Behavior of an axisymmetric shock wave near a cumulation point

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The pattern of the cumulation of an axisymmetric (but not cylindrical) converging shock wave is investigated theoretically. If dissipative processes are neglected, the cumulation is unbounded.

Converging shock waves are among the best known examples of cumulative flows and are related with important technical applications[1] as well as with interesting scientific problems.[2,3]

It was shown already in the early papers by Guderley, Landau, and Stanyukovich[4–6] that flow behind a converging shock wave having spherical or cylindrical symmetry is self-similar, while the amplitude and especially the velocity V of the wave increase without limit as the wave radius a tends to zero:

$$V \propto a^{-0,195} \tag{1}$$

(for a cylindrical wave and for an adibat exponent $\gamma = 7/5.$).

The important question of the limit of amplitude growth was repeatedly raised in the literature. Obviously, if the radius a becomes comparable with the mean free path, the dissipative processes on the wave front cannot be neglected. In a plasma, for example, allowance for the finite width of the front imposes a limit on the velocity of a converging wave.[7] The present paper does not deal with such small scales, and is restricted to the gasdynamic approximation.

Another mechanism that limits the wave amplitude was considered by Zababakhin (see Ref. 3 and 8 and the citations therein). In his opinion, cumulation of a spherical shock wave with a cylindrical one is due to the degenerate one-dimensional character of these flows, and superposition of a three-dimensional pertubation should limit the cumulation—the energy density will everywhere be finite. Favoring this assumption is, in particular, the fact that the Guderley-Landau-Stanyukovich solutions are unstable to small three-dimensional distortions of the front.[9–11]

This raises an important question: is unlimited cumulation of non-one-dimensional converging shock wave possible? Such a question was posed in an experimental paper[12] and answered in the affirmative. It was found that the wave produced by an annular discharge in a gas accelerates and becomes enhanced as the center of the ring is approached.

The present paper is devoted to a theoretical investigation of non-one-dimensional cumulative shock waves. It is shown that cumulation is a feature of converging axisymmetric—not necessarily cylindrical—waves and is unbounded if dissipative processes are disregarded. Such waves can be produced not only in annular discharges but also, e.g., in noncylindrical z-pinches.[13]

We use here the approximate theory developed by Chester, Chiswell, and Witham (CCW).[9] A function $\Phi(x,y,z)$ was introduced to describe the surfaces of the shock-wave fronts at each instant of time t in accordance with the relation

$$\Phi(x, y, z) + V_{s0}t = 0, \tag{2}$$

and an equation is obtained for this function

$$\operatorname{div}(M^{n+1}\operatorname{grad}\Phi) = 0, \tag{3}$$

$$M = |\operatorname{grad}\Phi|^{-1}. \tag{4}$$

Here V_{S0} is the speed of sound in the unperturbed gas, M is the local Mach number for the shock wave, and $n \approx 5.1$ for $\gamma = 1.4$. At present there is no consistent derivation of Eq. (3) from the gasdynamics equations, and the degree of its validity for each class of problem can be estimated only by comparison with exact solutions (such a comparison is given in Ref. 9 for Guderley-Landau-Stanyukovich flows).

It is assumed in the present paper that the CCW theory can be used also for two-dimensional axisymmetric converging shock waves. In particular, good quantitative agreement (within $\lesssim 4\%$) is obtained when the solutions obtained for the stability of spherical converging waves to non-one-dimensional perturbations in the CCW approximation[10] are compared with a rigorous gasdynamic approach.[11] Since all the results of the present paper were obtained, just as in Ref. 10, from an analysis of a linearized Eq. (3), the use of the CCW theory here seems justified.

1. BEHAVIOR OF EQUATION (3) NEAR A SYMMETRY AXIS

Clearly, to assess the feasibility of unbounded cumulation of an axisymmetric shock wave it is necessary to study the behavior of the solution of (3) near a symmetry axis. It is less obvious that it suffices to consider the solution in an infinitely small vicinity of an individual isolated point on the axis, and not, say, in the vicinity of a finite line segment. An investigation of the instability of a cylindrical wave to arbitrary axisymmetric perturbations shows that when a wave that deviates arbitrarily little from cylindrical approaches the axis, it reaches the latter in the general case just at individual isolated points (in contrast, e.g., to axisymmetric perturbations of a cylindrical z-pinch, when the neck produced on the axis can have an arbitrary length). We choose one such point as the origin $r = 0$, $z = 0$ of the coordinate frame and let $t = 0$ at the instant when the wave arrives at this point.

We proceed to study the possible types of local behavior

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of a shock wave in the vicinity of the origin. From the mathematical standpoint we deal here with a classification of the structurally stable zeros of the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\Phi}{\partial r}\left[\left(\frac{\partial\Phi}{\partial r}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]^{-(n+1)/2}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial\Phi}{\partial z}\left[\left(\frac{\partial\Phi}{\partial r}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]^{-(n+1)/2}\right\}=0, \quad (5)$$

since Φ vanishes at the point $r=0$, $z=0$ by virtue of the foregoing assumptions [see (2)].

We seek the solution of (5) in the vicinity of the origin in the form of a series in powers of (not necessarily integer) r and z:

$$\Phi=\Phi_0+\sum_{p+q>\alpha} b_{pq}z^p r^q, \quad (6)$$

where

$$\Phi_0=\sum_{p+q=\alpha} b_{pq}{}^0 z^p r^q \quad (7)$$

is the set of terms with smallest powers of α. It should be noted that Φ can be represented by the series (6) only if $t\leqslant 0$, inasmuch as at $t>0$, i.e., after the cumulation, secondary (intersecting) shock waves can be produced, and the CCW theory used here must be modified.

It is convenient to seek the function Φ_0 in the form

$$\Phi_0=R^\alpha \exp\left[\int F(\theta)\,d\theta\right], \quad (8)$$

where $R=(r^2+z^2)^{1/2}$ and $\sin\theta=r/R$. Substituting (8) in (5) we can separate the variables R and θ and obtain one first-order equation for $F(\theta)$. A qualitative analysis of this equation allows us to assume that there exist only three[1)] solutions free of singularities at least at $t<0$, viz.,

$$\alpha=1,\quad \Phi_0{}^{(1)}=R\cos\theta=z,$$
$$\alpha=1+1/n,\quad \Phi_0^{(2)}=R^{1+1/n}\sin^{1+1/n}\theta=r^{1+1/n}, \quad (9)$$
$$\alpha=1+2/n,\quad \Phi_0{}^{(3)}=R^{1+2/n}.$$

They correspond to one-dimensional flows: plane, converging cylindrical, and converging spherical shock waves.

As $R\to 0$, the terms of higher power in (6) are small compared with Φ_0, so that at small R one can seek a solution (5) in the form $\Phi=\Phi_0+\Phi_1$ and linearize (5) with respect to Φ_1. We emphasize that this approach differs from the linearization procedure used in investigations[9,10] of the stability of one-dimensional solutions to non-one-dimensional perturbations. The small parameter here is not the amplitude of the perturbation of the initial data, but the distance R to the null point.

We consider next the corrections to all three possible functions

1. $\Phi_0^{(1)}=C_0 z$. Here and elsewhere C stands for various positive constants. The equation for $\Phi_1^{(1)}$ is

$$n\frac{\partial^2}{\partial z^2}\Phi_1^{(1)}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi_1^{(1)}}{\partial r}\right). \quad (10)$$

Its solution as $R\to 0$ is

$$\Phi_1^{(1)}=-C_1(2z^2+nr^2), \quad (11)$$

so that

$$\Phi^{(1)}=C_0 z-C_1(2z^2+nr^2). \quad (12)$$

The surfaces $\Phi^{(1)}+V_{S0}t=0$ describe an axisymmetric convex shock wave propagating along the z axis. Another type of null point can be obtained by combining two such solutions, at $z\leqslant 0$ and $z\geqslant 0$. This type corresponds to encounter of two waves at the point $r=0$, $z=0$. We shall not dwell on this, since the feasibility of such a local behavior of shock waves is perfectly obvious.

2. $\Phi_0^{(2)}=C_0 r^{1+1/n}$. In this case

$$\frac{\partial^2}{\partial z^2}\Phi_1^{(2)}=nr^{1/n}\frac{\partial}{\partial r}\left(r^{-1/n}\frac{\partial\Phi_1^{(2)}}{\partial r}\right). \quad (13)$$

The solution of (13) takes as $R\to 0$ the form

$$\Phi_1^{(2)}=-C_1[(n-1)z^2+r^2]. \quad (14)$$

Consequently the flow near the investigated null point (we call this a "neck," since it constitutes, in particular, a nonlinear stage of instability of a cylindrical wave to neck-type perturbations) is described by the equation

$$\Phi^{(2)}=C_0 r^{1+1/n}-C_1[(n-1)z^2+r^2]. \quad (15)$$

With decrease of the minimum neck radius $a(t)$ the second curvature radius d of the shock-wave front, at points where $z=0$, also tends to zero (in accord with Ref. 12), with

$$d\propto a^{1/n}. \quad (16)$$

At small R, the equation for the boundary surface $\Phi^{(2)}=0$ is

$$r=[(n-1)C_1/C_0]^{n/(n+1)}z^{2n/(n+1)}. \quad (17)$$

It is a paraboloid $r\propto z^k$ with $1<k<2$. The shock-wave velocity at the vertex (at $z=0$) becomes infinite like

$$M=\left(\frac{n+1}{n}C_0 a^{1/n}-2C_1 a\right)^{-1}=\frac{n}{C_0(n+1)}a^{-1/n}\left[1+\frac{2n}{(n+1)}\frac{C_1}{C_0}a^{1-1/n}\right]. \quad (18)$$

The velocity increases more slowly than in the case of a cylindrical wave, but the principal terms are the same for $a\to 0$.

Let us examine the restrictions on the validity of the procedure used to linearize the equation for $\Phi_1^{(2)}$. The corresponding conditions

$$\left|\frac{\partial\Phi_1^{(2)}}{\partial r}\right|\ll\frac{n+1}{n}C_0 r^{1/n},\qquad \left|\frac{\partial\Phi_1^{(2)}}{\partial z}\right|\ll\frac{n+1}{n}C_0 r^{1/n} \quad (19)$$

are met at

$$r\ll(C_0/C_1)^{n/(n-1)},\qquad C_1(n-1)z^2\leqslant C_0 r^{1+1/n}, \quad (20)$$

i.e., in a region bounded by the parabola (17) (in other words, at $t\leqslant 0$), and r is small enough.

3. $\Phi_0^{(2)}=R^{1+2/n}$. In this case it turns out that the solution of the equation for the perturbation $\Phi_1^{(3)}$ decreases as $R\to 0$ more slowly than the zeroth approximation $\Phi_0^{(3)}$. This

means that a singularity such as a focus of a spherical converging wave is structurally unstable in the class of two-dimensional flows.

Thus, in the absence of secondary waves, the general behavior of an axisymmetric shock wave near the point of intersection with the symmetry axis can be the following: (a) wave propagation along the axis; (b) encounter of two shock waves (followed by formation of secondary waves), or (c) cumulative flow such as a neck with unbounded increase of the wave amplitude.

2. ANNULAR SHOCK WAVE

Consider the flow produced as a result of an annular discharge in a homogeneous immobile gas having a density ρ_0 and a pressure P_0. The energy input E_0 (assumed instantaneous for simplicity) is then distributed over a thin ring with a major radius R_0. An experimental investigation of just such a flow is described in Ref. 12, where particular attention is paid to enhancement of the shock wave near the center of the ring.

The results of the preceding section lead to the conclusion that wave propagation towards the ring center is described by a neck-type solution and is indeed cumulative. In the simple case

$$E_0 \gg P_0 R_0^3 \tag{21}$$

the wave can be regarded as strong up to the instant of the start of the cumulation. The parameters in Eqs. (15)–(18) can then be expressed to within numerical coefficients, and quantitative laws can be obtained for the variation of the amplitude and shape of the shock wave near the center.

The flow produced after the discharge (non-self-similar and non-one-dimensional) is determined by three dimensional constants, E_0, ρ_0, and R_0, and by one dimensionless constant γ. The neck produced at the center should be described by Eqs. (15)–(18), in which the dimensional constants are combinations of E_0, ρ_0, and R_0. We find as a result that the shock-wave form near the ring center is given by

$$(r/R_0)^{1+1/n} - \left[\frac{a(t)}{R_0}\right]^{1+1/n} = (\beta/R_0^2)\left[(n-1)z^2 + r^2 - a^2(t)\right], \tag{22}$$

where β is a dimensionless constant that depends only on γ. The wave velocity $v = |da/dt|$ satisfies the relation

$$V = \delta\left(\frac{E_0}{\rho_0 R_0^3}\right)^{1/2}\left(\frac{r}{R_0}\right)^{-1/n}\left\{1 + \frac{2n\beta}{n+1}\left(\frac{r}{R_0}\right)^{1-1/n}\right\}. \tag{23}$$

Here δ is one more unknown dimensionless constant. The values of β and δ can be obtained from experiment or by numerical calculation of the entire dynamics of the annular shock wave.

CONCLUSIONS

1. Cumulation of converging shock waves is not an exclusive property of one-dimensional flows. The presence of cumulation points on the symmetry axis is also a feature of two-dimensional axisymmetric waves.

2. The wave amplitude increases without limit near the cumulation point if dissipative processes are neglected. Such a flow is produced, in particular, in an annular discharge in a gas.

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[1)] This statement could not be proved rigorously.

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