{d\pi_{xx}^{(0)}}{dt} + 2\pi_{xx}^{(0)}\operatorname{div}\mathbf{V} - \pi_{xy}^{(0)}\operatorname{rot}_z\mathbf{V}\right]. \quad (A.4)$$

The remaining initial hydrodynamic equations of our problem are standard. These are the ion heat-balance equation

$$\frac{d}{dt}p_i + 2p_i\operatorname{div}\mathbf{V} + (\hat{\pi}\nabla)\mathbf{V} = 0, \quad (A.5)$$

where **V** is the ion velocity, and the equations of motion of the electrons and ions. It is assumed also that the quasineutrality condition is met.

From the ion and electron equations of motion, with allowance for Maxwell's equations, we obtain

$$\operatorname{rot}_z(m_i n d\mathbf{V}/dt + \nabla\hat{\pi}) = (\mathbf{B}\nabla)\operatorname{rot}_z\mathbf{B}/4\pi, \quad (A.6)$$

where $\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}$ is the total magnetic field. This equation is the closed-current equation, $\operatorname{div}\mathbf{j} = 0$. We represent the ion velocity in the form

$$\mathbf{V} = \mathbf{V}_E + \mathbf{V}_p + \mathbf{V}_\pi + \mathbf{V}_I, \quad (A.7)$$

where

$$\mathbf{V}_E = c[\mathbf{E}\mathbf{e}_z]/B_0, \quad \mathbf{V}_p = c[\mathbf{e}_z, \nabla p_i]/enB_0,$$
$$\mathbf{V}_\pi = c[\mathbf{e}_z, \nabla\pi]/enB_0, \quad \mathbf{V}_I = [\mathbf{e}_z, d_0\mathbf{V}_E/dt]/\omega_{Bi}, \quad (A.8)$$

$d_0/dt = \partial/\partial t + \mathbf{V}_E\nabla$, and $\mathbf{e}_z$ is a unit vector along z. Substituting (A.7) in (A.6), we obtain, at the required accuracy,

$$\frac{d_0}{dt}\Delta_\perp\varphi + \frac{c_A^2}{c}\hat{\mathscr{D}}_\parallel\Delta_\perp A + X = 0. \quad (A.9)$$

Here φ and A are defined by the relations

$$\mathbf{E}_\perp = -\nabla_\perp\varphi, \quad \mathbf{B}_\perp = [\nabla A, \mathbf{e}_z], \quad (A.10)$$

the subscript $\perp$ labels the vector components transverse to the equilibrium magnetic field $\mathbf{B}_0$, and $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The quantity X is defined as

$$X = \frac{B_0}{c}\operatorname{rot}_z\left\{\frac{d_0}{dt}(\mathbf{V}_p + \mathbf{V}_\pi) + ((\mathbf{V}_p + \mathbf{V}_\pi)\nabla)V_E + \frac{\nabla\pi}{m_i n_0}\right\}. \quad (A.11)$$

This quantity is due to ion-temperature effects.

We put $p_i = p_{0i} + \tilde{p}_i$, where p_{0i} and p_i are the equilibrium and wave parts of the ion pressure. For the homogeneous plasma considered by us, $\nabla p_{0i} = 0$, it follows from (A.5), when Eqs. (A.8) for $\mathbf{V}_E$ and $\mathbf{V}_I$ are taken into account in Eq. (A.3) for $\pi^{(0)}$, that

$$\tilde{p}_i = 2en_0\rho_i^2\Delta_\perp\varphi + \bar{p}, \quad (A.12)$$

where p satisfies the relation

$$d_0\bar{p}/dt = 0. \quad (A.13)$$

According to (A.13) the quantity p does not depend on the equilibrium ion temperature T_{0i}. Equation (A.13) itself is valid even for cold ions, $T_{0i} = 0$, i.e., in the case when $\bar{p}$ is the wave part of the ion pressure, $\bar{p} = \tilde{p}_i$. A nonzero solutions of (A.13), $\bar{p} \neq 0$, would mean then the presence of a pressure perturbation in a plasma with a zero equilibrium pressure, thereby contradicting the usual physical notions that $\tilde{p}_i$ is small compared with p_{0i}. Assuming the situation with $\bar{p} \neq 0$ to be far-fetched, we put $\bar{p} = 0$ and accordingly rewrite (A.12) in the form

$$\tilde{p}_i = 2en_0\rho_i^2\Delta_\perp\varphi. \quad (A.14)$$

Taking (A.14) into account, we transform Eq. (A.8) for $\mathbf{V}_p$ into

$$\mathbf{V}_p = 2c\rho_i^2[\mathbf{e}_z, \nabla\Delta_\perp\varphi]/B_0. \quad (A.15)$$

Using (A.15), as well as the expression (A.8) for $\mathbf{V}_E$ and Eq. (A.10) for $\mathbf{E}_\perp$, we obtain the following expression for the contribution of the terms with $\mathbf{V}_p$ to (A.11):

$$\operatorname{rot}_z\left\{\frac{d_0}{dt}\mathbf{V}_p + (\mathbf{V}_p\nabla)\mathbf{V}_E\right\} = \frac{2c\rho_i^2}{B_0}\frac{d_0}{dt}\Delta_\perp^2\varphi. \quad (A.16)$$

When calculating $\mathbf{V}_E$ from Eq. (A.8) it suffices to take into account only the components of the tensor $\pi^{(0)}$, in which we put $\mathbf{V} = \mathbf{V}_E$. We have then

$$\mathbf{V}_\pi = -\rho_i^2\Delta_\perp\mathbf{V}_E/2. \quad (A.17)$$

From this equation we find that the contribution of the terms with $\mathbf{V}_\pi$ to (A.11) is obtained from the relation

$$\operatorname{rot}_z\left\{\frac{d_0}{dt}\mathbf{V}_\pi + (\mathbf{V}_\pi\nabla)\mathbf{V}_E\right\} = -\frac{c\rho_i^2}{2B_0}\frac{d_0}{dt}\Delta_\perp^2\varphi. \quad (A.18)$$

We calculate now the contribution made to (A.11) by the term with we use here Eqs. (A.3) and (A.4), the expressions of form (A.8) for $\mathbf{V}_E$ and $\mathbf{V}_I$, and relation (A.14) for p_i. We get then

$$\operatorname{rot}_z\nabla\pi^{(0)} = -\frac{\rho_i^2 n_0 m_i c}{B_0}\left\{\frac{1}{2}\frac{d_0}{dt}\Delta_\perp^2\varphi\right.$$

$$+\frac{c}{B_0}\left[\nabla\frac{\partial\,\Delta_\perp\varphi}{\partial x_\alpha},\ \nabla\frac{\partial\varphi}{\partial x_\alpha}\right]_z\Bigg\}, \quad (A.19)$$

$$\mathrm{rot}_z\,\nabla\pi^{(1)}=-\frac{\rho_i^2 n_0 m_i c}{B_0}\Bigg\{\frac{1}{4}\frac{d_0}{dt}\Delta_\perp^2\varphi$$

$$-\frac{c}{B_0}\left[\nabla\frac{\partial\,\Delta_\perp\varphi}{\partial x_\alpha},\ \nabla\frac{\partial\varphi}{\partial x_\alpha}\right]_z\Bigg\}. \quad (A.20)$$

Equations (A.20) and (A.19) illustrate the remarkable fact that although the tensors $\pi^{(0)}$ and $\pi^{(1)}$ are formally of different order of smallness, their contribution to the wave equation (A.9) is of the same order. From (A.2), (A.19), and (A.20) it follows that

$$\mathrm{rot}_z\,\nabla\pi=-\frac{3}{4}\frac{\rho_i^2 n_0 m_i c}{B_0}\frac{d_0}{dt}\Delta_\perp^2\varphi. \quad (A.21)$$

Using (A.11), (A.16), (A.18), and (A.21) we reduce (A.9) to Eq. (2.1) of Sec. 2.

In the linear approximation, the use of (2.1) yields for the Alfven waves a dispersion equation that coincides with (1.1). This is evidence that Grad's hydrodynamics is applicable to the problem of linear Alfven waves, as noted in Sec. 1. If Braginskiĭ hydrodynamics is used, however, i.e., the $\pi^{(1)}$ contribution is neglected, a dispersion equation of the form (1.1) is obtained, but with ρ_i^2 preceded by 1 rather than 3/4.

We call attention also to the fact that the right-hand sides of Eqs. (A.19) and (A.20) for $\pi^{(0)}$ and $\pi^{(1)}$ contain nonlinear terms of rather complicated structure (with derivatives $\partial/\partial x_\alpha$). When the indicated contributions are summed these terms cancel each other, and are therefore missing from the wave equation (2.1). In the Braginskiĭ approximation these terms would be contained in the wave equation, which would have then a much more complicated structure than (2.1).

Note that a set of equations that coincides, in the case of $\rho_i^2\Delta_\perp$ [sic!] and if dissipation is neglected, with our system (2.1), (2.3), (2.4) was obtained heuristically in Ref. 17.

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