The theory of singular magnetic field compression in plasma systems

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The possibility of using magnetic flux compression into infinitesimal volumes by a cylindrically symmetric plasma liner to generate ultrahigh magnetic fields is investigated. Self-similar solutions are obtained for the single-fluid magnetohydrodynamic equations describing cylindrically symmetric plasma singular compression with a frozen-in axial magnetic field. It is shown that compression of the magnetic field in plasma can result in much high amplitudes of the magnetic field (for given kinetic energy of the plasma sheath) as compared with isotropic singular compression.

1. INTRODUCTION

Existing estimates[1,2] and the results of early experiments[3,4] suggest that the use of hollow plasma sheaths as liners for magnetic field compression is the only way of advancing to the next range of ultrahigh magnetic fields, i.e., fields between 20 and 100 MG, using existing laboratory techniques.

The zero-dimensional model, i.e., the so-called snowplow model, in which the liner is assumed to be a thin, perfectly conducting shell, provides a qualitative description of the dynamics of a plasma liner together with an initial magnetic field . This approach can yield simple qualitative estimates. The magnetic flux in the interior of the liner is conserved, and the liner is compressed under the influence of the azimuthal magnetic field $B_\varphi = 2I(t)/cR(t)$ due to the axial current $I(t)$ flowing through it (i.e., the liner is compressed by a Z pinch). It may be shown[2,3] that the compression regime in this model is determined by the values of the following two dimensionless parameters:

$$a = I_m^2 t_m^2/\mu c^2 R_0^2, \qquad b = (cR_0B_0/2I_m)^2, \tag{1}$$

where R_0 and B_0 are the initial values of the radius of the liner and the axial magnetic field in its interior, μ is the mass per unit length of the liner, I_m is the current amplitude, and t_m is the current rise time. The relative radial compression of the liner $\delta = R_0/R_f$ and the corresponding axial field amplitude $B_f = B_0\delta^2$ increase with increasing b. For finite current pulse lengths, each value of b has an optimum value of a, i.e., liner mass μ per unit length, for which the compression of the axial magnetic field is a maximum. Thus, for a sinusoidal current pulse, the optimal value $a \simeq 4$ corresponds to the compression range $B_f = 100B_0 - 400B_0$. This optimum compression occurs at time $t_1 \simeq 1.1t_m$. For the optimum choice of the mass per unit length, the sinusoidal current of amplitude I_m that compresses the axial magnetic field is only a few percent less effective than a constant current $I = I_m$ turned on at $t = 0$.

The zero-dimensional model leads to the following estimate for the maximum axial magnetic field compressed by the liner in the case of optimum compression:

$$B_f = \left(2\ln\frac{R_0}{R_f}\right)^{1/2}\frac{2I_m}{cR_f}. \tag{2}$$

We thus obtain a scaling that determines the liner current I necessary to generate a magnetic field B_f for a fivefold compression to final radius R_f, and the corresponding energy loss:

$$I \approx (23\ \text{MA})(R_f/1\ \text{mm})(B_f/100\ \text{MG}), \tag{3}$$

$$E \approx (6\ \text{MJ/cm})(R_f/1\ \text{mm})^2(B_f/100\ \text{MG})^2. \tag{4}$$

These sample model estimates are confirmed by exact analytic solutions for compression dynamics in a diffuse, hollow Z-pinch in a longitudinal magnetic field.[2] These solutions belong to the class of so-called self-similar solutions with uniform deformation, and describe uniform isotropic compression of plasma together with magnetic field, without energy redistribution among the plasma particles. During this compression, the wall thickness of the initially thin hollow plasma liner remains small compared with its radius. The scaling formulas, based on the self-similar solutions,[2,5] are therefore identical with (3) and (4).

Although for sufficiently small values of the parameter b we can formally allow values of the radial compression δ and final magnetic field that are as large as desired, these quantities actually have upper bounds imposed by the instability of the cylindrically symmetric implosion. In Z-pinch compression, we usually have $\delta \sim 8$–10, but it has been shown in Refs. 3 and 6 that an axial magnetic field will stabilize the compression of the plasma sheath of the Z-pinch, so that a stable 12–15-fold compression is observed. In the stable compression process, the main limitation for given δ is the presence of a "turning point" for the infinitesimal compression regime, since the pressure due to the axial magnetic field inside the plasma liner increases as $R(t)^{-4}$, i.e., more rapidly than the plasma pressure [$\sim R(t)^{-2\gamma}$, where $\gamma < 2$ is the adiabatic exponent] and the pressure due to the azimuthal magnetic field compressing the liner [$\sim R(t)^{-2}$].

Moreover, singular implosive compression regimes are known in hydrodynamics, e.g., axial convergence of shock waves, the collapse of hollow shells, and so on (see Ref. 7), in which the pressure rises without limit near the point of maximum compression. The infinite increase in the energy density in the cumulation process is due to the fact that the volume in which the energy is confined decreases more rapidly than the energy supplied to it. When the Reynolds number is high enough, viscous dissipation does not then restrict singulars compression.[7] It has been argued that the instabilities that arise during compression can destroy cylindrical symmetry while retaining axial symmetry, i.e., singular compression is again unrestricted. Experimentally, one ob-

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serves stable tenfold compression in cylindrical singular compressive implosion of a gas (see, for example, Ref. 8). This means that, when the pinch current is suitably distributed, it is possible to achieve ultrahigh compression of the magnetic field in which the magnetic field initially rises as $R(t)^{-2}$ during uniform compression by the thin plasma liner, followed by singular compression in which the magnetic field rises as $R(t)^{-\alpha}$ (it will be shown below at $\alpha<1$). The physical conditions for the latter compression stage require a plasma flow in which the energy introduced into the plasma volume is concentrated in the axial region. In this paper, we shall use examples of exact self-similar solutions of the magnetohydrodynamic equations to demonstrate the possibility of compression of the magnetic field by the implosion of plasma liners, and will describe their principal properties.

2. FORMULATION OF THE PROBLEM

The singular compressive flows in which we are interested are described by the equations of ideal magnetohydrodynamics of single-fluid single-temperature plasmas. Let us direct the z-axis along the magnetic field, and write these equations in terms of cylindrical coordinates in the case of cylindrically symmetric radial motion. The equations of continuity, induction, adiabaticity, and motion are

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rnu)=0, \tag{5}$$

$$\frac{\partial B}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rBu)=0, \tag{6}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+(\gamma-1)\frac{T}{r}\frac{\partial}{\partial r}(ru)=0, \tag{7}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{1}{8\pi}\frac{\partial B^2}{\partial r}+\frac{\partial P}{\partial r}=0 \tag{8}$$

where n is the particle number density of the plasma ($n=n_e=n_i$), u is the radial component of the plasma velocity, B is the axial component of the magnetic field, $\rho=m_i n$ is the mass density of the plasma, T is the plasma temperature in energy units ($T=T_e=T_i$), $P=2nT$ is the pressure, and γ is the adiabatic exponent (all numerical calculations were performed for $\gamma=5/3$).

To obtain the self-similar solutions, we write each of the variables as the product of a power function of the dimensionless compression $\alpha(t)=R(t)/R_0$ and a function of the self-similar coordinate $\xi=r/R(t)$ [$R(t)$ is the time-dependent spatial scale of the problem and $R_0\equiv R(t_0)$, where t_0 is a fixed, but otherwise arbitrary time):

$$u(r,\ t)=\dot{R}(t)\xi U(\xi), \tag{9}$$

$$n(r,\ t)=n_0\alpha(t)^{2\chi}N(\xi), \tag{10}$$

$$T(r,\ t)=T_0\alpha(t)^{-2\lambda}\Theta(\xi), \tag{11}$$

$$B(r,\ t)=B_0\alpha(t)^{\mu}H(\xi), \tag{12}$$

where χ, λ, and μ are, for the moment, arbitrary exponents, and n_0, T_0, and B_0 are arbitrary normalizing constants. Uniform self-similar flows are determined by taking the time dependence of the compression function in the form

$$\alpha(t)=|t/t_0|^{1/(\lambda+1)} \tag{13}$$

and the following relation between the indices:

$$\mu=\chi-\lambda. \tag{14}$$

The self-similar solutions form a two-parameter family with independent indices λ and χ. For our particular range of problems, solutions with $\lambda>0$ are particularly interesting and describe, as can be seen from (13), the unlimited increase in the compression rate as we approach the collapse time $t=0$ at which the radial scale $R(t)$ tends to zero. The exponents λ and χ are determined by the initial and boundary conditions in each particular case. For many problems, the choice $\lambda=\chi$ is of special significance and, as can be seen from (12) and (14), the scale of the axial magnetic field does not then depend on time. In the analysis of magnetic compression and plasma flow in Θ-pinches, this choice of parameters in analogous to the choice $\chi=0$ in pure gas-dynamic problems, in which the density scale is often taken to be time-independent.[7,9]

3. PROPERTIES OF SELF-SIMILAR SOLUTIONS DESCRIBING SINGULAR COMPRESSION

The methods for obtaining and investigating self-similar solutions of this class have been described in sufficient detail, and we shall not pause here to examine these questions (the reader is referred, for example, to Refs. 7, 9, and 10). We confine our attention to listing the main properties of the compressive solutions that we shall study.

1. In the first instance, we are interested in solutions in which singular energy concentration on the axis is not accompanied by a singular mass density. In other words, at the collapse time $t=0$, the plasma particles remain at finite distance from the axis. The corresponding solutions take the form of power-type asymptotic profiles of hydrodynamic variables at inifinity ($r/t\to\infty$):

$$u\sim r^{-\lambda},\quad n\sim r^{2\chi},\quad T\sim r^{-2\lambda},\ B\sim r^{\chi-\lambda}. \tag{15}$$

At the instant of collapse, $t=0$, the self-similar profiles are wholly of the power type, i.e., they are given by (15) for all $r>0$.

2. The requirement that the mass, energy, and current densities be integrable near the axis at the time of collapse is shown by (15) to lead to the inequalities

$$\chi+1>0,\quad \chi-\lambda+1>0 \tag{16}$$

for the self-similarity indices χ and λ. Hence, it follows that these densities give rise to divergences in the total mass and energy per unit length, and also the current, for $r\to\infty$. This means that self-similar solutions are meaningful only at finite distances from the axis. A solution is naturally cut off on the surface of a piston that maintains the compression. This can be, for example, a magnetic piston, i.e., the current sheath of the Z-pinch, in which the current is confined by the skin effect to the outer surface of the plasma liner. The law of motion of the piston is determined by the motion of the plasma particles in the self-similar solution, and is not of the power type. We note that singular compression is, in general, characterized by "loss of control": if the compression is initially maintained by a converging piston, then, after a certain time, the law of motion of the piston becomes unimportant and, in a finite neighborhood of the axis, the compression process reaches the limiting self-similar state.

3. When the axial magnetic field is present, the phase

space of the self-similar problem is no longer two-dimensional, as in gas dynamics,[9] and becomes three-dimensional.[10] Accordingly, the shape of the spectrum of eigenvalues (λ, χ) changes, i.e., while in gas dynamics the requirements of existence and analyticity of solution can often be used to isolate either a unique solution or a discrete spectrum of values of λ for given χ, here we are dealing with continuous eigenvalue spectra of λ and χ (Ref. 10). The values of the similarity indices for which the profiles obtained below belong to these continuous spectra are in no sense special. In particular, they cannot be obtained by dimensional analysis because we are confining our attention here to the so-called self-similar solutions of the second kind (see Ref. 7).

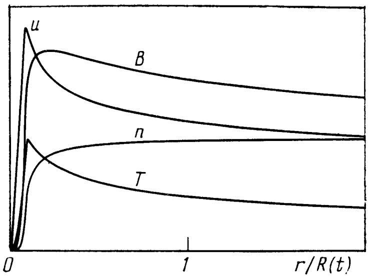

FIG. 2. velocity, magnetic field, density, and temperature profiles for the singular compression of a continuous column of hot plasma: $\lambda = 0.18$, $\chi = 0$.

4. BASIC TYPES OF SELF-SIMILAR SOLUTION

The solutions examined below describe the asymptotic compression of plasma with a frozen-in axial field that takes place inertially when the influence of the external pressure due to the azimuthal magnetic field of the Z-pinch, acting on the periphery of the liner, is no longer significant in comparison with the geometric concentration factor, i.e., convergence to the axis.

To construct the required solutions, we must proceed as in Ref. 10 and find the solutions of the set of ordinary differential equations for the self-similar profiles of the hydrodynamic variables $U(\xi)$, $N(\xi)$, (ξ), and $H(\xi)$, obtained by substituting (9)–(12) into the original set of equations (5)–(8). The asymptotic behavior of the solutions near singular points representing the state of the plasma near the axis, on the free surface, at infinity, and so on, is then examined analytically. The specific profiles of hydrodynamic variables are determined by numerical integration, using the known asymptotic behavior. In particular, the asymptotic behavior given by (15) is valid when $r \to \infty$ for all the profiles examined here.

Compression of a continuous column of cold plasma. A possible self-similar regime of singular compression of plasma with an entrained magnetic field is the compression of cold plasma ($\beta = 8\pi P/B^2 = 0$). Figure 1 shows profiles of the flow variables (here and henceforth, we shall be using arbitrary units). Near the axis ($r \to 0$), the asymptotic profiles are

$$u \sim r, \quad n \sim r^{2(2\chi+\lambda+1)/(1-\lambda)}, \quad B \sim r^{2(\chi+1)/(1-\lambda)}. \tag{17}$$

Isentropically compressed plasmas remain cold down to the point of collapse, at which $\beta \sim 1$ in the shock wave reflected from the axis. We note that, at the instant of collapse, the self-similar solution predicts that the magnetic field becomes infinite on the axis [see (15)] although this happens relatively slowly, i.e., according to (16), slower than r^{-1}.

Compression of a continuous column of hot plasma. The opposite limiting case occurs when hot plasma is compressed in an axial magnetic field and $\beta \gg 1$ throughout the volume of the plasma. Figure 2 shows the profiles for this case, and the corresponding asymptotic forms near the axis are

$$u \sim r, \quad n \sim r^{2(\gamma\chi+\lambda+1)/(\gamma-\lambda-1)}, \qquad T \sim r^2, \quad B \sim r^{\gamma(\chi+1)/(\gamma-\lambda-1)}. \tag{18}$$

Convergence of a shock wave to the axis. Self-similar solutions in which an MHD shock wave converges to the axis through the quiescent plasma in an axial magnetic field exist for $\lambda = \chi$. The undisturbed part of the plasma is characterized for such solutions by a uniform magnetic field $B = \text{const}$ and power-law density and temperature profiles that ensure uniform pressure:

$$n = \text{const}_2 \cdot r^{2\lambda}, \quad T = \text{const}_3 \cdot r^{-2\lambda}. \tag{19}$$

(In the special case of cold plasma, $\text{const}_3 = 0$ and the divergence in temperature near the axis is absent.) The corresponding self-similar profiles are shown in Fig. 3. For the converging shock wave, the acoustic and Alfven Mach numbers are now $M_{s1} = 4.4$, $M_{A1} = 4.0$, i.e., the density and magnetic field compression in this type of shock wave is $n_2/n_1 = B_2/B_1 = 2.9$, and the temperature discontinuity is $T_2/T_1 = 5.2$ (see Ref. 11). The adiabatic compression of the magnetic field behind the converging shock-wave front is 1.2, i.e., the magnetic field is compressed by this type of flow

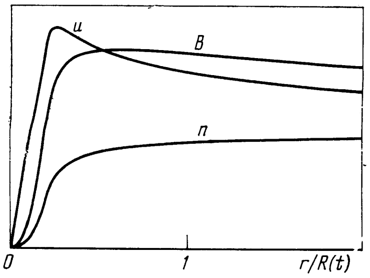

FIG. 1. Velocity, magnetic field, and density profiles for the singular compression of a continuous column of cold plasma: $\lambda = 0.15$, $\chi = 0$.

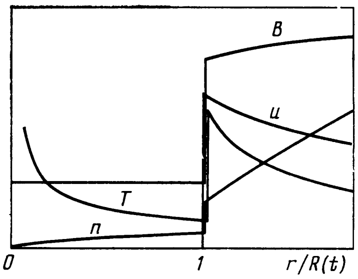

FIG. 3. Profiles of hydrodynamic variables for a shock wave converging to the axis: $\lambda = \chi = 0.5$. The point $r = R(t)$ corresponds to the converging shock wave front.

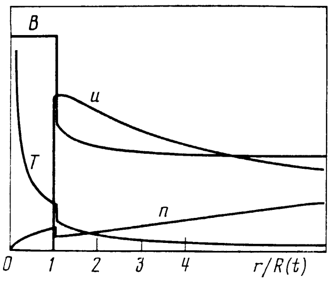

FIG. 4. Profiles of hydrodynamic variables for a shock wave reflected from the axis under the conditions of Fig. 3. The point $r = R(t)$ corresponds to the reflected shock wave front.

by a factor of 3.5. We note that the Mach numbers remain constant despite the amplification of the shock wave during the convergence process because the density profile (19) ensures the necessary increase in the velocity of the fast magnetoacoustic wave as the axis is approached.

Reflection of a shock wave from the axis. Singular self-similar solutions of this class admit continuation beyond the point of collapse ($t = 0$) at which shock waves reflected from the axis appear. When $t > 0$, the flow includes a region of compressing plasma, ahead of the reflected shock wave front, and a region of shock-compressed plasma between the front and the axis. Figure 4 shows the self-similar profiles describing the flow discussed in the last example for $t > 0$. In this case, $\lambda = \chi$, and the plasma behind the reflected shock wave front is at rest, the magnetic field is uniform, and the density and temperature profiles are given by (19) (where, of course, $\text{const}_3 > 0$). The compression of the magnetic field in the reflected shock wave ($M_{s1} = 1.2$, $M_{A1} = 2.0$) is 1.7, and the adiabatic compression ahead of the front is 1.4. In this example, therefore, the magnetic field on the axis, after the convergence and reflection of the shock wave, is greater by the factor $3.5 \times 2.3 = 8.0$ than that in the undisturbed plasma.

Collapse of an empty plasma sheath. This flow is analogous to the classical Rayleigh bubble collapse problem. The pressure and axial magnetic field vanish on the inner boundary of the plasma. This ensures that the boundary conditions on the free surface, i.e., the continuity of pressure and of the tangential component of the electric field, are satisfied. The corresponding profiles are shown in Fig. 5. As in the case of compression of a continuous column of cold or hot plasma, the magnetic field on the axis becomes infinite at the time of collapse. We note that there are no self-similar solutions describing the unrestricted concentration of the magnetic field in the empty cavity. Actually, because the magnetic flux in the cavity must be conserved, the magnetic field in the interior must increase as $R(t)^{-2}$, whereas the magnetic field in the plasma surrounding the cavity is shown by (16) to grow no faster than $R(t)^{-1}$, so that the total pressure cannot be continuous across the free surface. In other words, an arbitrarily small magnetic flux trapped in the cavity of the plasma liner will necessarily stop the compression, i.e., it will lead to the situation described by the zero-dimensional model. Singular compression of the magnetic field is possible only for the field frozen into the plasma.

Cumulation of a fast magnetoacoustic wave. Finally, we have the possibility of a peculiar self-similar compression of plasma with a trapped magnetic field, in which the compression front is a fast magnetoacoustic wave, i.e., a weak discontinuity propagating in stationary plasma and trapping it in a radial implosion. An analogous type of solution for singular compression in the absence of an external field has been investigated in the theory of laser fusion. The solution can be obtained, for example, by time-reversing the solution given in Ref. 12 (see also Ref. 13).

This regime occurs for $\lambda = \chi < 0$. The density and temperature profiles in the undisturbed plasma have the form given by (19) and, because $\lambda < 0$, the plasma density increases toward the axis. An increase in the mass of the plasma brought into motion by the compression wave slows down the wave, whose velocity vanishes at $t = 0$. Figure 6 shows the corresponding profiles of hydrodynamic variables.

The types of self-similar solution enumerated above do not exhaust all the possible solutions in this geometry. For example, we have not considered solutions describing plasma compression in a Θ-pinch (i.e., compression by an axial magnetic field that grows toward the periphery), solutions involving focusing the entire plasma mass on the axis, solutions with ionizing shock waves, and so on. Such solutions can be constructed by the same method, but this lies outside our framework here.

5. CONCLUSION

The self-similar solutions that we have obtained demonstrate the basic possibility of singular compression of an axial magnetic field together with the plasma. As $R(t)$ decreases, this compression proceeds more slowly than uniform compression. The singular compression process is

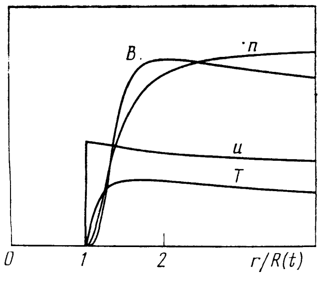

FIG. 5. Collapse of a hollow plasma sheath: $\lambda = 0.15$, $\chi = 0$. The point $r = R(t)$ corresponds to the inner boundary of the sheath.

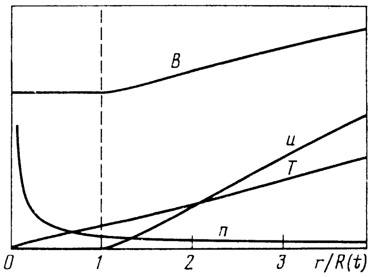

FIG. 6. Singular compression by a fast magnetoacoustic wave: $\lambda = \chi = -0.4$. The point $r = R(t)$ corresponds to the converging compression wave front, marked by the broken line. The hydrodynamic variables undergo a weak discontinuity on the front.

therefore more conveniently implemented on the final stage of compression of plasma with an entrained magnetic field. The transition to the limiting regime after completion of uniform compression can increase the final magnetic field by more than an order of magnitude. The stabilization of the first stage, i.e., uniform compression, by a combination of an axial magnetic field and the compressing aximuthal field of the Z-pinch, was demonstrated in Refs. 3 and 6, both theoretically and experimentally. The same stabilization factor operates on the final stage of singular compression. A complete discussion of dynamic stability at all stages of compression of a plasma together with magnetic field, including an analysis of the conditions imposed on the pinch current profile that are necessary to achieve singular compression, will be given in a separate paper.

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