# Topologically nontrivial loop defects in condensed media

I. A. Ovid'ko

*M. I. Kalinin Polytechnic Institute, Leningrad*

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A scheme is discussed for the topological classification of nonuniform states in condensed media whose degeneracy space depends on the characteristic length of the nonuniformities of the order parameter and external fields. An algorithm for calculating the topological charges of loop defects with nontrivial (continuous and containing point singularities) cores is exhibited. Loop defects in superfluid $^3$He and ferromagnets, and also the nonuniform states of a doughnut-shaped drop of a nematic liquid crystal, are investigated in the framework of the approach described.

## 1. INTRODUCTION

In recent years considerable attention has been paid to the study of nonuniform configurations of the order parameter of condensed media in the framework of the topological approach—a new theoretical method which uses the concepts of homotopic topology (see the reviews in Refs. 1–3). This approach has great value for the physics of nonuniform states of matter, since it makes it possible to display general regular features of the behavior of defects and other topological excitations in various systems differing substantially in their physical properties, e.g., in ordinary crystals, liquid crystals, superfluids, superconductors, and ferromagnets. By means of topological methods it is possible, in particular, to obtain a criterion for the stability of topological excitations, a classification of the excitations, the laws of their coalescence, and a description of the influence of topological excitations on the macroscopic properties of the condensed medium. As a result of the topological classification, excitations are divided into classes in such a way that passage from one class to another requires the creation of a discontinuity in the order-parameter field (this involves a considerable cost in energy), whereas for the transformation of excitations within one class a continuous deformation of the order parameter is sufficient (such a process usually does not involve the surmounting of an energy barrier).

In condensed media with a constant degeneracy space the topological excitations of general form are relative topological textures (RTT)—nonuniform states of a condensed medium whose order parameter in a certain part $M$ of the medium has a fixed distribution (e.g., as a result of the action of an external field).[4] Each RTT is described by a mapping

$$g : K \to V \tag{1}$$

(here $K$ is the volume occupied by the medium and $V$ is the degeneracy space of the medium), the restriction of which to $M$ is a specified continuous fixed mapping

$$g|_M = f : M \to V. \tag{2}$$

Particular cases of RTT are defects: singularities of the order parameter (in this case $M$ is the empty set), and particle-like solitons.[5] Other examples of RTT and an algorithm for calculating the topological charges of RTT by the methods of obstruction theory are considered in Ref. 4. One example of an RTT is depicted in Fig. 1a.

However, the concept of RTT, despite its considerable generality, is not adequate for the description of nonuniform states in condensed media whose degeneracy space is varying, i.e., the internal symmetries of the medium can be different at different points of the medium. This is usually associated with dependence of the degeneracy space on nonuniformities of the order parameter and with the influence of external fields. We introduce the concept of $(K, V; \tilde M, \tilde V)$ configurations, which are analogs of RTT for media with a varying degeneracy space. *Definition*: a $(K, V; \tilde M, \tilde V)$ configuration is a nonuniform state of a condensed medium occupying a volume $K$, such that in a certain part $\tilde M$ ($\tilde M \subset K$) of the medium order parameter can take values only in the subspace $\tilde V$ of the degeneracy space $V$ of the medium ($\tilde V \subset V$). This configuration is described by a continuous mapping $g: K \to V$, the restriction of which to $\tilde M$ is a continuous (nonfixed) mapping

$$g|_{\tilde M} = \tilde f : \tilde M \to \tilde V. \tag{3}$$

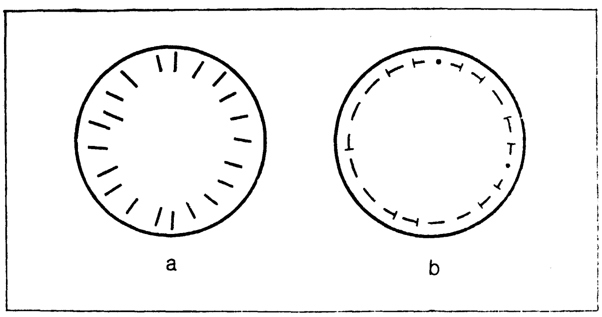


FIG. 1. Examples of nonuniform configurations in condensed media. a) Example of a relative topological texture—a nematic in a cylindrical vessel $K$. Near the side surface $M$ of the cylinder $K$ the molecules of the nematic are rigidly in the direction normal to the surface. The fixed mapping $f: M \to V$ describes the distribution (depicted in the figure) of the nematic molecules near the surface. b) An exmaple of a $(K, V; \tilde M, \tilde V)$ configuration—a nematic in a cylindrical vessel $K$. Near the side surface $\tilde M$ of the cylinder $K$ the molecules of the nematic can rotate only in the plane tangential to the surface, i.e., the region $\tilde V$ of variation of the orientation of the molecules is $S^1$—a circle. The degeneracy space in the rest of the cylinder is $V = RP_2$—the projective plane. The mapping $\tilde f: \tilde M \to \tilde V$ describing the distribution (depicted in the figure) of the nematic molecules near $\tilde M$ can be varied continuously.

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An example of such configurations is depicted in Fig. 1b. Other particular cases of $(K, V; \tilde{M}, \tilde{V})$ configurations are surface defects (here $K$ is the volume of the medium and $\tilde{M}$ is its boundary, with the singular core of the surface defect excised),[6] and defects and solitons with nontrivial cores (in this case $\tilde{M}$ is the region $K$ (the volume of the medium) with the core of the defect or soliton, respectively, excised).[7–9] Relative topological textures and $(K, V; \tilde{M}, \tilde{V})$ configurations practically exhaust all possible types of nonuniform states in condensed media.

To calculate the set $A$ of topological charges of $(K, V; \tilde{M}, \tilde{V})$ configurations we propose a general scheme, which is given in the Appendix. The purpose of the present paper is an investigation, in the framework of the scheme indicated above, one of the most interesting types of $(K, V; \tilde{M}, \tilde{V})$ configurations—loop defects with nontrivial (continuous and containing point singularities) cores.

## 2. LOOP DEFECTS WITH NONTRIVIAL CORES

We shall consider a loop defect with a continuous nontrivial core in a condensed medium with a varying degeneracy space. The core of such a defect is a filled torus (doughnut) (Fig. 2). Within the core the nonuniformities of the order parameter are large and as a consequence the degeneracy space $V$ describing the internal symmetries of the condensed medium in the core is richer than the degeneracy space $\tilde{V}$ characterizing the symmetries in the rest of the system. For definiteness we shall assume that the entire medium occupies a cubic volume $K$. Then the loop defect is a $(K, V; \tilde{M}, \tilde{V})$ configuration, where $\tilde{M}$ is the cube $K$ with the defect core excised.

A related object for the configuration under consideration is a simple loop singularity (LS) in the form of a combination of a line singularity and a point singularity. The topological charge of the LS is the set $(a,b)$, where the subcharges $a \in \pi_1(V)$ and $b \in \pi_2(V)$ characterize the properties (inherent to the LS) of the line singularity and point singularity, respectively.[2] The subcharge $a$ is specified by the distribution of the order parameter along a closed line contour $D$ surrounding the LS, and the subcharge $b$ is specified by the distribution of the order parameter over the two-dimensional closed surface (a torus with a glued middle) generated by rotation of the contour $D$ around the LS with the point $r_0 \in V$ held fixed; see Fig. 3 (for more details see Ref. 2).

In contrast to a loop singularity, in the continuous nontrivial core of a loop defect there are no singularities of the order parameter; instead, the degeneracy space $\tilde{V}$ "spreads" in the core to the large space $V$. Then, by analogy with an LS, we should expect that a loop defect is a combination of a line defect and a point defect with continuous cores, and, as a consequence, the topological charge of each loop defect, i.e., the set $(\alpha, \beta)$, is an element of the group

$$L^0(V, \tilde{V}) = \pi_2(V, \tilde{V}) \times \pi_3(V, \tilde{V}). \qquad (4)$$

Here those properties of a line defect with a continuous core that are inherent to the loop defect are characterized by the subcharge $\alpha \in \pi_2(V,\tilde{V})$, while the point-defect properties are characterized by the subcharge $\beta \in \pi_3(V,\tilde{V})$. An exact calculation (see the Appendix) confirms the above statements.

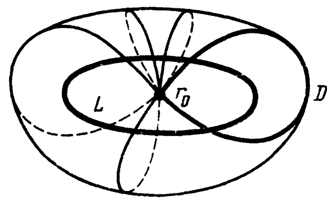


FIG. 3. Loop singularity—the thick line $L$. When the contour $D$ moves around the singularity (the point $r_0$ is stationary), the contour $D$ generates a two-dimensional closed surface—a torus with a glued middle.

The stability of a loop defect with a continuous core depends in various ways on the topological subcharges $\alpha$ and $\beta$. 1) If both subcharges $\alpha$ and $\beta$ are trivial ($\alpha = 0, \beta = 0$), the defect is topologically unstable, i.e., it can be completely eliminated (transformed into the uniform state) by means of a continuous deformation of the order parameter. 2) The subcharge $\alpha = 0$ and the subcharge $\beta$ is nontrivial. By means of continuous changes of the order parameter the defect can be broken, after which it is transformed into a point defect with a nontrivial core (characterized by $\beta$ only see Fig. 4). 3) The subcharge $\alpha$ is nontrivial and the subcharge $\beta = 0$. The loop defect is simply a line defect with a nontrivial core, folded into a ring. The defect cannot be broken, but, by decreasing the radius of the defect loop (contracting the defect), one can eliminate it completely (Fig. 5). 4) The subcharges $\alpha$ and $\beta$ are both nontrivial. The defect cannot be broken, but, by decreasing the radius of the defect loop, one can transform the defect into a point defect with a nontrivial core (Fig. 6). In this case the subcharge $\alpha$ is "lost." A similar result also obtains for ordinary loop singularities.[2]

There exists the interesting possibility of a transformation of loop defects of the type "defect with charge $(\alpha, \beta) \leftrightarrow$ defect with charge $(\alpha', \beta)$", where $\alpha' \neq \alpha$. This possibility is connected with changes of shape of the defects. For example, suppose that a loop $(\alpha, \beta)$-defect is first compressed into a point defect characterized entirely by the subcharge $\beta$ (the subcharge $\alpha$ is lost), and then transformed again into a loop

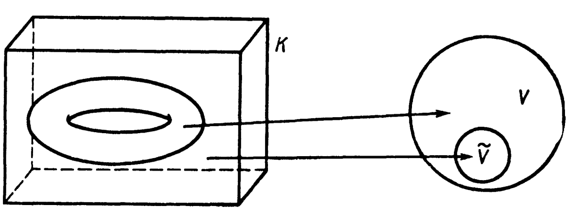


FIG. 2. Loop defect in a condensed medium occupying a cubic volume $K$; $V$ and $\tilde{V}$ are the degeneracy spaces of the medium in and outside the defect core, respectively ($\tilde{V} \subset V$).

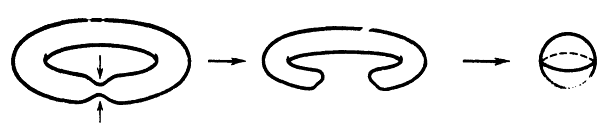

FIG. 4. The case when $\alpha = 0$ and $\beta$ is nontrivial. The loop defect can be broken and transformed into a point defect with a nontrivial core.

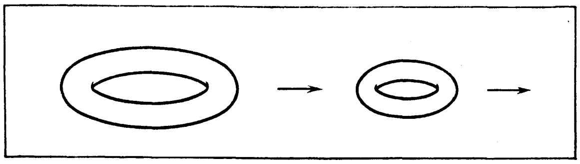

FIG. 5. The case when $\alpha$ is nontrivial and $\beta = 0$. The loop defect can be eliminated entirely by decreasing the radius of its loop.

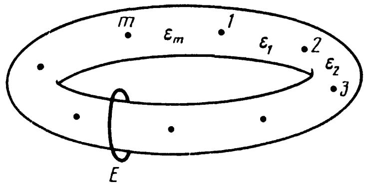


FIG. 7. Loop defect with $m$ internal singularities. The contour $E$ "encircles" the defect. Each subcharge $\varepsilon_i$ characterizes the distribution of the order parameter on the part of the core situated between the $i$th and $(i+1)$th singularities.

defect. In the latter transformation the defect again acquires a topological characteristic $\alpha'$, but the latter is arbitrary and does not have any connection with the subcharge of the initial loop defect.

In analogy with the case of ordinary line defects,[9] in the cores of loop defects point singularities can be encountered (Fig. 7). Topologically stable point singularities separate the parts of the core in which the distributions of the order parameter differ substantially (the homotopy classes of the mappings (specified by the order parameter) of sections of the parts separated by the singularities are different). Since the smallest possible number of different parts of a loop defect is equal to two, the number $m$ of topologically stable point singularities is $m \geqslant 2$. The characteristics of the "intracore" singularities are intimately related to the form of the group $\pi_2(V)$. At the same time, the loop defect also carries features of the point singularity situated in the medium with degeneracy space $\widetilde{V}$. This is reflected in the structure of the topological charge of a loop defect with $m$ intracore singularities. This charge, which (see the Appendix) is a set $(\gamma, \delta, \varepsilon_1, ..., \varepsilon_m)$, is an element of the group

$$L^m(V, \widetilde{V}) = \pi_2(\widetilde{V}) \times \mathrm{Im}(\pi_2(V, \widetilde{V}) \xrightarrow{\varphi} \pi_1(\widetilde{V})) \times [\mathrm{Ker}(\pi_2(V, \widetilde{V}) \xrightarrow{\varphi} \pi_1(\widetilde{V}))]^m. \quad (5)$$

Here $\gamma \in \pi_2(\widetilde{V})$, $\delta$ (an element of the group Im $\varphi$) is an image of the homomorphism $\varphi$, and $\varepsilon_i$ is an element of the group $(\mathrm{Ker}\,\varphi)_i$ (the kernel of the homomorphism $\varphi$), $i = 1, ..., m$; here $m \geqslant 2$, and $\varphi$ belongs to the exact sequence

$$\rightarrow \pi_r(\widetilde{V}) \rightarrow \pi_r(V) \rightarrow \pi_r(V, \widetilde{V}) \rightarrow \pi_{r-1}(\widetilde{V}) \rightarrow. \quad (6)$$

The subcharge $\gamma$ describes features of the point defect associated with $\pi_2(V)$, and the subcharge $\delta$, specified by the distribution of the order parameter along a closed contour $E$ "encircling" the defect core (Fig. 7), describes features of the line defect (in a medium with degeneracy space $\widetilde{V}$), both sets of features being inherent to the configuration investigated. The subcharge $\delta$ is contained among the elements of the group $L^0(V, \widetilde{V})$, and therefore (partially) characterizes as well a loop defect without point singularities in the core. Each subcharge $\varepsilon_i$ characterizes the distribution of the order parameter on that part of the core situated between the $i$th and $(i+1)$st point singularities (Fig. 7). In the terminology of Ref. 7, $\varepsilon_i$ is the linear-soliton index on the part from the $i$th to the $(i+1)$st singularity. The subcharges $\varepsilon_{i-1}$ and $\varepsilon_i$ together describe the $i$th point singularity associated with $\pi_2(V)$. If $\varepsilon_{i-1} = \varepsilon_i$, the $i$th singularity is topologically unstable.

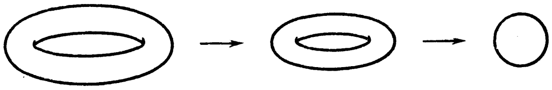

FIG. 6. The cases when $\alpha$ and $\beta$ are nontrivial. The loop defect can be compressed (by decreasing the radius of its loop) into a point defect with a nontrivial core.

A loop defect can be compressed to a point defect with a nontrivial core containing point singularities. In this case the configuration loses two subcharges: $\alpha$ and one of the $\varepsilon_i$. If $\delta = 0$ and $\varepsilon_j = 0$ for a certain value of $j$, then on the part of the core characterized by $\varepsilon_j$ the loop defect can be broken by means of a continuous deformation of the order parameter, and this also transforms the loop defect into a point defect (here the subcharges $\delta$ and $\varepsilon_j$ are lost).

### 3. EXAMPLES

*A) Ferromagnet with "easy-axis" anisotropy.* Outside the defect core the degeneracy space $\widetilde{V} = S^1$, and in the defect core we have $V = S^2$. Then

$$L^0(S^2, S^1) = Z \times Z \times Z = (Z)^3, \quad (7)$$

$$L^m(S^2, S^1) = (Z)^{m+1}, \quad (8)$$

where $Z$ is the group of the integers. As shown by (7), any loop defect with a continuous core can be characterized by a set $(z_1, z_2, z_3)$ of integers, each of which is a topological invariant, i.e., an invariant under continuous deformations of the order parameter of the ferromagnet (the magnetization vector). In this case the subcharge $\alpha = (z_1, z_2)$ and the subcharge $\beta = z_3$. Loop defects with point singularities in the core are classified, according to (8), by the sets $(z_1, ..., z_{m+1})$ of integers, in which $\delta = z_1, \varepsilon_i = z_{i+1}$, and $\gamma$ is always trivial, since the group $\pi_2(S^1)$ is trivial.

*B) Superfluid* $^3$He-A *in a magnetic field.* Outside the defect core the condition $\xi_D, \xi_M \ll \lambda$ is fulfilled, where $\lambda$ is the characteristic length of the nonuniformities of the order parameter, $\xi_D$ is the dipole length, and $\xi_M$ is the magnetic length. In the core the gradients of the order parameter are greater ($\lambda$ is smaller) and the condition $\xi_D \ll \lambda \lesssim \xi_M$ is fulfilled. Then the degeneracy space $V = SO(3)$ and $\widetilde{V} = S^1 \times S^1$ (Ref. 2). The group

$$L^0(SO(3), S^1 \times S^1) = (Z)^3. \quad (9)$$

The loop defects with a continuous core are classified by the sets $(z_1, z_2, z_3)$ of integers (in this case $\alpha = (z_1, z_2)$ and $\beta = z_3$). The group $\mathrm{Ker}[\pi_2(SO(3), S^1 \times S^1) \rightarrow \pi_2(S^1 \times S^1)]$

is trivial. Therefore, all the subcharges $\varepsilon_i = 0$, and this indicates the impossibility of the existence of topologically stable point singularities in the core of the defects described.

*C) Superfluid* $^3$He-A *in a weak magnetic field.* Outside the core the condition $\xi_D \ll \lambda \lesssim \xi_M$ is fulfilled, and in the core the condition $\lambda \lesssim \xi_D, \xi_M$ is fulfilled. The degeneracy space $V = [S^2 \times SO(3)]/Z_2$, and $\tilde{V} = SO(3)$ (Ref. 2). The group

$$L^0([S^2\times SO(3)]/Z_2,\ SO(3)) = (Z)^2. \tag{10}$$

The topological charge of a loop defect with a continuous core is the set $(z_1, z_2)$ of integers ($\alpha = z_1, \beta = z_2$). Loop defects with $m$ intracore singularities are classified by the sets $(z_1,...,z_m)$—the elements of the group

$$L^m([S^2\times SO(3)]/Z_2,\ SO(3)) = (Z)^m; \tag{11}$$

here the subcharges $\gamma$ and $\delta$ are trivial, and $\varepsilon_i = z_i$.

*D) Strongly nonuniform superfluid* $^3$He-A *in a magnetic field.* Outside the defect core the characteristic lengths satisfy the condition $\xi_M \ll \lambda \lesssim \xi_D$, and in the core they satisfy the condition $\lambda \lesssim \xi_M, \xi_D$. Then[2] we have

$$V = [S^2\times SO(3)]/Z_2, \quad \tilde{V} = [S^1\times SO(3)]/Z_2.$$

The topological charges of loop defects with continuous cores are sets $(z_1,z_2,z_3)$ that are elements of the group

$$L^0([S^2\times SO(3)]/Z_2,\ [S^1\times SO(3)]/Z_2) = (Z)^3, \tag{12}$$

with $\alpha = (z_1,z_2)$ and $\beta = z_3$. For loop defects with cores containing $m$ point singularities, the topological charges

$$(z_1, \ldots, z_{m+1}) \in L^m([S^2\times SO(3)]/Z_2,\ [S^1\times SO(3)]/Z_2) = (Z)^{m+1}; \tag{13}$$

here $\delta = z_1$, $\varepsilon_i = z_{i+1}$, and $\gamma$ is always trivial, since $\pi_2 \tilde{V} = 0$.

*E)* Finally we consider a *doughnut-shaped drop of a nematic liquid crystal*—an object similar in nature to the loop defects. The degeneracy space $V = RP_2$ of the nematic is the projective plane. On the drop boundary (a torus $S^1 \times S^1$) the degeneracy space is narrowed either to a point or to $\tilde{V} = S^1$. We shall study the latter case. The set of topological charges of the nonuniform states in the drop being described is the group (see the Appendix)

$$L^0(RP_2, S^1)\times \mathrm{Ker}(\pi_1(S^1) \xrightarrow{\chi} \pi_1(RP_2)) = (Z)^4, \tag{14}$$

where the homomorphism $\chi$ belongs to (6). If in the doughnut-shaped drop there are $m$ point singularities ($m \geqslant 2$), the nonuniform states of the drop are classified by the elements of the group

$$L^m(RP_2, S^1)\times \mathrm{Ker}(\pi_1(S^1) \xrightarrow{\chi} \pi_1(RP_2)) = (Z)^{m+2}. \tag{15}$$

## 4. CONCLUSION

In this paper we have introduced the concept of $(K, V; \tilde{M}, \tilde{V})$ configurations, which is necessary for the description of nonuniform states of general form in condensed media with a varying degeneracy space. A scheme for calculating the topological charges of the $(K, V; \tilde{M}, \tilde{V})$ configurations is proposed. It is shown that in condensed media topologically stable loop defects with nontrivial cores—one of the most interesting types of $(K, V; \tilde{M}, \tilde{V})$ configurations—can exist. Each loop defect with a continuous core is a combination of a line defect and a point defect, and is characterized, according to (4), by two topological subcharges $\alpha$ and $\beta$, where $\alpha \in \pi_2(V, \tilde{V})$ and $\beta \in \pi_3(V, \tilde{V})$. This gives rise to different stabilities of loop defects against attempts to break the defect and to compress it into a ball (a point defect with a nontrivial core; see Figs. 4–6) and, in addition, ensures the possibility of change in the subcharge $\alpha$ in the course of a transformation of the form "loop defect → point defect → loop defect". The presence of topologically stable point singularities in the core of a loop defect leads to a substantially different (in comparison with the case of defects with nontrivial cores) form of the topological charge characterizing the configuration (formula (5)).

In conclusion the author expresses his sincere gratitude to V. I. Vladimirov and A. E. Romanov for useful discussions.

## APPENDIX

1. The set $A$ of topological charges of $(K, V; \tilde{M}, \tilde{V})$ configurations can be represented in the form $A = B \times C$. Here $B$ is the set of homotopy classes of mappings $\tilde{f}: \tilde{M} \to \tilde{V}$ having the continuous extension $g: K \to V$, and $C$ is the set of homotopy classes of such extensions. To find $B$ one must use obstruction theory, the methods of which are described in, e.g., Refs. 4, 10, and 11. The procedure for calculating $C$ can be divided conveniently into two stages. First we choose the mapping $\tilde{f}$—a representative of some class from $B$. Assuming $\tilde{f}$ to be fixed, we seek by means of obstruction theory the set $C'$ of homotopy classes of the mappings $g: K \to V$ that are the extensions to $K$ of the fixed mapping $\tilde{f}$. In the second stage the condition that $\tilde{f}$ be fixed is lifted (now $\tilde{f}$ can vary within the limits of its homotopy class), and this leads, generally speaking, to the introduction of an equivalence relation between certain elements of the set $C'$. The set $C$ is the set of equivalence classes of the elements of the set $C'$.

2. We shall find the set $A = B \times C$ of topological charges of a loop defect with a continuous core (Fig. 2). The region $\tilde{M}$ is homotopically equivalent to a torus with a glued middle. Hence, the set of homotopy classes of the mappings $\tilde{M} \to \tilde{V}$ is the product $\pi_1(\tilde{V}) \times \pi_2(\tilde{V})$ (Ref. 2). Only those mapping which are homotopic to a constant mapping (mapping into a point) upon extension of the space $\tilde{V}$ to $V$ can be extended into the entire region $K$. Consequently,

$$B = \mathrm{Ker}\,(\pi_1(\tilde{V}) \to \pi_1(V)) \times \mathrm{Ker}\,(\pi_2(\tilde{V}) \to \pi_2(V)). \tag{A.1}$$

We shall find the set $C$ of homotopy classes of mappings $g: K \to V$ whose restriction $g|_{\tilde{M}}$ to $\tilde{M}$ belongs to some class from the set $B$ and can vary within the limits of this class. For this we consider first the set $C'$ of homotopy classes of the relative topological textures describable by the mappings $g: K \to V$ whose restriction $g|_{\tilde{M}}$ to $\tilde{M}$ is the fixed maping $\tilde{f}: \tilde{M} \to \tilde{V}$. By obstruction theory,

$$C' = H^2(K, \tilde{M};\ \pi_2(V)) \times H^3(K, \tilde{M};\ \pi_3(V)) = \pi_2(V) \times \pi_3(V), \tag{A.2}$$

where $H^r(K,\tilde{M};\pi_r(V))$ is the group of relative $r$-dimensional cohomologies with coefficients in the homotopy group $\pi_r(V)$. Let $C''$ be the set consisting of those elements of the group $C'$ that are homotopy classes of the mappings $h: K\to\tilde{V}$, the restriction $h|_{\tilde{M}}$ of the latter ;to $\tilde{M}$ being equal to the fixed $\tilde{f}$. The group

$$C''=H^2(K,\tilde{M};\ \mathrm{Im}(\pi_2(\tilde{V})\to\pi_2(V)))\times H^3(K,\tilde{M};\ \mathrm{Im}(\pi_3(\tilde{V})\to\pi_3(V)))=\mathrm{Im}(\pi_2(\tilde{V})\to\pi_2(V))\times\mathrm{Im}(\pi_3(\tilde{V})\to\pi_3(V)). \tag{A.3}$$

Since the space (cube) $K$ can itself be contracted to a point, mappings belonging to the classes $c''\in C''$ are not homotopic to a constant mapping only because the mapping $h|_{\tilde{M}}=\tilde{f}$ is fixed. Then replacement of the condition that the mapping $\tilde{f}$ be fixed by the condition that $f$ can vary within its class leads to the expression $C=C'/C''$ for the desired set $C$ (the set $C$ is the factor space of the group $C$ with respect to its subgroup $C''$). Taking into account the relations

$$\pi_r(V)/\mathrm{Im}(\pi_r(\tilde{V})\to\pi_r(V))=\mathrm{Ker}(\pi_r(V,\tilde{V})\to\pi_{r-1}(\tilde{V})), \tag{A.4}$$

$$\pi_r(V,\tilde{V})=\mathrm{Ker}(\pi_{r-1}(\tilde{V})\to\pi_{r-1}(V))\times\mathrm{Ker}(\pi_r(V,\tilde{V})\to\pi_{r-1}(\tilde{V})), \tag{A.5}$$

which follow from the exactness properties of the sequence (6), we obtain for $C$ the expression

$$C=\mathrm{Ker}(\pi_2(V,\tilde{V})\to\pi_1(\tilde{V}))\times\mathrm{Ker}(\pi_3(V,\tilde{V})\to\pi_2(\tilde{V})), \tag{A.6}$$

and for the set $A=B\times C$, the expression (4).

3. We shall find the set $A_m=B_m\times C_m$ of the topological charges of loop defects with $m$ intracore singularities (Fig. 7). The region $\tilde{M}$ is the same as in the preceding case. The region $K$ is a cube with $m$ internal points excised. The set of homotopy classes of the mappings $\tilde{M}\to\tilde{V}$ is $\pi_1(\tilde{V})\times\pi_2(\tilde{V})$. Upon extension of $\tilde{V}$ to $V$ the only mappings that cannot be extended into the region $K$ are those belonging to the homotopy classes characterized by the elements of $\mathrm{Im}(\pi_1(\tilde{V})\to\pi_1(V))$. Therefore,

$$B_m=\mathrm{Ker}(\pi_1(\tilde{V})\to\pi_1(V))\times\pi_2(\tilde{V})=\mathrm{Im}(\pi_2(V,\tilde{V})\to\pi_1(\tilde{V}))\times\pi_2(\tilde{V}). \tag{A.7}$$

We shall determine the sets $C'_m$ and $C''_m$ in the same way that we determined the sets $C'$ and $C''$ in the preceding case. According to obstruction theory,

$$C_m'=H^2(K,\tilde{M};\ \pi_2(V))\times H^3(K,\tilde{M};\ \pi_3(V))=(\pi_2(V))^m, \tag{A.8}$$

$$C_m''=(\mathrm{Im}(\pi_2(\tilde{V})\to\pi_2(V)))^m. \tag{A.9}$$

Then, taking (A.4) into account, we have

$$C_m=C_m'/C_m''=(\mathrm{Ker}(\pi_2(V,\tilde{V})\to\pi_1(\tilde{V})))^m. \tag{A.10}$$

From (A.5), (A.7), and (A.10) we obtain for $A_m=B_m\times C_m$ the formula (5).

4. The procedure for calculating the set $A^{do}=B^{do}\times C$ ($A^{do}=B^{do}\times C_m$) of topological charges of the nonuniform continuous (resp., with $m$ point singularities) states in a doughnut-shaped drop of a nematic differs from the procedure 2 (resp., 3) of this Appendix only in the analysis of the mappings $\tilde{M}\to\tilde{V}$. For a doughnut-shaped drop its boundary $\tilde{M}$ is a torus $S^1\times S^1$. The set of classes of the mappings $S^1\times S^1\to\tilde{V}=S^1$ is equal to the direct product $\pi_1(S^1)\times\pi_1(S^1)$ of homotopy groups, the elements of each of which are classes of mappings of the generating circles of the torus into $\tilde{V}=S^1$. Separating out from the given set only the classes of mappings that can be extended to the entire drop, we obtain the expressions

$$B^{do}=(\mathrm{Ker}(\pi_1(S^1)\to\pi_1(RP_2))^2)=B\times\mathrm{Ker}(\pi_1(S^1)\to\pi_1(RP_2)), \tag{A.11}$$

$$B^{do}=(\mathrm{Ker}(\pi_1(S^1)\to\pi_1(RP_2))^2)=B_m\times\mathrm{Ker}(\pi_1(S^1)\to\pi_1(RP_2)), \tag{A.12}$$

from which follow formulas (14) and (15).


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