Nonlinear interaction of pulsed and noise signals in nondispersive media

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The evolution of a perturbation that constitutes at the initial instant of time a sum of a pulsed regular signal and stationary noise is investigated on the basis of the Burgers nonlinear equation. Notice is taken of the effect of slowing down of the leading front of the pulse, of its spreading by turbulent viscosity, of the abrupt increase of the noise dispersion in the discontinuity region, and the decrease of the noise on the trailing edge of the pulse. Results of a numerical simulation are presented.

1. INTRODUCTION

The interaction between regular and noise signals in nonlinear nondispersive media is of great theoretical and applied interest. In acoustics, for example, many sources of intense noise, such as explosions, cavitation, or electric discharges, have both a regular and a noise component.[1] The ensuing phenomena are quite varied and depend strongly on the spatial scales and on the amplitudes of the interacting fields.

Propagation of finite-amplitude waves in nondispersive media is described by the known Burgers equation[2]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}. \qquad (1)$$

In terms of the reduced variables in Eq. (1), $u(x,t)$ is the vibrational velocity, (x,t) are the coordinate and the time, and μ is the coefficient of high-frequency viscosity. The Burgers equation is used as the model equation for hydrodynamic turbulence, and also describes the propagation of electromagnetic-wave in ferrites and in distributed rf lines, of magnetosonic waves in a plasma, and of intense acoustic waves in liquids and in gases.[1–4]

The problem of nonlinear interaction of noise with regular signals reduces to finding the statistical characteristics of the solution of Eq. (1) with the initial condition

$$u(x,\ t=0)=u_0(x)=\xi(x)+u_c(x), \qquad (2)$$

which constitutes a superposition of the noise field $\xi(x)$ and the regular field $u_c(x)$. In contrast to the linear problem of wave scattering by specified inhomogeneities, an important role is played here by the self-action of the components and by the inverse influence on the inhomogeneities. The nonlinearity of the medium leads to intense self action of the components of the initial perturbation and to their nonlinear distortion, up to formation of discontinuities. As a result, the properties of the waves change in the course of propagation, and this leads to different interaction mechanisms during the different stages.

The interaction of periodic perturbations with stationary noise in a linear nondispersive medium were heretofore investigated in the initial propagation stage, while there were still no discontinuities and the field could be adequately described within the framework of the Riemann equation,[5,6] during the intermediate stage when discontinuities were already present but did not yet coalesce,[7] during the stage of advanced discontinuities,[8] and finally in the concluding stage, when the wave propagation became linear as a result of energy absorption.[9]

We consider in the present paper the evolution of a mixed perturbation comprising a sum of a regular pulsed signal and stationary acoustic noise. It is known that the evolution of a pulsed signal differs fundamentally in many respects from that of a periodic one,[1,2] and therefore the interaction between a pulsed signal and noise also proceeds qualitatively differently from that between periodic and noise fields. This problem is timely also in view of the extensive use of high-power pulsed acoustic waves generated by detonation of underwater explosives.[10] Besides the analytic treatment, we present here the results of numerical simulation of the propagation of a mixed perturbation in a nonlinear medium during the stage of intense coalescence of shock fronts. Comparison of the data of a numerical experiment with the results of the theory has shown that the asymptotic theory developed in the present paper describes quite satisfactorily the nonlinear interaction between noise and regular signals in nonlinear nondispersive media.

2. INITIAL EQUATIONS AND QUALITATIVE PICTURE OF THE INTERACTION BETWEEN A PULSED REGULAR SIGNAL AND NOISE

Equation (1) can be reduced to a nonlinear Hopf-Cole transformation[11,12] to a linear diffusion equation, so that a solution of the initial equation can be obtained in quadratures:

$$u(x,t)=\int_{-\infty}^{+\infty}(x-y)\exp\left\{\frac{1}{2\mu}G(y;x,t)\right\}dy$$

$$\times\left\{t\int_{-\infty}^{+\infty}\exp\left\{\frac{1}{2\mu}G(y;x,t)\right\}dy\right\}^{-1}, \qquad (3)$$

$$(4)$$

$$G(y;x,t)=G_0(y)-(x-y)^2/2t,\qquad G_0(y)=-\int^{y}u_0(x)\,dx.$$

It follows from (1) that the relative influence of the nonlin-

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ear and dissipative effects in wave propagation is characterized by the acoustic Reynolds number $\mathrm{Re} = \sigma l/\mu$, where σ and l are respectively the characteristic amplitude of the characteristic spatial scale of the initial perturbation. We consider the case of infinite Reynolds numbers, when the initial field is subject to strong nonlinear distortions. As $\mu \to 0$ a contribution to the integrals in (3) is made only by a small vicinity of the point y, in which the function $G(y;x,t)$ has an absolute maximum. This enables us to write the solution of the Burgers equation in the form[11,13]

$$u(x,\ t)=[x-y(x,\ t)]/t, \tag{5}$$

where $y(x,t)$ is the coordinate of the absolute maximum of $G(y;x,t)$.

If a pure noise perturbation $\xi(x)$ is specified at the initial instant, with a characteristic amplitude $\sigma(\sigma^2 = \langle \xi^2 \rangle 0$ and with a spatial scale l, the function $y(x,t)$ in (5) is at $t > t_{\mathrm{in}} = l/\sigma$ a discontinuous piecewise-constant function of x with jumps at the discontinuity points $x = x_k$, and the continuous field $u(x,t)$ is transformed into a sequence of sawtooth pulses with equal slopes $u'_x = 1/t$ and with discontinuities at $x = x_k$. The coordinates of the discontinuities in a separate realization of the noise field are obtained from the condition $G(y_k;x_k,t) = G(y_{k+1};x_k,t)$ (Ref. 13):

$$x_k=V_kt+(y_k+y_{k+1})/2, \tag{6}$$

$$V_k=\frac{1}{\eta_k}\int\limits_{y_k}^{y_{k+1}} u_0(x)\,dx; \qquad \eta_k=y_{k+1}-y_k, \tag{7}$$

where η_k is the distance between the zeros of the field and V_k is the velocity of the kth discontinuity.

Owing to the random character of the initial field, the discontinuities move randomly and coalesce, thereby increasing the field's external scale $l(t)$, equal to the characteristic distance between the zeros. Note that it is convenient to describe the coalescence of the discontinuities by using the analogy with a gas of noninteracting randomly moving inelastically colliding particles,[13] with the particle velocity v_k and its mass m_k expressed in terms of the discontinuity velocity V_k and amplitude ΔV_k in the form

$$v_k\equiv V_k,\quad m_k=\Delta V_kt=y_{k+1}-y_k=\mathrm{const}.$$

The coalescence of the discontinuities turns out to be equivalent to absolutely inelastic particle collisions in which the mass and momentum conservation laws are satisfied. The particle velocity, and hence also the rate of their coalescence, is determined by the behavior of the structure function $G_0(y)$ (Ref. 4). We consider the case when the fluctuations of $G_0(y)$ are bounded:

$$\langle G_0^2\rangle=\sigma_*^2=\sigma^2l^2<\mathrm{const}.$$

The characteristic distance $l(t)$ between the particles, and hence also the particle mass $m_k \approx l(t)$, the characteristic velocity $v_k \approx V_k$, and the momentum $p = m_k v_k$ vary with time as

$$m(t)\approx l(t)\approx(\sigma_*t)^{1/2},\quad v\approx(\sigma_*/t)^{1/2},\quad p\approx\sigma_*. \tag{8}$$

We consider now the evolution of a regular pulsed perturbation in a nonlinear medium, assuming that $u_{\mathrm{m}}(x)$ differs from zero over a finite interval $x\in[0,X_{\mathrm{m}}]$, and

$$\int\limits_{-\infty}^{+\infty} u_{\mathrm{m}}(x)\,dx=P>0.$$

If A_{m} is the characteristic amplitude of the pulse, then any finite perturbation acquires at $t > t_{\mathrm{m}} = X_{\mathrm{m}}/A_{\mathrm{m}}$ the universal triangular form[2]

$$u_{\mathrm{m}}(x,t)=\begin{cases} x/t, & 0<x<X_{\mathrm{r}}, \\ 0, & x<0,\quad x>X_{\mathrm{r}}, \end{cases} \tag{9}$$

$$X_{\mathrm{r}}=X_{\mathrm{r}}(t)=(2Pt)^{1/2},\quad V_{\mathrm{r}}=dX_{\mathrm{r}}/dt=(P/2t)^{1/2}, \tag{10}$$

i.e., the information on the fine structure of the initial perturbation is completely forgotten in the medium. The motion of the discontinuity can then be interpreted as the motion of a regular discrete heavy particle in an immobile substance having a continuous density $\rho = 1$ (Ref. 4). As it moves, the heavy particle gathers matter from the interval $x\in[0,X_{\mathrm{r}}]$, its mass $M_{\mathrm{r}} = X_{\mathrm{r}}$ increases, and the law (19) of its motion is a consequence of the momentum conservation law $X_{\mathrm{r}} V_{\mathrm{r}} = P$.

We discuss now qualitatively the interaction of the regular and noise perturbations over times $t > t_{\mathrm{in}}, t_{\mathrm{m}}$, when these fields have already acquired the universal sawtooth form. It follows from the foregoing analysis that the laws governing the growth of the scale X_{r} of the regular perturbation (10) and of the characteristic scale of the noise field (8) have the same time dependence, so that the character of the evolution of the mixed perturbation is determined by the relation between the total pulse P of the regular perturbation and the characteristic noise-field $p \approx \sigma$. At $p \gg P$, when $l(t) \gg X_{\mathrm{r}}(t)$, the characteristic noise amplitude $\Delta U_{\mathrm{n}} \sim l(t)t$ is much larger than the regular-perturbation amplitude $\Delta U_{\mathrm{r}} = X_{\mathrm{r}} t$, so that the regular perturbation is practically suppressed by the noise perturbation in this case.

If the inverse relation $P \gg p (P \gg \sigma_*)$ holds, the problem of the interaction of the signal with the noise is equivalent to the problem of motion of a heavy particle with initial momentum P in a gas of randomly moving light particles whose average velocity is zero, while the characteristic mass m, the velocity v, and the momentum p vary with time in accordance with (8). In each inelastic collision event, the momentum of the heavy particle, its velocity, and its mass vary insignificantly, in virtue of the condition $p \ll P$. Indeed, the mass M_{r} of the heavy particle is approximately equal to X_{r} and it increases in small discrete steps $\Delta \mathrm{M}_{\mathrm{r}} \approx \mathrm{m}(t) \approx \mathrm{l}(t) \ll \mathrm{X}_{\mathrm{r}} = \mathrm{M}_{\mathrm{r}}$. It follows from the momentum conservation law that as a result of one collision the particle velocity also changes by a small amount $\Delta V_{\mathrm{r}} m v_k / M_{\mathrm{r}} \ll v_k \ll V_{\mathrm{r}}$. The coordinate of the heavy particle is therefore described approximately, as before, by expression (10), but owing to the collisions with randomly moving light particles the coordinate will fluctuate about its mean value.

Thus, if the condition $P \gg \sigma$ is met, the regular pulsed perturbation will retain on the average its coherent structure (9), and the main effect of the noise will be manifested by a "turbulent" broadening of the shock front. We obtain below asymptotic expressions for the probabilistic distribution of the field of the mixed perturbation $u(x,t)$ and for its mean value $\langle u(x,t)\rangle$ and variance $\sigma_u^2(x,t)$, and discuss a number of more subtle effects that arise when a regular pulsed perturbation interacts with stationary noise in a nondispersive nonlinear medium.

3. ONE-POINT PROBABILITY DENSITY AND MOMENTS OF THE VELOCITY FIELD

We proceed to an asymptotic analysis of the evolution of the statistical characteristics of the mixed perturbation $u(x,t)$ during the stage of interactions of fully developed discontinuities. It is shown in Refs. 8, 13, and 14 that the approximate behavior of the nonlinear wave during this stage is determined by the form of the energy spectrum $S(k)$ of the initial field at low wave numbers. In our formulation of the problem, $\xi(x)$ is a Gaussian stationary noise signal whose spectrum at zero is $S(0) = 0$, so that the action $\tilde{G}_0(y)$

$$\left(\tilde{G}_0(y) = -\int^{y}\xi(x)\,dx\right)$$

is a statistically homogeneous Gaussian process with variance

$$\langle \tilde{G}_0^2\rangle = \sigma_*^2 = \sigma^2 l^2 = \int_{-\infty}^{+\infty} dk S(k)/k^2. \tag{11}$$

At $t \gg t_{\rm in} = 1/\sigma$ the parabola contained in the functional $G(y;x,t)$ (4) is a smooth function of y compared with the initial action $G_0(y)$. The absolute maximum of $G(y;x,t)$ is then chosen from among a large number of local maxima $\tilde{G}_0(y)$ of a noise field, and the statistical characteristics of the mixed perturbation can be analyzed by the asymptotic theory of overshoots of random processes.[14,15]

We confine ourselves next to an asymptotic analysis of the statistical characteristics over times $t \gg t_{\rm m}, t_{\rm in}$, at which, as shown above, a regular pulsed perturbation is transformed into a universal triangular pulse with slope $u_x' = 1/t$ and with a discontinuity at the point $X_{\rm r} = (2Pt)^{1/2}$, and the information about its fine structure is lost. If $X_{\rm m} < l$, i.e., the initial spatial scale of the regular pulse is less than the characteristic scale of the noise, one can replace, without loss of generality, the smooth increase of the regular component of the initial action $G_0(y)$ from 0 to P in the region $y \in [0, X_{\rm m}]$ by an action discontinuity of amplitude P at the point $y = 0$. The total number of times that the level H is crossed by $G(y;x,t)$, needed to determine the one-point probability of the mixed perturbation,[8,14] can then be represented by a sum of two terms

$$N_\infty(H) = N(H;\ [-\infty,\ \infty]) = N_1(H;\ [-\infty,\ 0]) + N_2(H;\ [0,\ \infty]), \tag{12}$$

which take the form, since the noise field is Gaussian,[15]

$$N_1(H;[-\infty,0]) = \frac{\lambda}{2\pi}\int_{-\infty}^{0}\exp\left\{-\frac{[H+(x-y)^2/2t]^2}{2\sigma_*^2}\right\}dy, \tag{13}$$

$$N_2(H;[0,\infty]) = \frac{\lambda}{2\pi}\int_0^\infty \exp\{-[H+PE(y)+(x-y)^2/2t]^2/2\sigma_*^2\}dy, \tag{14}$$

where $E(y)$ is the unit step function and $\lambda = 1/l$.

Since the mixed perturbation is inhomogeneous and contains therefore a regular pulse localized in the region $x \in [0, X_{\rm r}(t)]$, it is convenient to analyze the statistical characteristics by subdividing the entire interval x into a number of subregions. We consider initially the evolution of the one-point probability distribution in the region $-\infty < x < X_{\rm r}(t) = (2Pt)^{1/2}$. If the condition $P \gg \sigma_*$ is met, it follows from (13) and (14) that $N_1 \gg N_2$ and $N_\infty(H) \approx N_1(H;[-\infty,0]$ in this interval and one can obtain for the one-point probability distribution of a mixed perturbation, using the procedure described in detail in Refs. 8 and 14, the expression

$$W(u;x,t) = \begin{cases} \dfrac{\exp[-u^2/2b^2(t)]}{b(t)\Phi(-x/l(t))}, & u \geq \dfrac{x}{t}, \\ 0, & u < x/t, \end{cases}$$

$$b^2(t) = l^2(t)/t^2 = \sigma_*^2/H_0 t, \qquad l^2(t) = \sigma_*^2 t/H_0, \tag{15}$$

$$\Phi(x) = \int_{-\infty}^{x}\exp(-t^2/2)\,dt,$$

where $l(t)$ is an asymptotic expression for the characteristic external scale of the noise field during the stage of coalescence of developed discontinuities, while H_0 is the solution of the transcendental equation

$$\frac{\lambda}{2\pi}\left(\frac{\sigma_*^2 t}{H_0}\right)^{1/2}\exp[-H_0^2/2\sigma_*^2] = 1, \qquad H_0 \approx \sigma_*(\ln\gamma(t))^{1/2}, \quad \gamma(t) = \sigma\lambda t/(2\pi)^2. \tag{16}$$

Using (15), we obtain for the mean value $\langle u(x,t)\rangle$ and for the variance $\sigma_u^2(x,t)$ of the mixed perturbation

$$\langle u(x,t)\rangle = \frac{b(t)}{\Phi(-x/l(t))}\exp\left[-\frac{x^2}{2l^2(t)}\right], \tag{17}$$

$$\sigma_u^2(x,t) = b^2(t)\left[1 + \frac{x}{l(t)}\frac{\exp[-x^2/2l^2(t)]}{\Phi(-x/l(t))} - \frac{\exp[-x^2/l^2(t)]}{\Phi^2(-x/l(t))}\right]. \tag{18}$$

The mutual influence of the regular pulse and the noise as they interact nonlinearly leads thus to a transformation of the one-point probability distribution of the mixed perturbation $u(x,t)$. In the region where the pulse hardly affects the statistical characteristics of the noise, $x \ll -l(t)$, it follows from (15) that the probability distribution $u(x,t)$ is Gaussian with zero mean value and with variance $\sigma_u^2 = b^2(t)$, the same as obtained in Ref. 14, where the one-point probability distribution of stationary noise was obtained for developed discontinuities. With increase of x, when the presence of the regular pulse begins to manifest itself, the one-point distribution is transformed into a truncated Gaussian distribution.

This lowers the variance of the mixed perturbation and leads to a nonzero mean value in the region $x<0$.

One must single out particularly the region of the trailing gently sloping edge of the regular pulse, $l(t)<x \ll (2Pt)^{1/2}$. Using the asymptotic representation for the function $\Phi[-x/l(t)]$ with large values of the argument, the expression for the probability distribution of the absolute-maximum coordinate can be reduced to the form

$$w(y;x,t)=\begin{cases} xl^{-2}(t)\exp[xy/l^2(t)], & y\leqslant 0,\\ 0, & y>0.\end{cases} \qquad (19)$$

At $x/l(t)\gg 1$ the distribution $w(y;x,t)$ becomes exponential and localized in a narrow region $\Delta y\sim l^2(t)/x$ on the y axis. Taking (5) into account, we obtain from (19) for the one-point distribution of the mixed perturbation $u(x,t)$

$$W(u;x,t)=\begin{cases} \dfrac{xt}{l^2(t)}\exp\left[\dfrac{x^2}{l^2(t)}\right]\exp\left[-\dfrac{uxt}{l^2(t)}\right], & u\geqslant \dfrac{x}{t}\\ 0, & u<x/t\end{cases}, \qquad (20)$$

and for the mean value and the variance we have the following asymptotic expressions:

$$\langle u(x,t)\rangle=\frac{x}{t}\left[1+\left(\frac{l(t)}{x}\right)^2\right], \qquad (21)$$

$$\sigma_u^2(x,t)=\frac{b^2(t)}{(x/l(t))^2}. \qquad (22)$$

Consequently, the variance of the noise on the trailing edge of the regular pulse decreases quadratically with increasing x, $\sigma_u^2\sim x^{-2}$ while the mean value approaches asymptotically the universal form (9) of the regular pulse.

The dependence of the statistical characteristics of the mixed perturbation on x in the region $-\infty<x<(2Pt)^{1/2}$ can be qualitatively explained by using the analogy, described above, with quasiparticles. Let a heavy particle (discontinuity of the regular signal) having a momentum $P>0$ be located at the initial instant $t=0$ in a point with coordinate $x=0$ in a gas of randomly moving noninteracting light particles (discontinuities of the noise field, and let the velocity of the particle be directed along the positive x axis. This regular perturbation introduces a substantial asymmetry in the one-dimensional problem. When the heavy particle is displaced (the discontinuity moves) to the right along the x axis, it drags along all the light particles from the interval $x\in[0,X_r(t)]$ with which it collided inelastically. At the same time, the only light particles that can penetrate into the region $x\in[0,X_r(t)]$ are those located at the instant $t=0$ in the region $x<0$ and having a momentum $p>0$, i.e., an initial velocity directed along the x axis. The asymmetry introduced by the motion of the heavy particles upsets the dynamic equilibrium of the light particles. Thus, only particles with momentum $p>0$ are located in the region $x\in[0,X_r(t)]$ at the instant $t>0$, and particles with momentum $p<0$ flow continuously out of the region $-l(t)<x<0$. This loss of particles is not compensated for, since the heavy particle acts as an inpenetrable "barrier" for light particles with negative momentum and located at $x>0$ at the instant $t=0$. The heavy particle absorbs in the course of its motion the possible competitors for replacement of the produced vacancies (light particles with $p<0$) and its mass is accordingly increased. It can be concluded from this qualitative consideration that violation of the dynamic equilibrium of the light particles leads to an excess of particles having a positive momentum in the region $-l(t)<x<(2Pt)^{1/2}$. This leads to the appearance of a nonzero mean field and to a decrease of the noise variance within this region.

Consider now the statistical characteristics of the mixed perturbation $u(x,t)$ at $x\sim(2Pt)^{1/2}$, i.e., in the region where the shock front of the regular pulse is located. At $x\sim(2Pt)^{1/2}$ the terms in (12) become of the same order, and a contribution to the total number of crossings of the level H is made by the local maxima of the functional $G(y;x,t)$, which are located in narrow regions L_1 and L_2 on the negative and positive y axes ($y(L_1\in(-l^2(t)/x,0)$, $L_2\in((2Pt)^{1/2}-l(t),\ (2Pt)^{1/2}+l(t))$. In this case one of the two dominant velocity values is realized with a definite probability, and one can obtain for the probability distribution of the coordinates of the absolute maxima of $G(y;x,t)$ the expression

$$w(y;z,t)=\frac{1}{\alpha[2\pi l^2(t)]^{1/2}}\left\{\exp\left[-\frac{X_r^2 z}{l^2(t)}\right]\right.$$
$$\left.\times\exp\left[\frac{X_r y}{l^2(t)}\right]E(-y)+\exp\left[-\frac{(X_r(1+z)-y)^2}{2l^2(t)}\right]\right\}, \qquad (23)$$
$$\alpha=\left[\frac{l^2(t)}{2\pi X_p^2}\right]^{1/2}\exp\left[-\frac{X_r^2 z}{l^2(t)}\right]+1,$$
$$X_r=(2Pt)^{1/2},\quad l^2(t)=\sigma_*^2 t/H_0,\quad z=(x-X_r)/X_r.$$

Using (23) and (5), it is convenient to represent the expressions for the mean value and variance of the mixed perturbation $u(x,t)$ in the form

$$\langle u(x,t)\rangle=\frac{x+(l^2(t)/X_r)}{2t}\left\{1-\mathrm{th}\left[\frac{X_r}{2l^2(t)}(x-\tilde{X}_r)\right]\right\}, \qquad (24)$$

$$\sigma_u^2(x,t)=\frac{l^2(t)}{\alpha t^2}+\frac{X_r^2}{4t^2\,\mathrm{ch}^2\left[\dfrac{X_r}{2l^2(t)}(x-\tilde{X}_p)\right]}, \qquad (25)$$

where

$$\tilde{X}_r=X_r\left[1-\frac{l^2(t)}{X_r^2}\ln((2\pi)^{1/2}X_r/l(t))\right].$$

Thus, the mutual influences of the regular and noise components in the discontinuity region of the regular pulse lead to an anomalous increase of the noise variance and to a turbulent broadening of the shock front; the value of the broadening can be found from (24) and (25) to be

$$\Delta x\sim l^2(t)/X_r. \qquad (26)$$

In addition, when an intense pulse propagates in a stationary-noise field, an effect takes place similar to that observed when a regular pulse propagates in a nonlinear viscous medium. That is to say, the shock front moves at a velocity lower than that of the discontinuity of a regular pulse propagating in a nonlinear medium in the absence of noise at infinite Reynolds numbers. It follows from (24) and (25) that within one and the same time the shock front of the pulse negotiates in a low-viscosity nonlinear medium, in the absence of noise, a distance $x = X_r$, while the influence of the noise (of the turbulent viscosity) lowers the velocity of the discontinuity, which is displaced during the same time by a distance

$$x=\widetilde{X}_r=X_r\{1-[l(t)/X_r]^2 \ln [(2\pi)^{1/2} X_r/l(t)]\}.$$

Consequently, propagation of a pulse in a stationary-noise field in a nonlinear medium is equivalent in a certain sense to propagation of a regular pulse in a viscous medium with a high-frequency viscosity coefficient $\mu_\epsilon = \sigma_*^2/2H_0$. Note that the concept of turbulent viscosity was invoked in Refs. 4 and 8 to describe the interaction of a regular periodic signal with noise.

In the region $X > (2Pt)^{1/2}$ we have $N_2 > N_1$ in (12) and the local maxima of $G(y;x,t)$ are located on the positive y axis. The pulse has practically no effect on the statistical characteristics of the noise, and to find the method described in Ref. 14 is perfectly suitable for finding the probability distribution of the mixed perturbation. In this case, the one-point probability density of the mixed perturbation is Gaussian with zero mean value and variance $\sigma_u^2 = b^2(t)$.

We conclude by briefly formulating the main conclusions that follow from the foregoing analysis. In contrast to the interaction between a regular periodic signal and stationary noise, when a pulsed signal and noise propagate simultaneously, no disruption of the coherent structure of the pulse in the nonlinear medium is produced by the noise modulation at $t \gg t_{in}$, t_m. The mean field at $t \gg t_{in}$, t_m is a triangular pulse of scale $(2Pt)^{1/2}$ with a smeared-out shock front $\Delta x \sim l^2(t)/X_r$ and a smeared-out transition region of the trailing edge, compared with a pulse propagating in a nonlinear medium in the absence of noise at infinite Reynolds numbers. The influence of the noise is also manifested by a slower velocity of the shock front of the regular pulse. The noise variance, remaining constant and equal to $b^2(t)$ outside the pulse, increases on the trailing edge of the regular pulse and increases anomalously in the region of the shock front. All these effects are reflected in the results, presented in the next section, of a numerical simulation of the nonlinear interaction of a regular pulse with a stationary Gaussian noise during the stage of interaction of fully developed discontinuities.

4. RESULTS OF NUMERICAL SIMULATION OF THE PROPAGATION OF A REGULAR PULSE AGAINST THE BACKGROUND OF STATIONARY NOISE

To analyze the evolution of nonlinear waves whose behavior is described by the Burgers equation (1), we propose in the present paper a synthesized approach based on invoking the result of numerical simulation, using the determination of the coordinate of the absolute maximum of $G(y;x,t)$ to find the statistical characteristics of the field $u(x,t)$ from the analytic expression (5). The problem of determining the characteristics of the field $u(x,t)$ consists of the following: first, calculation of the initial-action integral $G_0(y)$; second, summing $G_0(y)$ and the parabola $\beta = -(x-y)^2/2t$, i.e., finding the functional $G(y;x,t)$; third, determining the coordinate $y(x,t)$ of its absolute maximum; last, substituting $y(x,t)$ in (5) to obtain the field $u(x,t)$ at the point (x,t). This procedure of constructing an analytic solution by invoking results of numerical simulation yields the profiles of the nonlinear waves during the stage of interaction of developed discontinuities and to analyze them statistically, without considering beforehand all the preceding propagation stages.

We use as the example in our analysis a high-power unipolar pulse $u_m(x) = P\delta(x)$ against the background of a Gaussian stationary noise $\xi(t)$, in a nondispersive nonlinear medium. We assume that the noise spectrum at zero spatial frequency is zero, so that the variance of the initial action is bounded. We consider the evolution of the field $u(x,t)$ for times $t \gg t_{in}$, when discontinuities are produced in the nonlinear-wave profile and the characteristic distance between them $l(t)$ is much larger than the initial correlation radius ρc of the stationary noise. The absolute maximum of $G(y;x,t)$ is chosen from a large number of independent local maxima of the initial action. This allows us to carry out the numerical analysis in steps $\Delta \gg l$ and simulate the process $G_0(y)$ by a discrete sequence of independent Gaussian random numbers having equal variances

$$\widetilde{G}_0(m\Delta)=\sigma_*\xi_m, \quad m=0, 1, 2, \ldots, X/\Delta. \tag{27}$$

It was assumed in the numerical experiment that $\langle \xi_i \rangle = 0$, $\langle \xi_i^2 \rangle = 1$, $\sigma_* = 0{,}1$. The integral of the pulsed perturbation was simulated by a unit step function of amplitude $P = 1\,(P \gg \sigma)$ at the point $x = 1.5$. The initial field $u_0(x)$ was simulated in discrete x steps $\Delta = 0.05$ over a realization length $X = 5$, and the statistical characteristics of the field $u(x,t)$ were investigated for different times t. The reduction of the numerical simulation was based on $N = 1000$ realizations of the field $u(x,t)$ and for a single point (x,t).

Figure 1 shows one realization of a mixed perturbation $u(x,t)$ at two instants, $t_1 = 1$, and $t_2 = 2$. The field $u(x,t)$ is a sequence of sawtooth pulses with equal slope $u_x' = 1/t$. It follows from the theory and is easily seen from Fig. 1 that when the time is increased the discontinuities move randomly with velocities V_k (7), the zeros of the field (7) remain immobile, and the slopes of the sections between the discontinuities decrease.

Figures 2 and 3 show respectively the results of a numerical simulation for a mean value $\langle u(x,t) \rangle$ and variance $\sigma_u^2(x,t)$ at $t_1 = 1$ and $t_2 = 2$. For comparison, Fig. 2 shows also, for the same times, the profile of a regular pulse propagating in a nonlinear medium without the noise component.

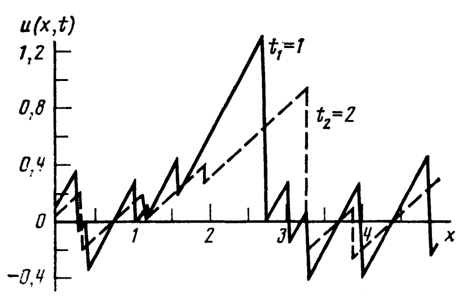

FIG. 1. Realization of the random field $u(x,t)$.

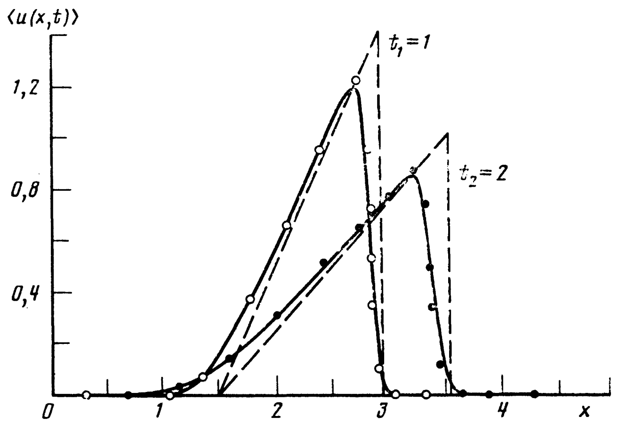

FIG. 2. Evolution of the mean value $\langle u(x,t)\rangle$ ($N = 1000$, $\sigma_* = 0.1$, $P = 1$); ○—theoretical results for $t_1 = 1$; ●—theoretical results for $t_2 = 2$; the dashed lines show the profile of the regular pulse in the absence of noise.

The number of averaged field realizations is $N = 1000$. It can be seen from Figs. 2 and 3 that the results of the numerical simulations confirm qualitatively the predictions of the asymptotic theory developed above (the smearing of the regular-pulse boundaries on account of turbulent viscosity, the deceleration of the shock front, as well as the decrease of the variance on the trailing edge and its anomalous increase in the shock-front region).

The results of the numerical simulation were compared with the theory not only qualitatively but also quantitatively. To this end, the characteristics of the noise were analytically calculated for a discrete model, in which the noise was simulated by a set of discrete random independent quantities of equal variance σ_*^2 and discrete steps Δ. The results have shown that in the case of a discrete model the statistical characteristics of the mixed perturbation are described by Eqs. (17), (18), (24), and (25), but with a characteristic noise-field scale l_d determines by the variance σ_*^2 and by the discrete step spacing Δ

$$l_d^2(t)=\sigma_*^2 t/H_d, \quad H_d\simeq\sigma_* \ln^{1/2}(\sigma_* t/\Delta^2). \tag{28}$$

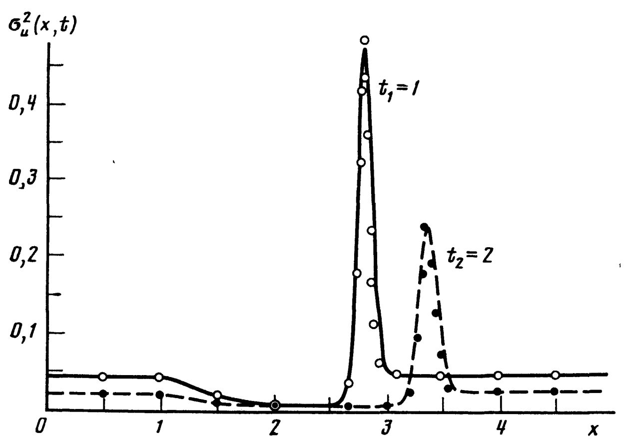

FIG. 3. Evolution of the variance $\sigma_u^2(x,t)$ ($N = 1000$, $\sigma_* = 0.1$, $P = 1$): ○—theoretical results for $t_1 = 1$, ●—theoretical results for $t_2 = 2$.

Comparison of the experimental $l_e(t)$ and the theoretical l_d for the external turbulence scale at the values $\Delta = 0.05$ and $\sigma_* = 0.1$ yields $l_d(t=1) = 0.228$; $l_e(t=1) = 0.2006$; $l_d(t=2) = 0.303$; $l_e(t=2) = 0.283$. The theoretical values of the characteristic noise scale exceed thus the experimental. The discrepancy is due to the asymptotic character of the developed theory ($t \gg t_{in}$), and the relative error $P_{err}(t)$ decreases with t; thus, $P_{err}(t=1) \approx 13\%$; $P_{err}(t=2) \approx 9\%$.

Recognizing that the form of the probability distribution of mixed excitation over long times is universal and is determined by two scales, $X_r(t)$ and $l(t)$, we compared the shapes of the theoretical and experimental mean-value and variance plots. The scale $l(t)$ for the theoretical equations was chosen to be the characteristic noise scale obtained from experiment. It can be seen from (15) and (18), in the region where noise has practically no influence on the pulse, the noise variance is constant at $\sigma_u^2 = l^2(t)/t^2$, so that the external scale $l(t)$ can be uniquely determined from the experimentally obtained noise variance. The theoretical mean value $\langle u(x,t)\rangle$ and variance $\sigma_u^2(x,t)$ calculated from Eqs. (17), (18), (24), and (25) are represented by the marker points in Figs. 2 and 3. Theory and experiment are practically in full agreement in the region of the trailing edge of the regular pulse and deviate insignificantly ($P_{err} \approx 5\%$) in the region of the shock front.

Comparison of the theory with the numerical simulation leads thus to the conclusion that an asymptotic theory, including one that employs the notion of turbulent viscosity,[8] describes satisfactorily the statistical characteristics of a mixed perturbation with fully developed discontinuities.

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