Dissipation function for elastoplastic media

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A standard formal derivation is offered for the dissipation function, the rate of heat evolution in a unit volume, for a medium in which a static shear stress and an irreversible (plastic) strain are possible. An expression [expression (10)] is derived for the tensor of the rate of plastic strain. It is shown that the effect stems exclusively from the shear stress in the medium. This effect, if it occurs, is linear in the rates and can have a decisive inverse effect on the process of plastic deformation. The possibility of an avalanche fracture of the medium when the initial temperature decreases is studied in a simple example.

1. Irreversible processes which occur in a liquid and give rise to an evolution of heat in the volume and to an increase in the entropy are associated with a molecular transport of momentum and energy. Added to the corresponding fluxes are components proportional to the gradients of the velocity and temperature distributions, multiplied by viscosity coefficients and thermal conductivities.[1] The expression for the dissipation function is quadratic in these gradients. In a solid, in addition to the expression which directly generalizes the results for a liquid,[2] the heat which arises from plastic deformations characterized by hysteresis leads to a fundamentally new and frequently dominant effect. While the elastic strain is related in an unambiguous way to the thermodynamic state of the object, the plastic strain is a function of the process. The microscopic mechanism for plastic deformation involves a motion of dislocations and is determined by the dislocation density and the rate of change of this density, both of which are tensor quantities.[2,3]

In §2 we offer a formal derivation of a general expression for the dissipation function, containing specific effects which are linear in the rates of plastic strain. We show that these effects always stem exclusively from the shear part of the stress tensor. In §3 we take up an example which illustrates the governing effect of the heat which is evolved during plastic deformation on the course of the process.

2. We use Eulerian coordinates for the calculations, and we retain the notation of Ref. 1 to the maximum extent possible. The complete system of equations in continuum mechanics can be formulated as conservation laws: continuity equations for the mass, momentum, and energy,

$$\frac{\partial \rho}{\partial t}+\frac{\partial j_k}{\partial x_k}=0, \quad j_k=\rho v_k, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial \Pi_{ik}}{\partial x_k}=0, \quad \Pi_{ik}=\rho v_i v_k-\sigma_{ik}, \tag{2}$$

$$\frac{\partial E}{\partial t}+\frac{\partial Q_k}{\partial x_k}=0 \tag{3}$$

(Π_{ik} is the momentum flux tensor, and σ_{ik} is the stress tensor), supplemented by the equation of state of the medium. The latter equation can be found by differentiating the corresponding thermodynamic potential with respect to its independent variables. The most convenient approach is to choose for these variables parameters whose constancy characterizes the thermodynamic equilibrium: the temperature T, the velocity v_i, and the stress tensor σ_{ik}. The hydrodynamic approximation corresponds to the case in which the gradients of these quantities are small. The thermodynamic identity for the potential of a unit mass from these variables,

$$d\mu=-s\,dT-V_{ik}d\sigma_{ik} \tag{4}$$

formally determines a purely thermodynamic state function, the specific-volume tensor, whose trace is equal to the reciprocal of the density:

$$V_{ik}=-\partial\mu/\partial\sigma_{ik}, \quad V_{ii}=1/\rho. \tag{5}$$

We will discuss the relationship between this tensor and the derivatives of the displacement vector of points in the object below.

The energy density of a unit mass is found by means of Legendre transformations:

$$\begin{gathered}\varepsilon=\mu+Ts+V_{ik}\sigma_{ik}=\mu-T(\partial\mu/\partial T)-\sigma_{ik}(\partial\mu/\partial\sigma_{ik}),\\ d\varepsilon=T\,ds+dR.\end{gathered} \tag{6}$$

Here we have introduced the streamlined notation

$$dR=\sigma_{ik}dV_{ik} \tag{7}$$

for the change in the elastic energy of a unit mass in an adiabatic process.

Expressions for the dissipation function and the energy flux density are found from the continuity equation for the energy, (3), for which the time derivative of the energy density of a unit volume, $E=\rho(\varepsilon+v^2/2)$, is expanded and transformed with the help of Eqs. (1), (2) and (6):

$$\begin{aligned}\frac{\partial E}{\partial t}&=\left(\varepsilon+\frac{v^2}{2}\right)\frac{\partial\rho}{\partial t}+v_i\left(\frac{\partial\rho v_i}{\partial t}-v_i\frac{\partial\rho}{\partial t}\right)+\rho T\frac{\partial s}{\partial t}+\rho\frac{\partial R}{\partial t}\\
&=-\left(\varepsilon+\frac{v^2}{2}\right)\frac{\partial\rho v_k}{\partial x_k}+v_i\frac{\partial\sigma_{ik}}{\partial x_k}-v_i\frac{\partial\rho v_i v_k}{\partial x_k}\\
&\quad-v_i^2\frac{\partial\rho}{\partial t}+\rho T\left(\frac{\partial s}{\partial t}+v_k\frac{\partial s}{\partial x_k}\right)\\
&\quad-\rho T v_k\frac{\partial s}{\partial x_k}+\rho\left(\frac{\partial R}{\partial t}+v_k\frac{\partial R}{\partial x_k}\right)-\rho v_k\frac{\partial R}{\partial x_k}.\end{aligned}$$

By grouping terms and using some identities, we can pull out a divergence and a total derivative with respect to the time:

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$$\frac{\partial E}{\partial t}=-\frac{\partial}{\partial x_k}\rho v_k\left(\varepsilon+\frac{v^2}{2}\right)+\frac{\partial}{\partial x_k}v_i\sigma_{ik}-\sigma_{ik}\frac{\partial v_i}{\partial x_k}$$
$$+T\left(\rho\frac{\partial s}{\partial t}+\rho v_k\frac{\partial s}{\partial x_k}+s\frac{\partial\rho}{\partial t}+s\frac{\partial \rho v_k}{\partial x_k}\right)+\rho\frac{dR}{dt}$$

The energy flux should be found from

$$Q_k=\rho v_k(\varepsilon+v^2/2)-v_i\sigma_{ik}. \tag{8}$$

Using Eq. (3) for energy conservation, we find an expression for the dissipation function: the total time derivative of the entropy of a unit volume, $S=\rho s$, multiplied by the temperature:

$$T\frac{dS}{dt}=\sigma_{ik}\left(\frac{\partial v_i}{\partial x_k}-\rho\frac{dV_{ik}}{dt}\right). \tag{9}$$

Here we have used expression (7) for dR.

Remarkably, the expression in parentheses on the right is orthogonal to the unit tensor, so that it has a vanishing trace ($V_{ii}=1/\rho$), and we can write

$$\frac{\partial v_k}{\partial x_k}-\rho\frac{d}{dt}\frac{1}{\rho}=0.$$

The work of hydrostatic pressure thus always leads to reversible changes in the density, and the expression for the dissipation function contains only the shear parts of these tensors:

$$T\frac{dS}{dt}=\tau_{ik}\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)-\rho\frac{dV_{ik}^0}{dt}\right], \tag{10}$$
$$\tau_{ik}=\sigma_{ik}-\frac{1}{3}\delta_{ik}\sigma_{ll},\qquad V_{ik}^0=V_{ik}-\frac{1}{3}\delta_{ik}V_{ll}=-\frac{\partial\mu}{\partial\tau_{ik}}.$$

The quantity in square brackets in (10) phenomenologically determines a tensor of the rate of plastic strain. Since the velocity of a point is the total derivative of its displacement with respect to the time, by multiplying (10) by dt we find a readily interpreted relation: The increment in the heat is equal to the work performed by shear forces minus the elastic internal energy of the shear stress which has been stored in the medium, $dR_{\text{sh}}=\tau_{ik}dV_{ik}$:

$$TdS=dQ=dA_{\text{sh}}-dB_{\text{sh}}\,. \tag{11}$$

Expression (11) vanishes identically only if the strain tensor at each point in the medium determines the stress state of the medium unambiguously. In this case, dissipative effects are usually related exclusively to the viscosity and heat conduction.

3. Let us demonstrate the possibility that these effects can have a dominant influence on the process of plastic deformation in a simple example, the compression of a sample by a piston in a cylinder with rigid, heat-insulating wall.

There are many empirical criteria for the beginning of irreversible deformation—criteria which do not give us the details of the microscopic picture.[3] We choose the simplest of these criteria: the attainment of the "plasticity threshold," i.e., the limiting value of the modulus of the shear stress tensor $\tau_{ik}^2(T)$, which depends only on the temperature (we are ignoring hardening by pressure). Experimental data indicate that this quantity is usually very small in comparison with the elastic moduli of a solid; i.e., the thermodynamic functions can be found in the approximation of Hooke's law. In particular, the explicit expression for V_{ik} in terms of σ_{ik} can be found in linear elasticity theory by differentiating the expansion of the potential in powers of the invariants of the stress tensor (this expansion depends on three constants):

$$\mu=\varepsilon_0-\frac{\sigma_{ll}}{3\rho_0}-\frac{\sigma_{ll}^2}{18\rho_0K_0}-\frac{\tau_{ik}^2}{4\rho_0G_0}, \tag{12a}$$
$$V_{ik}=\frac{\delta_{ik}}{3\rho_0}\left(1-\frac{p}{K_0}\right)+\frac{\tau_{ik}}{2\rho_0G_0}, \tag{12b}$$
$$\varepsilon=\mu+\sigma_{ik}V_{ik}=\varepsilon_0+\frac{\sigma_{ll}^2}{18\rho_0K_0}+\frac{\tau_{ik}^2}{4\rho_0G_0}. \tag{12c}$$

The hydrostatic pressure $p=-\sigma_{ll}/3$ and the shear stress τ_{ik} have been separated out of the stress tensor; ρ_0 is the density of the undeformed medium; and K_0 and G_0 are the bulk modulus and shear modulus of the underformed medium. The expansion of the elastic energy begins with quadratic terms. In the absence of a plastic deformation, $\rho_0V_{ik}=u_{ik}$ is the strain tensor, and (10) vanishes identically, since we have

$$du_{ik}/dt={}^1/_2(\partial v_i/\partial x_k+\partial v_k/\partial x_i).$$

In the case of single-sided compression of a cylindrical sample parallel to its axis, there are no displacements in the transverse directions, x and y. We denote by

$$u=(l-l_0)/l_0 \tag{13a}$$

the relative decrease in length, and we denote by

$$\sigma=-\sigma_{zz}=p+\tau, \tag{13b}$$
$$\sigma_{xx}=\sigma_{yy}=-p+\tau/2 \tag{13c}$$

components of the stress tensor. The latter equality is an unambiguous consequence of the requirement $\tau_{ll}=0$. The stored elastic energy of a unit volume, (12c), is

$$\rho_0(\varepsilon-\varepsilon_0)=p^2/2K_0+3\tau^2/8G_0. \tag{13d}$$

In the region of elastic compression, the specific load on the piston is related to the displacement by the single-sided compression modulus:

$$\sigma=-(K_0+4G/3)u. \tag{14a}$$

Added to the volume decrease is a shear part associated with the rigidity of the wall:

$$\tau=4G_0u/3. \tag{14b}$$

When the shear stress reaches the critical level for the initial temperature T_1, $\tau^*_{(T_1)}=4G_0u^*/3$, plastic phenomena arise, and the heating of the sample becomes important through the temperature dependence of the threshold shear stress. The thermal effect should be determined from (11), but to illustrate the point we write the complete energy balance equation:

$$TdS=C_0dT=\sigma du-d\rho_0(\varepsilon-\varepsilon_0).$$

We assume that the heat capacity $C_0=TdS/dT$ is constant; in condensed matter, it is relatively independent of the process.

Substituting in (13b), and expanding (13d), we see that (11) holds (the effect is determined exclusively by the shear). We find[17])

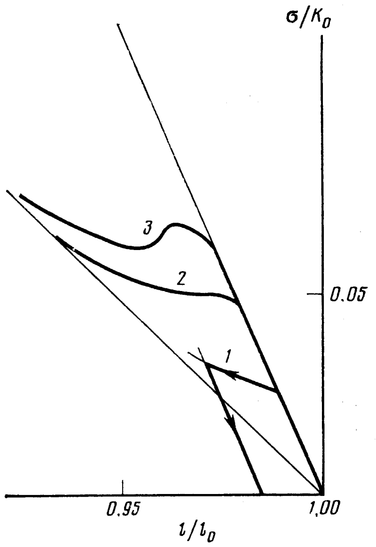

FIG. 1. Force in a piston versus its displacement. 1—Plastic deformation and hysteresis; 2—critical regime; 3—avalanche fracture.

$$\left(C_0+\frac{3\tau}{4G_0}\frac{d\tau}{dT}\right)dT=\tau\,du.$$

This equation can be integrated easily:

$$u-u^*=\frac{3}{4G_0}[\tau_{(T)}-\dot{\tau}_{(T_1)}]+C_0\int_{T_1}^{T}\frac{dT}{\tau(T)}\,. \qquad (15)$$

Along with (13b), it determines the relationship between the strains and stresses in the plastic region in terms of the temperature, as a parameter.

Figure 1 shows some representative curves of $\sigma(u)$ for the very simple function

$$\tau=\tau_0[1+(T/T_0)^2]^{-1}, \qquad (16)$$

which conveys the decrease in the limiting shear stress with increasing temperature. All the solutions have the asymptotic behavior $\sigma=Ku$ at $\sigma>\tau$, with a slope smaller than (14a). The slope $d\sigma/du$ in the dependence on T_0, τ_0, and T_1 can be either negative or positive, so that we have nonmonotonic curves, similar to van der Waals isotherms. The nonmonotonic curves lie above the monotonic curves in this figure. At a sufficiently high initial temperature T_1, the curves are monotonic, plastic effects come into play in a weak deformation, and these effects are manifested solely by hysteresis. The reduction of the loading should follow line (14a), which corresponds to the elastic law, with a residual deformation.

At low temperatures T_1 the regime is not monotonic; it may be unstable and may lead to nonuniform avalanche destruction of the object, possibly involving the appearance of fracture surfaces. A study of this process is an exceedingly complicated problem involving simultaneous solution of the equations of motion and the heat-conduction equation with a heat source determined by expression (10).

[1)]If $d\tau/dT<0$, the effective heat capacity decreases (!) by an amount equal to the magnitude of the change in the shear elastic energy with the temperature: $C_0\to C_0+d[3\tau^2(T)/8G_0]/dT$.

[1]L. D. Landau and E. M. Lifshitz, Gidrodinamika, Nauka, Moscow, 1986 (Hydrodynamics, Pergamon, New York).
[2]L. D. Landau and E. M. Lifshitz, Teoriya uprugosti, Nauka, Moscow, 1965 (Theory of Elasticity, Addison-Wesley, Reading, Mass., 1970).
[3]G. E. Mase, *Continuum Mechanics*, McGraw-Hill, New York, 1970 (Russ. transl. Mir, Moscow, 1974).

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