Dynamics of solitary dissipative vortices: vortex lattices and their stability

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It is shown that the existence of vortex structures in a viscous liquid can be explained within the framework of the concept of solitary dissipative vortices. A minimal model is proposed in which the structural elements of the theory are the Burgers and Sullivan vortices. The vortex–vortex interaction dynamics is studied. It is shown that the vortices can arrange themselves in ensembles and form periodic structures (lattices). The evolution of a dissipative lattice is of a self-similar nature, and asymptotically goes over into the regime of rigid-body rotations. A selection rule, according to which a triangular lattice is energetically the most advantageous lattice, is formulated. The results obtained are valid for finite circular lattices of sufficiently large size. It is shown that such circular lattices form a sequence through which a limiting transition to an infinite lattice can be realized if the size is increased. The dispersion law for the vibrations in a dissipative lattice is found. It is found that only damped vibrations occur in a triangular lattice, and that there exists an upper wavelength limit λ_0 such that vibrations with $\lambda \geqslant \lambda_0$ do not occur.

INTRODUCTION

A classical example of self-organization is the formation of ordered spatial structures in a convectively unstable liquid at sufficiently large Rayleigh numbers.[1] This phenomenon can be explained within the framework of the typical—in synergetics—wave concept of a dominant mode (order parameter).[2] Quite recently, a similar phenomenon in the form of highly ordered vortex structures (lattices and rings) was discovered in the course of an investigation of convection in a rotating liquid.[3,4] And although a number of the results obtained in this experiment at high angular velocities admit of an interpretation on the basis of the Chandrasekhar wave theory,[5] the wave concept cannot explain the observed linear angular-velocity dependence of the vortex number in the region of low angular velocities. A similar dependence had earlier been found by Hopfinger and his coworkers[6] in an experiment in which intense vortices were excited in a rotating liquid by very small-scale turbulence produced by a vibrating lattice.

It turns out that this universality can be explained within the framework of an alternative (to the wave) concept of solitary dissipative vortices. The existence of the latter is indicated also by the appearance in a number of experiments of irregular vortex structures[3,6–8] and the observation of elementary interaction processes between the individual vortices in the course of a transition from a regular to an irregular lattice.[3]

The role of such fundamental vortex objects can be played by the theoretically best-understood solitary dissipative stationary vortices, first described by Burgers[9,10] and Sullivan.[11] These vortices are further distinguished by the fact that they are translationally invariant. The principal mechanism underlying the maintenance of dissipative vortices in the steady state is the balance of the viscous diffusion and the radial flow at each level.

In the present paper the reason for the interest in such vortex objects is that they can serve as structure elements in the study of collective effects: acts of interaction, formation of ordered structures, etc.

A general approach to such problems is through adiabatic perturbation theory under the assumption that the characteristic dimensions of the vortices are small compared to the characteristic distance between them. In this approach the structure of the vortices is computed in the lowest order of the theory, while the nature of their evolution is computed in the next approximations. Variants of this theory have been employed in the construction of soliton,[12] superconductivity,[13] and quantum field[14] theories.

Let us emphasize that the dynamical systems considered in the paper reflect a number of general properties characteristic of open dissipative systems of the hydrodynamic type, and, consequently, are largely system-independent. It is therefore to be hoped that dissipative vortices will turn out to be useful outside hydrodynamics, just as ordinary vortex filaments turned out to be important objects of investigation in the theories of superfluidity, superconductivity, and magnetized plasmas.

1. CHOICE OF MODEL; DISSIPATIVE VORTICES

Let us consider the motion of a viscous incompressible liquid in the half space $z \geqslant 0$ with a rigid lower boundary $z = 0$ in the case when this motion is described by the Navier-Stokes equation:

$$\dot{\mathbf{u}}+(\mathbf{u}\nabla)\mathbf{u}+w\partial_z\mathbf{u}=-\nabla P+\vartheta\Delta\mathbf{u}, \tag{1.1}$$

$$\dot{w}+\mathbf{u}\nabla w+w\partial_z w=-\partial_z P+\vartheta\Delta w, \tag{1.2}$$

$$\nabla\mathbf{u}+\partial_z w=0, \quad w|_{z=0}=0. \tag{1.3}$$

Here $\mathbf{u}$ and w are respectively the horizontal and vertical components of the velocity, P is the kinematic pressure, i.e., the usual liquid pressure divided by the density, and ϑ is the

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coefficient of viscosity. Let us note that here and in what follows all the vector quantities lie in the horizontal plane $\mathbf{x} = (x,y)$.

We shall study the special class of flows for which the horizontal velocity component $\mathbf{u}$ does not depend on the vertical coordinate z. For such flows we find directly from the incompressibility equation and the boundary condition (1.3) that

$$\nabla \mathbf{u} = -D, \qquad w = zD, \tag{1.4}$$

where D is the divergence of the two-dimensional $\mathbf{u}$ field. Substituting the representation for the vertical velocity into the equations of motion (1.1) and (1.2), and separating the variables, we obtain the system of equations

$$\dot{\mathbf{u}} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p + \vartheta\Delta\mathbf{u}, \tag{1.5}$$

$$\dot{D} + \mathbf{u}\nabla D + D^2 = \vartheta\Delta D + c. \tag{1.6}$$

Here c is the separation parameter and $p = P - cz^2/2$ is that part of the total pressure which does not depend on z.

Since it is convenient to construct the subsequent theory in terms of D and the vorticity field $\omega = \operatorname{curl} \mathbf{u}$, let us rewrite Eq. (1.5) in the form

$$\dot{\omega} + \mathbf{u}\nabla\omega - D\omega = \vartheta\Delta\omega. \tag{1.7}$$

This vortex-evolution equation differs from the analogous equation obtained in the case of a two-dimensional incompressible liquid only by the $D\omega$ term, which describes the effect of the stretching of the vortex tubes by the vertical currents.

As shown in Refs. 9–11, Eqs. (1.6) and (1.7) admit of solutions in the form of stationary, localized, axisymmetric, and dissipative vortices. The first type—the Burges vortex—is described by the expressions

$$D = 2a, \qquad \omega = \frac{\varkappa a}{2\pi\vartheta}\exp\left(-\frac{a\mathbf{x}^2}{2\vartheta}\right),$$
$$\mathbf{u} = -a\mathbf{x} + \frac{[\mathbf{n},\mathbf{x}]}{\mathbf{x}^2}\frac{\varkappa}{2\pi}\left[1 - \exp\left(-\frac{a\mathbf{x}^2}{2\vartheta}\right)\right], \tag{1.8}$$

where $\varkappa$ and $a = c^{1/2}/2 > 0$ are free parameters characterizing respectively the intensity and dimension of the vortex $\mathbf{n}$ and the vertical unit vector. The second type—the Sullivan vortex—is described by the expressions

$$D = 2a + d, \quad d = -6a\exp\left(-\frac{a\mathbf{x}^2}{2\vartheta}\right), \quad \omega = \frac{\varkappa a}{2\pi\vartheta} I'\left(\frac{a\mathbf{x}^2}{2\vartheta}\right),$$
$$\mathbf{u} = -a\mathbf{x} + 6\vartheta\frac{\mathbf{x}}{\mathbf{x}^2}\left[1 - \exp\left(-\frac{a\mathbf{x}^2}{2\vartheta}\right)\right] - \frac{[\mathbf{n}\mathbf{x}]}{\mathbf{x}^2}\frac{\varkappa}{2\pi} I\left(\frac{a\mathbf{x}^2}{2\vartheta}\right). \tag{1.9}$$

Here we have used the notation:

$$I(x) = H(x)/H_\infty, \qquad H(x) = \int_0^x dt\,\exp\left(-t + 3\int_0^t \frac{1-e^{-\tau}}{\tau}d\tau\right),$$
$$H_\infty = H|_{x=\infty}, \qquad I' = \partial_x I(x).$$

It is not difficult to see from a comparison of (1.8) and (1.9) that the Sullivan vortex possesses a more complicated structure than the Burgers vortex. This manifests itself in the presence of a cylindrical core at whose walls the radial component of the velocity changes sign. Outside the core the motion of the liquid is qualitatively the same as in the Burgers vortex, but inside it the liquid rises in the vicinity of the walls of the core and subsides in the central part. For the radius of the core we have the estimate $\rho_c = 1.68(2\vartheta/a)^{1/2}$; the nature of the flow and the corresponding profiles for the Sullivan vortex are depicted in Figs. 1 and 2.

Let us emphasize that, of all the known hydrodynamic models possessing steady-state localized dissipative vortex

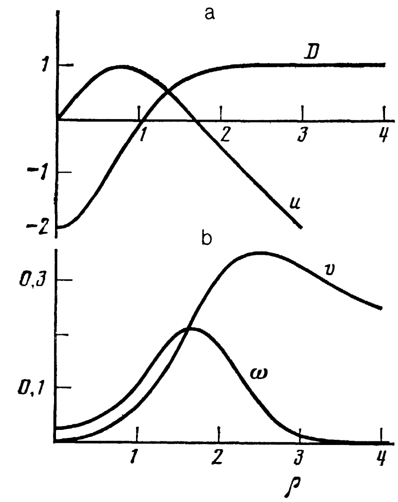

FIG. 1. Radial profiles for the Sullivan vortex: a) the divergence D in units of $2a$ and the radial component u of the velocity in units of $(2a\vartheta)^{1/2}$; b) the tangential component v of the velocity in units of $\varkappa(2a\vartheta)^{1/2}/4\pi\vartheta$ and the vorticity ω in units of $\varkappa a/2\pi\vartheta$; everywhere ρ is in units of $(a/2\vartheta)^{1/2}$.

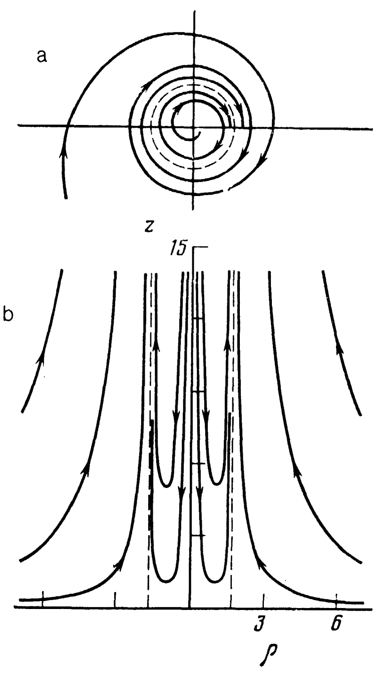

FIG. 2. Motion of the liquid in a dissipative Sullivan vortex: a) top view; b) side view [ρ and z are in units of $(a/2\vartheta)^{1/2}$]. The dashed line corresponds to the cylindrical surface at which the radial velocity changes sign. The horizontal flow structure was computed with $\varkappa/4\pi\vartheta = 90$.

solutions, the present model is minimal in the set of requisite properties. Similar solutions have been obtained for more complicated models with allowance for the thermal conductivity and buoyancy of the liquid.[15,16]

2. EQUATIONS OF MOTION OF AN ENSEMBLE OF DISSIPATIVE VORTICES

Let us construct the equations describing the dynamics of an ensemble of weakly interacting dissipative vortices, choosing as the structure element of the theory the Sullivan vortex (1.9).

To begin with, let us explicitly separate out in the divergence field the uniform-divergence background $2a$. Further, taking account of the translational-invariance properties of Eqs. (1.6) and (1.7), we represent the divergence and vorticity fields for an ensemble of N vortices localized at different points $x_n = (x_n, y_n)$ in space in the form

$$D=2a+\sum_{n=1}^{N} d_n+\delta, \qquad \omega=\sum_{n=1}^{N}\omega_n+\zeta,$$

$$d_n=-6a\exp\left[-\frac{a}{2\vartheta}(\mathbf{x}-\mathbf{x}_n)^2\right], \quad \omega_n=\frac{\varkappa_n a}{2\pi\vartheta}I'\left(\frac{a}{2\vartheta}(\mathbf{x}-\mathbf{x}_n)^2\right). \tag{2.1}$$

Here δ and ζ are deformation corrections due to the interaction of the vortices.

In order to obtain the corresponding representation for the velocity, let us assume that the uniform-divergence background $2a$ is the collective result of independent contributions from all the dissipative vortices. In this case, as is easy to show, we have for the uniform-divergence velocity component $\mathbf{u}_a$, which satisfies the condition $\nabla\mathbf{u}_a = -2a$, the expression

$$\mathbf{u}_a=\sum_{n=1}^{N}\gamma_n(\mathbf{x}_n-\mathbf{x}) \tag{2.2}$$

with the condition

$$\sum_{n=1}^{N}\gamma_n=a, \tag{2.3}$$

where the γ_n, like the $\varkappa_n$, are free parameters of the theory.

Thus, the resultant velocity field can be represented in the following form:

$$\mathbf{u}=\mathbf{u}_a+\sum_{n=1}^{N}\mathbf{u}_n+\mathbf{v}. \tag{2.4}$$

Here $\mathbf{v}$ is a correction to the velocity field ($\nabla\mathbf{v} = -\delta$ and curl $\mathbf{v} = \zeta$) and $\mathbf{u}_n$, the contribution to the velocity field from the nth dissipative vortex, is given in accordance with (1.9) by the expression

$$\mathbf{u}_n=6\vartheta\frac{\mathbf{x}-\mathbf{x}_n}{(\mathbf{x}-\mathbf{x}_n)^2}\left(1+\frac{d_n}{6a}\right)+\frac{[\mathbf{n},\mathbf{x}-\mathbf{x}_n]}{(\mathbf{x}-\mathbf{x}_n)^2}\frac{\varkappa_n}{2\pi}I\left(\frac{a}{2\vartheta}(\mathbf{x}-\mathbf{x}_n)^2\right). \tag{2.5}$$

We shall assume that the characteristic dissipative-vortex dimension $\sigma = (2\vartheta/a)^{1/2}$ is much smaller than l, the characteristic dissipative-vortex separation. Then we have the estimate

$$\varepsilon=(2\vartheta/al^2)^{1/2}\ll 1, \tag{2.6}$$

which guarantees that the vortex–vortex interaction is weak. We shall construct the asymptotic theory in terms of this parameter.

Neglecting the exponentially small terms of the order of $\exp(-\varepsilon^{-2})$, let us write out the asymptotic representations, useful in the vicinity of the nth vortex, for the $\mathbf{u}$, D, and ω fields:

$$\mathbf{u}=\mathbf{u}_n-a(\mathbf{x}-\mathbf{x}_n)+\mathbf{V}_n+\mathbf{U}_n+\mathbf{v},$$

$$D=2a+d_n+\delta, \qquad \omega=\omega_n+\zeta, \tag{2.7}$$

where $\mathbf{V}_n$ and $\mathbf{U}_n$ have the meaning of velocities induced at the location of the nth vortex by all the remaining vortices, and are given by the expressions

$$\mathbf{V}_n=\sum_{m=1}^{n}\gamma_m(\mathbf{x}_m-\mathbf{x}_n),$$

$$\mathbf{U}_n=\sum_{m=1}^{N}{}'\left(6\vartheta\frac{\mathbf{x}-\mathbf{x}_m}{(\mathbf{x}-\mathbf{x}_m)^2}+\frac{\varkappa_m}{2\pi}\frac{[\mathbf{n},\mathbf{x}-\mathbf{x}_m]}{(\mathbf{x}-\mathbf{x}_m)^2}\right). \tag{2.8}$$

Here and below a summation sign with a prime indicates that the term with the number $m = n$ is omitted in the summation.

Let us substitute (2.7) into the basic equations (1.6) and (1.7), assuming the free parameters $\mathbf{x}_n$ and $\varkappa_n$ to be time dependent, and let us make the substitution $\mathbf{x} - \mathbf{x}_n \to \mathbf{x}$, which is equivalent to our going over the moving reference system fixed to the center of the nth vortex. As a result, after the mutual cancellation of the terms describing the nth undeformed vortex, we are left with the following system of equations for the deformations:

$$\dot{\delta}+\nabla(\mathbf{u}_n\delta+\mathbf{v}d_n)+6a\delta+4d_n\delta-\vartheta\Delta\delta$$

$$=(\dot{\mathbf{x}}_n-\mathbf{V}_n-\mathbf{U}_n)\nabla(d_n+\delta)-\mathbf{v}\nabla\delta-\delta^2, \tag{2.9}$$

$$\dot{\zeta}+\nabla(\mathbf{u}_n\zeta+\mathbf{v}\omega_n)-\vartheta\Delta\zeta$$

$$=(\dot{\mathbf{x}}_n-\mathbf{V}_n-\mathbf{U}_n)\nabla(\omega_n+\zeta)-\nabla(\mathbf{v}\zeta)+\frac{\dot{\varkappa}_n}{\varkappa_n}\omega_n, \tag{2.10}$$

which we shall solve with the initial condition $\mathbf{v}|_{t=0} = 0$, which indicates the absence of deformation corrections at the initial moment of time.

Equations (2.9) and (2.10) describe the nonlinear response of the corrections to the effective forces that determine the corresponding inhomogeneous terms. Taking account of the radial symmetry of the linearized homogeneous problem, we choose as an orthogonal basis the system of functions $\exp(ik\varphi)$, where φ is the polar angle and $k = 1,2,\dots$. Then the following expansion is valid:

$$(\dot{\mathbf{x}}_n-\mathbf{V}_n-\mathbf{U}_n)\nabla\begin{bmatrix} d_n \\ \omega_n \end{bmatrix}=\sum_{k=0}|\mathbf{x}|^k(q_k e^{i(k+1)\varphi}+\text{ c.c.})\,\partial_{|\mathbf{x}|}\begin{bmatrix} d_n \\ \omega_n \end{bmatrix}, \tag{2.11}$$

$$q_0=\dot{z}_n^*+\sum_{m=1}^{N}{}'\left(\gamma_m z_{mn}^*-\frac{\mu_m}{z_{mn}}\right), \qquad \mu_m=6\vartheta-i\frac{\varkappa_m}{2\pi}, \tag{2.12}$$

$$q_k=\sum_{m=1}^{N}{}'\frac{\mu_m}{z_{mn}^k}, \qquad k\geqslant 1, \qquad z_{mn}=z_m-z_n, \qquad z_n=x_n+iy_n,$$

where the expansion coefficients q_k depend on the time implicitly through the vortex coordinates x_n and y_n.

Since the ω_n and d_n are localized on the scale $\sigma = (2\vartheta/a)^{1/2}$ and the z_{mn} have a characteristic scale of l, it can easily be verified that the formal series (2.11) is at the same time a perturbation expansion in the small parameter ε. This circumstance allows us, if we seek the solutions for δ, ζ, and $\mathbf{v}$ in the form

$$\begin{bmatrix} \mathbf{v} \\ \delta \\ \zeta \end{bmatrix} = \sum_{k=0} \varepsilon^k e^{ik\varphi} \begin{bmatrix} v_k \\ \delta_k \\ \zeta_k \end{bmatrix} + \text{c.c.}, \qquad (2.13)$$

to immediately draw the conclusion that the k th angular mode is excited only in k th order perturbation theory, and is of a localized nature. The hierarchy in the generation of the deformation modes allows us to greatly simplify the problem if we limit ourselves to the first approximation of the theory.

Thus, retaining the terms of up to first order in ε in (2.13), we find from (2.9) and (2.10) that

$$(\partial_t + \hat{L}_n) \begin{bmatrix} \delta \\ \zeta \end{bmatrix} = \mathbf{q}_0 \nabla \begin{bmatrix} d_n \\ \omega_n \end{bmatrix} + \frac{\dot{\varkappa}_n}{\varkappa_n} \begin{bmatrix} 0 \\ \omega_n \end{bmatrix}, \qquad (2.14)$$

where $\mathbf{q}_0 = (\operatorname{Re} q_0, \operatorname{Im} q_0)$ and $\hat{L}_n$ is a linear stability operator for a solitary vortex.

The choice of a solitary dissipative vortex as the structure element in the asymptotic theory being developed here implicitly presupposes its stability. There is at present no direct proof of this fact, which is based on an analysis of the spectral properties of the operator $\hat{L}_n$. Nevertheless, Foster and Duck's[17] numerical analysis of the stability of the dissipative Long vortex[18] can serve as an indirect indication of the stability of the vortices in question, at least against two-dimensional perturbations.

If, as assumed, the operator $\hat{L}_n$ is stable, then the only dangerous effect is the secular growth of the zero-modes, i.e., the eigenfunctions of the operator $\hat{L}_n$ that correspond to the zero eigenvalue. The explicit form of the zero-modes can be determined without specifying the form of the operator $\hat{L}_n$. In the case when the exact solutions and symmetry groups of the basic steady-state equations are known, the zero-modes are obtained by differentiating the solutions with respect to the group parameters.[13,14,19] Using the fact that Eqs. (1.6) and (1.7) are invariant under the group of similarity transformations of the vorticity field and the translation group with group parameters $\mathbf{x}_n$ and $\varkappa_n$, and the fact that they possess exact solutions, namely, the solutions (1.9), we find the corresponding set of zero-modes: $(\nabla d_n, \nabla\omega_n)$, $(0, \omega_n/\varkappa_n)$.

Since the right-hand side of (2.14) is, by construction, a linear combination of the zero-modes, the condition for the nonoccurrence of deformation corrections reduces to the requirement that

$$\dot{\varkappa}_n = 0, \qquad (2.15)$$

$$q_0 = 0. \qquad (2.16)$$

From (2.12) and (2.16) we finally obtain the equations of motion for an ensemble of dissipative Sullivan vortices:

$$\dot{z}_n^* = \sum_{m=1}^{N}{}' \left(\frac{\mu_m}{z_{nm}} - \gamma_m z_{nm}^* \right), \qquad \mu_m = 6\vartheta - i\frac{\varkappa_m}{2\pi}. \qquad (2.17)$$

Equations (2.17) have a simple physical meaning, and describe the motion of each of the vortices under the action of velocities induced by the remaining vortices. Furthermore, the first term on the right-hand side, a term which describes the mutual repulsion ($\operatorname{Re}\mu = 6\vartheta > 0$) and the rotation of the vortices, is effective at small scales, while the second, which describes the attraction ($\Sigma\gamma_m = a > 0$), is effective at large scales.

As a qualitative analysis of (2.9) and (2.10) shows, the procedure for eliminating the zero-modes in the next orders of the perturbation theory leads to corrections only in Eq. (2.16), which determines the vortex trajectories, the equation (2.15) for the intensities being satisfied exactly in all orders in ε. And what is more, it can be shown that the corrections in (2.16) occur in the perturbation theory orders (in ε) not lower than the fifth, and that they are due to the resonant interaction of the second and third harmonics. This shows how the deformation of the vortices affects their trajectories in the course of their evolution.

The consistent use of the zero-mode elimination method requires knowledge of all the eigenfunctions and eigenvalues of both the direct and the adjoint stability operator if we do not limit ourselves to the first approximation. In particular, knowledge of the adjoint zero-modes is required for the elucidation of the fundamental structure of the equations describing the vortex trajectories.

It is relatively easy to find the adjoint zero-modes for the Burgers vortices, the evolution of which is described by Eq. (2.10) with $d_n = 0$, $\delta = 0$, and ω_n and $\mathbf{u}_n$ determined in accordance with (1.8). In this case the zero-modes ω_n and $\nabla\omega_n$ correspond to the adjoint zero-modes 1 and $\mathbf{x}$, which, as can easily be verified, form an orthogonal basis. Successively multiplying (2.10) by the adjoint zero-modes, and integrating over $\mathbf{x}$, we find the equations

$$\dot{\varkappa}_n = 0, \qquad \dot{\mathbf{x}}_n = \int [\mathbf{V}_n - \mathbf{U}_n](\omega_n + \zeta)\,d\mathbf{x} \Big/ \int (\omega_n + \zeta)\,d\mathbf{x}. \quad (2.18)$$

Here $\mathbf{V}_n$ and $\mathbf{U}_n$ are given as before by the formulas (2.8), but we must in this case set $\vartheta = 0$ in the expression for $\mathbf{U}_n$. The structure of the second equation in (2.18) shows that it has the meaning of a vortex-momentum conservation law, and shows that the trajectory of a Burgers vortex is a nonlinear functional of specific hydrodynamic fields.

We can, by limiting ourselves to the first approximation, easily obtain from (2.18) an equation describing an ensemble of dissipative Burgers vortices. The equation thus obtained then differs from (2.17) only by the formal requirement that $\vartheta = 0$ in the definition of μ_m, which implies the absence of mutual repulsion between the Burgers vortices at small scales. Such a characteristic leads to a situation in which, as a result of the attraction, all the vortices in the long run dissolve. For this reason, there exists no stationary configuration composed of two or more Burgers vortices.

Below we shall analyze the equations (2.17) for the case in which $\operatorname{Re}\mu_m = 6\vartheta \neq 0$. This case corresponds to the Sullivan vortices. The transition to the Burgers vortices can be effected in the final answers through passage to the $\vartheta \to 0$ limit. Let us note that in this case the radius ρ_c of the core of a Sullivan vortex tends to zero, and that this core degeneracy allows us in the description of the collective effects of the Sullivan vortices to go over to the Burgers vortices, whose evolution, unlike their structure, does not depend on the viscosity.

3. INTERACTION BETWEEN TWO DISSIPATIVE VORTICES

In order to form an idea about the nature of the interaction between dissipative vortices, let us consider the equation of motion (2.17) for two Sullivan vortices:

$$\dot{z}_1^* = -\gamma_2(z_1^* - z_2^*) + \frac{\mu_2}{z_1 - z_2}, \quad \dot{z}_2^* = -\gamma_1(z_2^* - z_1^*) + \frac{\mu_1}{z_2 - z_1}. \qquad (3.1)$$

Let us introduce the following new coordinates: $z = z_1 - z_2$ and $Z = (z_1\mu_1^* + z_2\mu_2^*)/(\mu_1 + \mu_2)^*$, which respectively describe the relative motion of the vortices and the motion of their "center of gravity." Then from (3.1) it follows that

$$\dot{z}^* = -az^* + \mu/z, \quad \dot{Z}^* = Az^*, \qquad (3.2)$$

where $a = \gamma_1 + \gamma_2$, $\mu = \mu_1 + \mu_2$, and $A = (\gamma_1\mu_2 - \gamma_2\mu_1)/\mu$. Integrating, to start with, the first equation in (3.2), we find the relative motion. In terms of $\xi = |z|^2$ and $\theta = \arg z$, we obtain

$$\xi = \xi_c + (\xi_0 - \xi_c)e^{-2at}, \qquad (3.3)$$

$$\theta = \theta_0 + \lambda \ln[\xi(\xi_0 - \xi_c)/\xi_0(\xi - \xi_c)]. \qquad (3.4)$$

Here we have adopted the notation

$$\xi_0 = \xi|_{t=0}, \quad \theta_0 = \theta|_{t=0}, \quad \xi_c = \mathrm{Re}\,\mu/a = 12\vartheta/a,$$

$$\lambda = -\mathrm{Im}\,\mu/2\,\mathrm{Re}\,\mu = (\varkappa_1 + \varkappa_2)/48\pi\vartheta.$$

Let us emphasize that $\xi_0 \geqslant \xi_c$. For the Sullivan vortices this inequality reflects the presence of a core, while for the Burgers vortices it is satisfied automatically, since $\xi_c = 0$ when $\vartheta = 0$.

Using (3.3) and (3.4), we find from the second equation in (3.2) the law governing the motion of the center of gravity:

$$Z = z_0 \frac{A^*}{2a} \frac{\xi_0 - \xi}{\xi_0 - \xi_c} F_1\left(1, -\frac{1}{2} - i\lambda, 1 + i\lambda; 2; \frac{\xi_0 - \xi}{\xi_0}, \frac{\xi_0 - \xi}{\xi_0 - \xi_c}\right) + Z_0, \qquad (3.5)$$

where F_1 is the Appel hypergeometric function of two variables, $Z_0 = Z|_{t=0}$, and $z_0 = \xi_0^{1/2} e^{i\theta_0}$. The law governing the absolute motion can be obtained from the obvious relations

$$z_1 = Z + (\mu_2/\mu)^* z, \quad z_2 = Z - (\mu_1/\mu)^* z. \qquad (3.6)$$

According to (3.3)–(3.6), the dissipative Sullivan vortices move along spirals and come closer together in the course of the evolution, asymptotically approaching, as $t \to \infty$, the regime

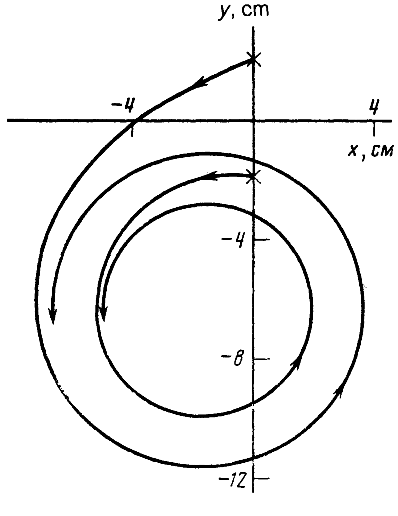

FIG. 3. Evolution of two dissipative vortices. The initial positions of the vortices are marked by sign x. The trajectories were computed with the parameters: $\vartheta = 10^{-2}$ cm^2/sec; $\gamma_1 = 2\times10^{-2}$ sec^{-1}; $\gamma_2 = 6\times10^{-2}$ sec^{-1}, $\varkappa_1/2\pi = -0.6$ cm^2/sec; and $\varkappa_2/2\pi = 0.9$ cm^2/sec.

$$z_1 = Z_c + (\mu_2/\mu)^* \xi_c^{1/2} \exp\{i(\theta_c + 2\lambda at)\},$$

$$z_2 = Z_c - (\mu_1/\mu)^* \xi_c^{1/2} \exp\{i(\theta_c + 2\lambda at)\},$$

$$\theta_c = \theta_0 + \lambda \ln(\xi_c/\xi_0), \qquad (3.7)$$

which is revolution with angular velocity $2a\lambda$ along concentric circles of radii $\xi_c^{1/2}|\mu_{1,2}/\mu|$ about a stationary center Z_c:

$$Z_c = Z|_{\xi=\xi_c} = Z_0 + z_0 \frac{A^*}{2a\lambda} F\left(1, -\frac{1}{2} - i\lambda, 1 + i\lambda; \frac{\xi_0 - \xi_c}{\xi_0}\right). \qquad (3.8)$$

Here F is the Gauss hypergeometric function. Typical behavior of the trajectories is depicted in Fig. 3.

As follows from (3.8), for $\lambda \neq 0$ the displacement of the center of gravity is finite, while for $\lambda = 0$, which corresponds to a vortex pair ($\varkappa_1 = -\varkappa_2$), the displacement is infinite. In the latter case (3.5) can be expressed in terms of elementary functions:

$$Z = Z_0 + z_0 \frac{A^*}{a}\left[1 - \left(\frac{\xi}{\xi_0}\right)^{1/2} + \frac{1}{2}\left(\frac{\xi}{\xi_0}\right)^{1/2} \times \ln \frac{(1 + (\xi/\xi_c)^{1/2})(1 - (\xi_0/\xi_c)^{1/2})}{(1 + (\xi_0/\xi_c)^{1/2})(1 - (\xi/\xi_c)^{1/2})}\right], \qquad (3.9)$$

where $\xi(t)$ is given by the formula (3.3).

The corresponding solutions for the Burgers vortices can be obtained from the solutions (3.3) and (3.4) through passage to the $\vartheta \to 0$ limit. The result will be

$$\xi = \xi_0 e^{-2at}, \quad \theta = \theta_0 + \frac{\mathrm{Im}\,\mu}{2a\xi_0}(1 - e^{2at}), \qquad (3.10)$$

$$Z = Z_0 + \frac{z_0 A^*}{2a} \beta^{1/2} e^{i(\beta + \pi/4)}\left[\Gamma\left(-\frac{1}{2}, i\beta\right) - \Gamma\left(-\frac{1}{2}, i\beta\frac{\xi_0}{\xi}\right)\right]. \qquad (3.11)$$

Here $\beta = \mathrm{Im}\,\mu/2a\xi_0$ and $\Gamma(\alpha,x)$ is the incomplete gamma function. The resulting displacement of the center of gravity will be given by the expression (3.11) without the second term in the square brackets (it vanishes for $t \to \infty$).

4. VORTEX LATTICES

As has already been noted in the Introduction, vortices can order themselves in lattices. In the case of a finite simple lattice having a symmetric shape, and composed of identical vortices, with $\mu_m = \mu = 6\vartheta - i\varkappa/2\pi$, the equations of motion (2.17) reduce to form

$$\dot{z}_{mn}^* + a z_{mn}^* = \mu \sum_{k=-M}^{M} \sum_{l=-N}^{N} (z_{mn} - z_{kl})^{-1}, \tag{4.1}$$

$$z_{mn} = 2m\omega_1 + 2n\omega_2, \quad \mathrm{Im}(\omega_2/\omega_1) > 0. \tag{4.2}$$

Here $2\omega_1$ and $2\omega_2$ are the lattice constants and M and $N(k)$ are fixed by the lattice shape and dimensions. In essence, (4.2) constitutes a requirement that the lattice maintain its crystal properties in the course of the evolution. This assumption is justified for the internal points of vortex crystals that are sufficiently large, so that the effect of the boundary deformations[20] on these points can be neglected.

For such an internal point z_{mn}, i.e., one sufficiently far from the boundary, we have the asymptotic relation

$$\sum_{k=-M}^{M} \sum_{l=-N}^{N} (z_{mn} - z_{kl})^{-1} = \int_{-M}^{M} dy \int_{-N(y)}^{N(y)} (z_{mn} - 2\omega_1 x - 2\omega_2 y)^{-1}\, d\mathrm{x}. \tag{4.3}$$

This relation can be rigorously proved if take account of the mutual concellation of the contributions from vortices symmetrically located with respect to the point z_{mn}. As a result, the dominant contribution to the sum is made by the remaining vortices located far from z_{mn}, and in the limit of an infinite lattice (4.3) is an identity.

Let us make in the integral (4.3) the change of variables

$$x' = 2\,\mathrm{Re}(\omega_1 x + \omega_2 y), \quad y' = 2\,\mathrm{Im}(\omega_1 x + \omega_2 y),$$

which corresponds to a transition from an oblique-angled (x,y) coordinate system to a Cartesian (x',y') system. We obtain for the right member of (4.3) the expression

$$\frac{1}{4\,\mathrm{Im}(\omega_1^* \omega_2)} \int_G \frac{dx'\,dy'}{z_{mn} - x' - iy'}. \tag{4.4}$$

The domain G of the integration is determined by the shape of the vortex crystal. It is well known that, when G is a circle,

$$\int_G \frac{dx'\,dy'}{z_{mn} - x' - iy'} = \pi z_{mn}^*. \tag{4.5}$$

Using (4.2)–(4.5), we find that (4.1) is satisfied for any m and n if

$$\dot{\omega}_i^* + a\omega_i^* = \mu\pi\omega_i^*/4\,\mathrm{Im}(\omega_1^*\omega_2), \quad i = 1,\ 2. \tag{4.6}$$

Equations (4.6) conserve the quantity $\tau = \omega_2/\omega_1$, which characterizes the shape of the unit cell, and, when this circumstance is taken into account, they reduce to the simpler form

$$\dot{\omega}_1^* = -a\omega_1^* + \tilde{\mu}/\omega_1, \quad \omega_2 = \tau\omega_1, \tag{4.7}$$

where $\tilde{\mu} = \pi\mu/4\,\mathrm{Im}(\tau)$. Since (4.7) coincides with the first equation in (3.2), the evolution of the lattice has the same character as the relative motion of two vortices.

Consequently, Burgers-vortex lattices collapse according to the law (3.10). Only Sullivan-vortex lattices go over asymptotically, as $t \to \infty$, into a stationary regime of rigid-body rotation with angular velocity Ω. The unit-cell area $4\,\mathrm{Im}(\omega_1^*\omega_2)$ then assumes the limiting value S. It follows directly from (4.6) that, in the stationary regime,

$$a - i\Omega = \pi\mu/S, \tag{4.8}$$

from which we obtain

$$\Omega = \varkappa/2S, \quad S = 6\pi\vartheta/a. \tag{4.9}$$

The first expression in the Feynman formula,[21] which relates the angular velocity with the vortex density $g = 1/S$, while the second indicates the absence of mass flow. Indeed, the specific mass flow rate, i.e., the flow rate per unit cell is given by the quantity

$$J = 2aS + \int d\,d\mathrm{x},$$

where, in accordance with (1.9), the first term takes account of the contribution to the total flow rate from the uniform-divergence component of the velocity field, while the second takes account of the contribution from the local velocity-field component. Evaluating the integral, we obtain

$$J = 2aS - 12\pi\vartheta,$$

from which the condition $J = 0$ for zero flow rate furnishes the second expression in (4.9).

Like vortex lattices in an ideal liquid, stationary noncirculating lattices constructed from identical dissipative vortices are inequivalent in the sense that their specific energy characteristics depend on the type of lattice.[22] This circumstance allows us to solve the problem of the choice of the lattice on the basis of energy principles. Furthermore, it turns out that, in the stationary-rotation regime, the flows induced by ideal and dissipative lattices with the same lattice constant are similar in terms of the complex velocity. And if the vortex intensity in an ideal lattice is, without loss of generality, assumed to be equal to 2π, then the similarity parameter is equal to μ.

Further, let us limit ourselves to the consideration of infinite lattices. According to Sec. 2, the system of equations (4.1) describes potential flow associated with the complex velocity

$$v^*(z) = -az^* + \mu \sum_{m,n} (z - z_{mn})^{-1}. \tag{4.10}$$

We can, by using the representation

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left[\frac{1}{z - z_{mn}} + \frac{1}{z_{mn}} + \frac{z}{z_{mn}^2} \right]$$

for the Weierstrass function,[23] carry out the summation on the right-hand side of (4.10) explicitly. We obtain

$$v^* = -az^* + \mu[\zeta(z) + \alpha z], \tag{4.11}$$

where $\alpha = -\Sigma' z_{mn}^{-2}$, and is determined only by the type of lattice. Notice that $\alpha = 0$ only for a square ($\tau = i$) or a triangular ($\tau = e^{i\pi/3}$) lattice.

Let us consider the regime of rigid-body rotation. In a reference system rotating with angular velocity Ω we obtain for the velocity v_r after taking account of the formula (4.8) the expression

$$v_r^* = v^* + i\Omega z^* = \mu[\zeta_0(z) - \pi g z^*]; \qquad (4.12)$$

here $\zeta_0(z) = \zeta(z) + \alpha z$ is a designation introduced for convenience of comparison with Ref. 22.

Comparing (4.12) with the analogous expression, (12), in Ref. 22, we easily see that the two expressions differ only because of the presence of the factor μ in (4.12). This equivalence allows us to generalize the results obtained in Refs. 22 and 24 to the case of dissipative lattices. The lattice-selection rule formulated in Ref. 22 holds here, since the condition $\alpha = 0$ is, as before, a necessary condition for the specific energy, as a function of the parameter τ, to have an extremum for a fixed unit-cell area. Of the square and triangular lattices satisfying this condition, the triangular lattice is energetically the more advantageous lattice.

To analyze its stability, let us derive the equations of motion for weak perturbations in a dissipative, noncirculating lattice. Using the similarity principle formulated above, we can suitably generalize Eq. (12) in Ref. 24, an equation which describes small oscillations in an ideal lattice in a rotating reference system. As a result we obtain

$$\dot{\mathbf{c}} = \mu^*[A^*\mathbf{c}^* - \pi g \mathbf{c}]. \qquad (4.13)$$

Here $\mathbf{c} = \{c_{mn}\}$ denotes the displacements of the vortices in the rotating reference system, and the explicit form and spectral properties of the operator A are given in Ref. 24, where it is shown that the eigenvector $c_{mn} = \exp\{i(m\varphi + n\psi)\}$ corresponds to the eigenvalue

$$B(k) = {}^1/_2[3\alpha - \zeta_0'(k) - (\zeta_0(k) - \pi g k^*)^2]. \qquad (4.14)$$

Note that $k = \pi^{-1}(\omega_2\varphi - \omega_1\psi)$ is the spectral parameter, and that the perturbation $\{c_{mn}\}$ can be considered to be a circularly polarized plane wave with wavelength iS/k.

It is easy to reduce Eq. (4.13) to the real form

$$\ddot{\mathbf{c}} + 2\pi g \operatorname{Re} \mu \dot{\mathbf{c}} + |\mu|^2[(\pi g)^2 - A^*A]\mathbf{c} = 0. \qquad (4.15)$$

Let us seek solutions in the form $c \sim e^{\delta t}$. The dispersion equation for the vibrational spectrum in a dissipative lattice is

$$\delta^2 + \delta 2\pi g \operatorname{Re} \mu + |\mu|^2[(\pi g)^2 - |B|^2] = 0, \qquad (4.16)$$

whence

$$\delta = -\pi g \operatorname{Re} \mu \pm [|B\mu|^2 - (\pi g \operatorname{Im} \mu)^2]^{1/2}. \qquad (4.17)$$

Thus, the dissipative lattice is stable if $|B| \leqslant \pi g$. And what is more, for $\pi g |\operatorname{Im} \mu/\mu| \leqslant |B| \leqslant \pi g$ a regime of pure damping develops, while for $|B| < \pi g |\operatorname{Im} \mu/\mu|$ we have a regime of damped oscillations. In the opposite case, i.e., when $|B| > \pi g$, the lattice is unstable.

A comparison with ideal lattices[24] shows that, other things being equal, any stable situation for ideal lattices will be all the more stable for dissipative lattices, and, conversely, any unstable situations turns out to be even more unstable. From this standpoint a triangular dissipative lattice (like an ideal triangular lattice) is stable. But it should be noted that a dissipative triangular lattice, unlike the ideal lattice, cannot support undamped oscillations, which disappear altogether in the long-wave approximation, since a pure damping regime arises when $0 < |k| < |k_0|$. The limit k_0 at which the oscillatory regime is cut off is given by the equation $|B(k_0)| = \pi g |\operatorname{Im} \mu/\mu|$, and depends on the ratio $\vartheta/\varkappa$. For $\vartheta/\varkappa \ll 1$ and $gk^2 \ll 1$, we find, using the asymptotic representation[24] $|B|^2 = (\pi g)^2(1 - \pi g |k|^2)$, that

$$|k_0| = 12(\pi/g)^{1/2}\vartheta/\varkappa. \qquad (4.18)$$

Furthermore, we emphasize that the norm $(\Sigma |c_{mn}|^2)^{1/2}$ of weak perturbations in a dissipative triangular lattice decreases like $e^{t \operatorname{Im} \delta}$, whereas in the ideal lattice it increases (albeit slowly) like $\ln^{1/2}(t\varkappa g/2)$.

CONCLUSION

As is well known, the presence of sources of vertical currents and angular momentum in a viscous liquid is a universal condition for the formation of intense localized vortices.[25,26] In the present paper we have shown that the ordered and disordered flow regimes in such open dissipative systems can be described within the framework of a minimal model having as its structure elements the solitary dissipative Burgers and Sullivan vortices. In such an approach the question of the formation of the sources themselves and of the specific dependences of the free parameters $\varkappa$, γ, and a of the theory on the external controlling parameters of an experiment remains open. It is clear that, to answer this question, we must construct more complicated models that will contain the minimal model as a component. Besides this, the structural element itself may be more complicated and dissipative vortices that take account of, for example, the magnetic, heat-conducting, and other properties of the medium are entirely possible.

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