Soliton in a randomly varying medium

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(Submitted 29 October 1985; resubmitted 27 August 1986)
Zh. Eksp. Teor. Fiz. **92**, 831–840 (March 1987)

A formalism for studying solitons in randomly varying media is proposed. This formalism is based on the introduction of a soliton distribution function in the form of a path integral. The probabilities approach proposed here is followed to obtain a physical picture of the formation of solitons of the nonlinear Schrödinger equation in disordered media. The results are applied to Langmuir solitons in an inhomogeneous plasma.

INTRODUCTION

As we know, solitons—spatially localized, static or moving solutions of nonlinear equations—can arise in many systems. When solitons form in real systems there are usually inhomogeneities which influence wave processes. In an inhomogeneous plasma, for example, localization of Langmuir waves (plasma waves) may occur.[1] Just how the random potential set up by the inhomogeneity influences the appearance and stability of solitons has not yet been studied, however.

In the present paper we study spatially fixed solitons which arise in an inhomogeneous medium. The new probabilistic approach which we have developed can be taken to determine both the stability and the probability for the appearance of a soliton. This approach is based on introducing in the theory a function $F(H)$, the probability for the formation of a soliton with a given value of the Hamiltonian H, and a magnitude Ω, the eigenvalue of the stationary equation which determines the soliton solution.

We know that a soliton in a homogeneous system corresponds to a completely definite value of the Hamiltonian H, which is either positive ($H>0$) or negative ($H<0$). According to Derrick,[2] stable solitons form if $H<0$, and unstable solitons if $H>0$. According to Ref. 3, however, the inequality $H<0$ is by itself a necessary and sufficient condition for the existence of a stable soliton, while the inequality $H>0$ is only a sufficient condition for an instability of the soliton, provided that soliton solutions exist.

If $H>0$, it is always necessary to test whether the necessary condition $\Omega<0$ for the existence of a soliton is satisfied.[3] In an inhomogeneous system, in contrast, there is a large finite probability that both positive and negative values of H will correspond to a soliton; i.e., the whole continuous spectrum of H. Here the criteria for the stability and existence of a soliton become more complicated. The functional dependence $H=H(\Omega)$, which is a linear function in a homogeneous system by virtue of the virial theorem, becomes nonlinear in this case. Analyzing the behavior of the functions $F(H)$ and $H=H(\Omega)$ with the help of the theory of singularities (elementary catastrophe theory),[4] we can draw a picture of the "metamorphoses" of the stability of a soliton as its parameters Ω and H change.

To solve the problem it is convenient to appeal to the analogy between self-localization in crystals and nonlinear wave processes.[5,3] Pursuing this analogy, we first determine the stationary points of the Hamiltonian H which correspond to solitons. After finding these points, we associate the Hamiltonian of the nonlinear equation with a surface. Absolute minima of this surface correspond to stable solitons, while other extrema correspond to unstable or metastable solitons. The slopes of this surface correspond to arbitrary wave packets. Since the slopes of a surface of this sort are not defined exactly, we have a continuous set of surfaces, all of which have the same stationary points. A wave packet with a Hamiltonian value H and a radius r corresponds to some point on one of these surfaces. If H is less than H_0—that value of the Hamiltonian which corresponds to an absolutely unstable soliton—then the wave packet will spread out or collapse, depending on the particular slope of the surface on which this point lies. If the radius of the packet is greater than the radius corresponding to an unstable soliton, the packet will spread out; in the opposite case, it will collapse. The critical value of the radius determining the evolution of a packet is in this case the radius corresponding to a stationary saddle point which separates slopes of the surface. If $H>H_0$, the direction of the evolution of the wave packet cannot be determined in this approach. A detailed time evolution of the behavior of wave packets can of course be determined by directly solving the dynamic equations. This role (the role of a nucleating center for collapse) of absolutely unstable solitons was discovered in the dynamics of wave packets in Refs. 5 and 3.

If, as a result of the collapse (or spreading), the wave packet reaches the opposite slope, the direction of its evolution changes, and it begins to spread out (or to collapse). In this manner, the behavior of wave packets may become an oscillatory behavior of spreading out and collapsing. Knowing the stationary points corresponding to different solitons, one can thus determine the direction of the evolution of wave packets in a qualitative way in several cases.

In this paper we will analyze in detail stationary points corresponding to solitons of the nonlinear Schrödinger equation in a random "white-noise" potential. In particular, we find that in three-dimensional space solitons with $H>0$ and with $H<0$ arise with roughly equal probabilities. Stable and unstable solitons with $H<0$ exist simultaneously. Accordingly, the dynamics of wave packets is in this case more varied than in a homogeneous system,[6,5,3] in which there is a unique, absolutely unstable soliton with $H>0$. A soliton of this sort in a homogeneous plasma is a nucleating region[5] for plasma collapse.[6]

In Section §1 we present the general formalism for describing solitons in an inhomogeneous system. This method is applicable exclusively to Hamiltonian systems.

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In Section §2 we consider as an example solitons of the nonlinear Schrödinger equation in an inhomogeneous white-noise structure. We examine the one-dimensional and three-dimensional cases.

We conclude with a summary of the results and a discussion of the applicability of the examples covered in this paper to three-dimensional Langmuir solitons in an inhomogeneous plasma.[5]

1. GENERAL FORMALISM

We assume that we have Hamilton's equations in the form[1)]

$$i(\partial U/\partial t)=\delta H/\delta U^*, \tag{1}$$

where the Hamiltonian H is a function of $U(\mathbf{x})$ and $V(\mathbf{x})$, where $U(\mathbf{x})$ is a complex wave field, while $V(\mathbf{x})$ is a random potential which can have an arbitrary correlation

$$\langle V(\mathbf{x})V(\mathbf{x}')\rangle=f(\mathbf{x}-\mathbf{x}'). \tag{2}$$

Alternatively, and equivalently, the probability for the formation of the potential $V(\mathbf{x})$ is

$$W[V(\mathbf{x})]=N_0\exp\left\{-\frac{1}{2}\int d^d\mathbf{x}\,d^d\mathbf{x}'\,V(\mathbf{x})g(\mathbf{x}-\mathbf{x}')V(\mathbf{x}')\right\}, \tag{3}$$

where the normalization factor N_0 is determined from the condition

$$\int W[V(\mathbf{x})]DV(\mathbf{x})=1. \tag{4}$$

This is a path integral. The functions $g(x)$ and $f(x)$ are related by

$$\int g(\mathbf{x}-\mathbf{x}')f(\mathbf{x}'-\mathbf{x}'')d^d\mathbf{x}'=\delta(\mathbf{x}-\mathbf{x}''). \tag{5}$$

We introduce the soliton distribution function (the distribution of soliton states) in a random potential. By definition, this distribution function is

$$F(H)=\int DV(\mathbf{x})\delta(H-H[V(\mathbf{x})])W[V(\mathbf{x})], \tag{6}$$

where $H=H[V(\mathbf{x}),U(\mathbf{x}),U^*(\mathbf{x})]$ is the Hamiltonian of Eq. (1). The function $F(H)$ represents the probability for the formation of a soliton with a given value of the Hamiltonian[2)] H. The function $U(\mathbf{x})$ in (6) is a soliton solution of Eq. (1) in the given realization of the random potential $V(\mathbf{x})$. Precisely the same function $F(H)$ has been introduced previously[7] in order to study the self-localization of excitons and current carriers in disordered systems. In the case of self-localization, however, one studies only absolutely unstable stationary points, which correspond to a self-localization barrier. The function $F(H)$ represents the probability for the appearance of a barrier of a given height H.

To calculate the function $F(H)$ we first need to find soliton solutions $U(\mathbf{x})$ for all realizations of the random potential $V(\mathbf{x})$. Such solutions are determined from an equation[3] of the type

$$\delta H/\delta U^*=\Omega(\delta N/\delta U^*), \tag{7}$$

where Ω is an eigenvalue, and N is that integral of Eq. (1) which expresses the conservation of the number of particles:

$$N=\int|U(\mathbf{x})|^2d^d\mathbf{x}.$$

We then need to evaluate the path integral (6) of $V(\mathbf{x})$. This is evidently an impossible task in general, so we will evaluate the integral of $V(\mathbf{x})$ by the method of steepest descent. The parameter of the steepest descent should be identified separately for each specific case.

We write the Hamiltonian H as

$$H[|U(\mathbf{x})|,V(\mathbf{x})]=H_0[|U(\mathbf{x})|]+H_1[|U(\mathbf{x})|,V(\mathbf{x})], \tag{8}$$

where for simplicity we choose H_1 in the form

$$H_1=\int|U(\mathbf{x})|^2V(\mathbf{x})d^d\mathbf{x}. \tag{9}$$

Using the definition of the Dirac δ-function $\delta(\mathbf{x})$, we write (6) as

$$F(H)=-i\int_{-i\infty}^{+i\infty}\frac{dt}{2\pi}\int DV(\mathbf{x})\exp\left(tH-tH_0-t\int|U(\mathbf{x})|^2V(\mathbf{x})d^d\mathbf{x}\right)\times W[V(\mathbf{x})]. \tag{10}$$

The equations for the optimal $t=t_e$ and $V=V_e$ are then found from variation of the argument of the exponential function in (10) with respect to t and $V(x)$ These equations are

$$V_e(\mathbf{x})=-t_e\int|U(\mathbf{x})|^2f(\mathbf{x}-\mathbf{x}')d^d\mathbf{x}', \tag{11}$$

$$H=H_0-t_e\iint|U(\mathbf{x})|^2f(\mathbf{x}-\mathbf{x}')|U(\mathbf{x}')|^2d^d\mathbf{x}\,d^d\mathbf{x}'. \tag{12}$$

Substituting (11) into (7), we find a soliton solution $U(\mathbf{x})$ as a function of t_e. Using (12), we then express t_e in terms of the value of the Hamiltonian H: $t_e=t_e(H)$. Finally, we find with exponential accuracy that the distribution function of the solitons can be expressed as a function of the value of the Hamiltonian H as follows:

$$F(H)\propto\exp\left(t_eH-t_eH_0+\frac{t_e^2}{2}\times\iint|U(\mathbf{x})|^2f(\mathbf{x}-\mathbf{x}')|U(\mathbf{x}')|^2d^d\mathbf{x}\,d^d\mathbf{x}'\right). \tag{13}$$

Let us use this approach to examine some very simple examples.

2. SOLITONS OF THE NONLINEAR SCHRÖDINGER EQUATION

We assume that we have a nonlinear Schrödinger equation describing a nonlinear process in a medium with absolutely random fluctuations of the potential $V(\mathbf{x})$:

$$\langle V(\mathbf{x})V(\mathbf{x}')\rangle=B\delta(\mathbf{x}-\mathbf{x}'), \tag{14}$$

where B is the constant of the white noise. The Hamiltonian in Eq. (1) is

$$H=\int\frac{|\nabla U|^2}{2m}d^d\mathbf{x}-c\int\frac{|U(\mathbf{x})|^4}{2}d^dx+\int V(\mathbf{x})|U(\mathbf{x})|^2d^d\mathbf{x}, \tag{15}$$

where m^{-1} and c are the dispersion and nonlinearity constants. This equation has many applications in plasma physics, solid state physics, and biophysics.

To calculate the function $F(H)$ in (6) by the method of steepest descent, we need to distinguish a parameter. We introduce the transformation $U(\mathbf{x})\to N^{1/2}\Psi(\mathbf{x})$, which by convention reduces the function $U(\mathbf{x})$ to the wave function of a quantum-mechanical particle, since we have $\int|\Psi|^2 d^d\mathbf{x} = 1$ (see Refs. 5 and 3 for details). We then introduce some scale transformations which put the Hamiltonian in dimensionless form:

$$\Psi(\mathbf{x}) \to r_W^{-d/2}\Psi(\mathbf{x}/r_W), \quad V(\mathbf{x}) \to (W_0/N)V(\mathbf{x}),$$

where $r_W = (mcN)^{-1/(2-d)}$ and $W_0 = N/mr_W^2$ are the length scale and energy of a soliton in a homogeneous medium.[3,5] Expression (15) then becomes

$$H = W_0 h = W_0\left[\int \frac{|\nabla\Psi|^2}{2} d^d\mathbf{x} - \int \frac{|\Psi|^4}{2} d^d\mathbf{x} + \int V(\mathbf{x})|\Psi|^2 d^d\mathbf{x}\right]. \quad (16)$$

If we apply the same transformations and also transformations of the type $H_0 \to W_0 h_0$ and $t \to W_0 r_W^3 t/B$, to the argument of the exponential function in integral (10), the parameter P,

$$P = W_0^2 r_W^3/B, \quad (17)$$

is singled out in the exponential function. For $P > 1$, we can use the method of steepest descent to evaluate integral (10). Evaluating (10) by this method, we find

$$F(H) = F(0)\exp\left(P\left[t_3 h - t_3 h_0 + \frac{t_3^2}{2}\int|\Psi(\mathbf{x})|^4 d^d\mathbf{x}\right]\right), \quad (18)$$

where

$$h_0 = \int \frac{|\nabla\Psi|^2}{2} d^d\mathbf{x} - \int \frac{|\Psi|^4}{2} d^d\mathbf{x}, \quad (19)$$

and the wave function $\Psi(\mathbf{x})$ is determined from the solution of an equation like (7):

$$-\Delta\Psi/2 - \varepsilon|\Psi|^2\Psi = \Omega\Psi. \quad (20)$$

Here $\varepsilon = 1 + t_e$, and Ω is an eigenvalue. The parameter t_e is found from an equation like (12):

$$h = \int\left[\frac{|\nabla\Psi|^2}{2} - \left(\frac{1}{2} + t_3\right)|\Psi|^4\right] d^d\mathbf{x}, \quad (21)$$

where $\Psi(\mathbf{x})$ is the solution of (20).

Equations (17)–(21) thus determine the function $F(H)$. Using these equations we can calculate $F(H)$. We introduce the notation

$$T = \int \frac{|\nabla\Psi|^2}{2} d^d\mathbf{x}, \quad Y = \int |\Psi|^4 d^d\mathbf{x}. \quad (22)$$

Multiplying Eqs. (20) by Ψ^*, and integrating over the entire space, we find

$$\Omega = T - \varepsilon Y. \quad (23)$$

In terms of this notation, (21) becomes

$$h = T - (\varepsilon - 1/2)Y. \quad (24)$$

For any solutions of Eq. (20), virial relations hold among the quantities h, T, Y, and Ω:

$$T = \frac{d}{4}\varepsilon Y, \quad \Omega = \frac{(d-4)}{4}\varepsilon Y,$$

$$h = \left[\frac{(2d-6) + \varepsilon(d-4)}{4}\right]Y. \quad (25)$$

The method for deriving these relations is described in detail in Refs. 5 and 3. Virial relations of this sort were originally introduced for nonlinear equations by Pekar in the theory of polaritons.[8] Using these relations, we find the following expressions for $\ln|F(H)/F(0)|$:

$$-R = \ln|F(H)/F(0)| = Pt_e^2 Y/2. \quad (26)$$

Let us consider the case in which the dimensionality of the space is $d = 1$. The eigenvalue of Eq. (20) is well known in this case (Refs. 5 and 7, for example):

$$\Omega = -\tfrac{1}{4}\varepsilon^2. \quad (27)$$

Writing $\varepsilon = 1 + t_e$, we find

$$h = -(3\varepsilon - 2)\varepsilon/12, \quad (28a)$$

$$R = P(\varepsilon - 1)^2\varepsilon/6. \quad (28b)$$

The length scale of the soliton is $r_s \propto \varepsilon^{-1}$. The parameter ε in (27), (28) varies from 0 to $+\infty$. Only in this case do Eq. (20) and thus Eq. (1) have soliton solutions.

Let us consider some limiting cases. The limit $c \to 0$ or, equivalently, $W_0 \to 0$ means that the nonlinearity vanishes from the Hamiltonian. The function $F(H)$ with $N = 1$ then exactly characterizes the tail of the state density of a quantum-mechanical particle (an electron or exciton) in a random potential:

$$F(H) \propto \exp(-(-3H)^{3/2} 2m/B),$$

where m is the mass of the electron and H is the binding energy. This expression was originally derived by Lifshitz.[9] It also follows from this analogy[5,3] that in the case $c \neq 0$ the function $F(H)$ can characterize not only a soliton but also the state density of a self-localized quantum-mechanical particle (an electron or exciton) in a random potential. In this case H corresponds to the adiabatic energy of the electron (or exciton) in the field of phonons (cf. Ref. 10). In the limit $B \to 0$ the function R tends toward $+\infty$. An exceptional case is represented by a small neighborhood of $H = -W_0$, which corresponds to the value of the Hamiltonian of a soliton in a homogeneous medium. In this case we evidently have $F(H) \propto \delta(H + W_0)$.

We can classify these solutions, i.e., we can distinguish in (28) states corresponding to stable solitons and states corresponding to absolutely unstable solitons. The role played by the latter in the dynamics of a system was discussed in Refs. 5 and 3.

Let us examine the functional dependence $h = h(\varepsilon)$ (Fig. 1). As ε is increased from 0, the function $h(\varepsilon)$ increases monotonically and reaches a maximum positive value of 1/3. After this point, $h(\varepsilon)$ decreases monotonically. From (27) we find $\varepsilon = \varepsilon(\Omega)$ and substitute it into (28a).

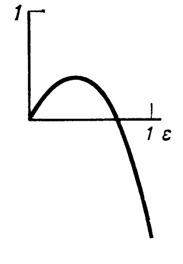

FIG. 1. Hamiltonian versus the parameter ε, which characterizes the scale of the system consisting of the soliton and the inhomogeneity.

We find an analog of the virial theorem, $h = h(\Omega)$, for a soliton which arises in a random structure. Since Ω decreases monotonically with increasing ε, while h takes on both positive and negative values, both stable and unstable solitons can form in a random structure, according to Derrick's criterion.[2] Stable solitons correspond to $h < 0$, and unstable solitons to $h > 0$. At small values of ε, the random potential characterizing ε is a potential "hill" for a soliton localized at it. This hill destroys the soliton. At $\varepsilon = 1$, the potential hill is replaced by a potential well. It is easy to see that in this case Ω and h correspond to a soliton in a homogeneous medium. At large values of ε and, correspondingly, large values of $|\Omega|$ and h, the function h is $h = \Omega$. This relation indicates that in this limit the soliton becomes greatly deformed and differs little from ordinary waves localized in a random potential.

We turn now to the behavior of the logarithm of the probability function, R, for the formation of a soliton with a given value of the Hamiltonian h: $R = R(\varepsilon)$. This behavior is shown in Fig. 2. As ε increases from 0, the function $R(\varepsilon)$ increases monotonically to its maximum value $R_{\max} = 2P/81$, which it reaches at the point $\varepsilon = 1/3$. The function $h = h(\varepsilon)$ reaches its maximum value at the same point. The function R then falls off monotonically, so that the soliton formation probability increases. At the point $\varepsilon = 1$ the function R satisfies $R \equiv 0$, and the probability attains a maximum. The function $R(\varepsilon)$ then again increases monotonically as $\varepsilon \to \infty$.

To see which types of solitons form as $R(\varepsilon)$ behaves in this complicated way we need to examine the functional dependence $R(H)$ which is shown in Fig. 3. We see a "beak"—Whitney's cusp singularity—with vertex at $\varepsilon = 1/3$. As ε is increased, a bifurcation occurs in the behavior of a soliton at $H > 0$ when this point is passed, according to catastrophe theory.[4] For $\varepsilon > 1/3$, the soliton is an unstable saddle point. At $1/3 < \varepsilon < 2/3$, the soliton is metastable. In this case, that minimum in the function space which corresponds to the soliton is not absolute and is separated from a homogeneous state by a barrier. The parameter ε serves as a typical value of the reciprocal of the size of a soliton which is localized in a random potential well. This size is actually the size scale of the inhomogeneity. For $0 < \varepsilon < 1$ the typical sizes of the inhomogeneities at which solitons localize are greater than the size of a soliton in a homogeneous medium. For $\varepsilon > 0$ we find the opposite situation: The sizes of the fluctuation wells are smaller than the size of a soliton in a homogeneous medium. In this case we have $H < 0$, and for $\varepsilon > 2/3$ the solitons are stable. This result actually means that the solitons localize only in narrow potential wells. Derrick's theorem is not correct for $h > 0$, since several types of stationary states arise in this case.

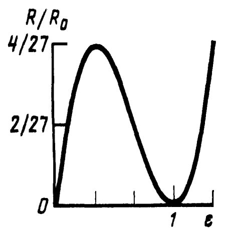

FIG. 2. Logarithm of the soliton formation probability, R, versus the size scale of the soliton, ε.

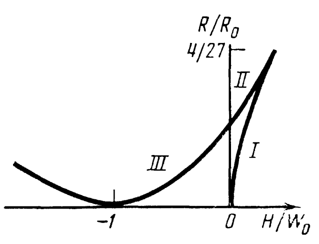

FIG. 3. R versus the Hamiltonian H.

Let us summarize the results for solution (28) found from this analysis:

1) For $0 < \varepsilon \leqslant 1/3$, the solution corresponds to absolutely unstable solitons.

2) For $1/3 < \varepsilon < 2/3$, it corresponds to metastable solitons.

3) For $2/3 \leqslant \varepsilon < +\infty$, it corresponds to absolutely stable solitons.

We now consider a space of dimensionality $d = 3$. In this case the eigenvalue of the ground state of Eq. (20) is[5]

$$\Omega = -44.4/\varepsilon^2. \tag{29}$$

Writing $\delta = \varepsilon^{-1}$ in this case, we find, in contrast with the $d = 1$ case,

$$\Omega = -44.4\delta^2, \tag{30a}$$

$$h = -44.4\delta^2(1-2\delta), \tag{30b}$$

$$R = R_0\delta(\delta-1)^2, \tag{30c}$$

where $R_0 = 88.8P$. The same equations, with $N = 1$, have been found in a calculation of the tails of the state density of excitons (or electrons) with allowance for their interactions with phonons.[10] It was found in that study that the interaction with phonons may cause the state-density tail to be cut off sharply. The tail of the state density of a free exciton (or electron) is also described by (30), but after we take the limit $c \to 0$ and set $N = 1$ (cf. Ref. 9).

Let us classify the solitons which are embodied in (30). The size scale of the system consisting of the soliton and the inhomogeneity, r_s, is $r_s \propto \delta^{-1}$. Soliton solutions correspond to a variation in ε from 0 to $+\infty$. Let us exmine the behavior $h = h(\delta)$ (Fig. 4). As δ increases from 0 to 1/3, $h(\delta)$ decreases monotonically, to its minimum value $W_{\min} = -44.4/27$. It then increases monotonically to $+\infty$. We find $\delta = \delta(\Omega)$ from (30a) and substitute it into (30b). We find an analog of the virial theorem for the most probable soliton which arises in a random structure. For $h > 0$, unstable solitons arise, according to Ref. 2. For $h < 0$, both stable and unstable solitons can arise (Fig. 5). Extremum 1 in Fig. 5 corresponds to the former, and extremum 2 to the later. At

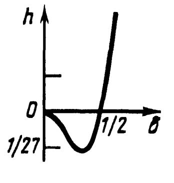

FIG. 4. The soliton Hamiltonian h in the three-dimensional case versus the size sale of the system consisting of the soliton and the inhomogeneity, δ.

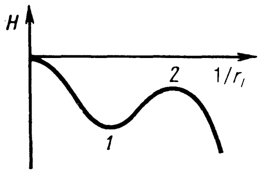

FIG. 5. The Hamiltonian H versus the reciprocal of the localization radius of a wave packet r_l^{-1}

small values of δ, i.e., when the system consisting of the soliton and the inhomogeneity is large, the minimum 1 corresponds to a stationary solution. Since the soliton corresponding to the extremum 2 is unstable, Derrick's criterion for solitons in inhomogeneous media does not hold. There exists a critical value of δ at which these two extrema (1 and 2) merge. This value corresponds to an unstable soliton. The instability of this soliton can lead to collapse.

This picture can be seen more clearly by analyzing the function $R = R(\delta)$, which is shown in Fig. 2. The regions in which this function increases and decreases correspond to solitons of different types. We write R as a function of h: $R = R(h)$. The nature of this function is shown in Fig. 3. A characteristic feature is a "beak," Whitney's cusp singularity. The vertex of the beak correspond to the point $\delta = 1/3$. For $0 \leqslant \delta \leqslant 1/3$ we have $h < 0$, as can be seen from Figs. 2–4. For $1/3 < \delta < 1/2$ we have $h < 0$, but this part of the $h(\delta)$ curve corresponds to part II of the function $R(H)$ (Fig. 3). According to the theory of singularities in functions (elementary catastrophe theory),[4] as we pass the vertex of the peak, going from branch I to branch II (Fig. 3), the extremum corresponding to the minimum becomes an extremum corresponding to a maximum. The maximum corresponds to a stationary saddle point, which determines a soliton. Branches II and III correspond to the same stationary point, regardless of whether we have $h > 0$ or $h < 0$. This stationary point corresponds to an unstable soliton and illustrates a violation of Derrick's stability criterion[2] even in the case $h < 0$, i.e., in a case in which it holds in homogeneous systems.

As a result we find the following in solution (30): stable solitons for $0 < \delta < 1/3$ and absolutely unstable solitons for $1/3 \leqslant \delta \leqslant +\infty$.

CONCLUSION

A distribution function of solitons (of soliton states) with a given value of the Hamiltonian H has been introduced here. As an example, we have calculated this function for a nonlinear Schrödinger equation in a medium with a completely random potential (a white noise). In an inhomogeneous medium we are of course talking about only the predominant type of solitons corresponding to the given value of H.

In a homogeneous medium, a soliton of the nonlinear Schrödinger equation is stable in the one-dimensional case. In an inhomogeneous medium, the stable solitons which correspond to $H < 0$ are accompanied by unstable solitons (saddle points) and metastable solitons of the nonlinear Schrödinger equation, which correspond to a value $H > 0$. The distinction between stable and metastable solitons here is drawn by analogy with homogeneous systems. Large sizes correspond to unstable solitons here.

In three dimensions, a soliton of the nonlinear Schrödinger equation is unstable in a homogeneous medium. In an inhomogeneous medium, in contrast, the unstable solitons corresponding to $H > 0$ are accompanied by stable and unstable (saddle-point) solitons of the nonlinear Schrödinger equation, which correspond to a negative value of the Hamiltonian H ($H < 0$). Here large-scale inhomogeneities correspond to stable solitons. With decreasing radius of the inhomogeneity, the soliton loses its stability.

The picture which has been drawn here of the behavior of solitons of the nonlinear Schrödinger equation with $d = 3$ as the soliton parameters Ω and H are varied can be applied to an inhomogeneous plasma, of the type usually produced in a tokamak. In three dimensions, a three-dimensional Langmuir soliton in a homogeneous plasma is unstable.[5] An inhomogeneous plasma is characterized by cavities, which appear in a completely random fashion. The sizes of these cavities are also arbitrary. Depending on the particular cavity at which a soliton localizes, the value of the Hamiltonian H, which represents the energy of the soliton in this problem, can be either positive or negative. Those solitons which localize at cavities whose radius is large in comparison with the radius of the soliton in the homogeneous medium have a negative energy ($H < 0$) and are stable. As the cavity radius decreases, the soliton may lose its stability even under the condition $H < 0$. The resulting instability may lead to a Langmuir collapse.[6] Alternatively, this instability may have the consequence that as an unstable soliton spreads out it "selects" a large cavity and converts into a stable soliton.

Stationary states (solitons) determine a surface (or a family of surfaces) which is known in catastrophe theory as a "Whitney cusp." The evolution of a wave packet depends on the position with respect to the surface of the point which corresponds to the value of the Hamiltonian W and the packet radius r_W. If this point is above the surface, however, it is not possible to determine the direction in which the wave packet will evolve.

The main conclusion which can be drawn from this analysis is thus that stable Langmuir solitons should arise at large-radius cavities. In other words, spatially localized, long-lived bunches of Langmuir waves (plasma waves) should be observed experimentally at large inhomogeneities in a plasma.

The method presented in this paper can be used to analyze the behavior of solitons in an inhomogeneous medium when the solitons are described by other equations with power-law nonlinearities, e.g., the Kadomtsev-Petviashvili equation.

I wish to thank V. E. Zakharov, A. V. Mikhaĭlov, and, especially, É.I. Rashba for useful discussions.

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Translated by Dave Parsons