Nonrelativistic electrons in the field of two strong electromagnetic waves

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We consider the motion of a nonrelativistic electron in the field of two strong monochromatic light waves propagating counter to each other. The wave function of the electron is obtained by using a quasiclassical approximation and perturbation theory. An expression is obtained for the gain of a weak test wave by using such wave functions.

INTRODUCTION

The motion of an electron in the field of a monochromatic light wave is described by the well-known Volkov function. Exact solutions of the relativistic wave equations were obtained in Refs. 1 and 2 for the motion of an electron in certain cases of plane-wave fields. The motion of an electron in the field of two light waves propagating counter to each other (standing wave), however, cannot be solved exactly.

Electron diffraction by a standing wave was considered by perturbation theory in Ref. 3. Perturbation theory was likewise used in Ref. 4 to investigate the modulation of a beam of relativistic electrons.

We consider in the present paper a quasiclassical approximation for the description of the motion of a nonrelativistic electron in the field of pump waves. The wave functions of the nonrelativistic electrons in the field of pump waves (standing wave) are obtained using a quasiclassical approximation. The corrections to these wave functions, necessitated by the rescattering effect (the electron absorbs a photo from one wave and gives it up to the other by induced emission) are obtained by perturbation theory. The electron energy is not altered by the rescattering, but the momentum is changed by an amount $\mathbf{p}' = \mathbf{p} + 2\hbar\mathbf{k}$, where $\mathbf{p}'$ and $\mathbf{p}$ are the electron momenta in the initial and final states, respectively, while $\mathbf{k}$ is the pump-wave momentum. The electron-momentum change can be accompanied by emission in the IR band. Using the wave functions derived, we obtain the gain of a weak test wave in this frequency band. We show that the gain is an optimum if the angle θ between $\mathbf{p}$ and $\mathbf{k}$ is close to $\pi/2$.

GENERAL RELATIONS

The Schrödinger equation for an electron in the field of two strong electromagnetic waves is of the form

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{r},t)+i\hbar\frac{e}{mc}[\mathbf{A}_1(\varphi_1)+\mathbf{A}_2(\varphi_2)]\nabla\Psi(\mathbf{r},t)+\frac{e^2}{2mc^2}[\mathbf{A}_1^2(\varphi_1)+\mathbf{A}_2^2(\varphi_2)]\Psi(\mathbf{r},t)+\frac{e^2}{mc^2}\mathbf{A}_1(\varphi_1)\mathbf{A}_2(\varphi_2)\Psi(\mathbf{r},t), \quad (1)$$

where $\mathbf{A}_1(\varphi_1)$ and $\mathbf{A}_2(\varphi_2)$ are the vector potentials of the two electromagnetic waves, respectively; $\varphi_1=\omega_1 t-\mathbf{k}_1\mathbf{r}$, $\varphi_2=\omega_2 t-\mathbf{k}_2\mathbf{r}$ (ω_1,ω_2 are respectively the frequencies of the first and second electromagnetic waves, and $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors of these waves); m is the electron mass.

We seek the solution of Eq. (1) in the form

$$\Psi(\mathbf{r},t)=\frac{1}{V^{1/2}}\exp\left[i\frac{\mathbf{pr}}{\hbar}-i\frac{Et}{\hbar}\right]\times\exp\left[i\frac{S_1(\varphi_1)}{\hbar}+i\frac{S_2(\varphi_2)}{\hbar}\right]\Psi_{int}(\mathbf{r},t), \quad (2)$$

where V is the normalization volume.

Here p and E are the momentum and kinetic energy of the electron:

$$S_1(\varphi_1)=\frac{1}{m\omega_1}\int^{\varphi_1}\left[\frac{e}{c}\mathbf{A}_1(\varphi_1)\mathbf{p}-\frac{e^2}{2c^2}\mathbf{A}_1^2(\varphi_1)\right]d\varphi_1, \quad (3)$$

$$S_2(\varphi_2)=\frac{1}{m\omega_2}\int^{\varphi_2}\left[\frac{e}{c}\mathbf{A}_2(\varphi_2)\mathbf{p}-\frac{e^2}{2c^2}\mathbf{A}_2^2(\varphi_2)\right]d\varphi_2. \quad (4)$$

Substitution of relations (2), (3), and (4) in (1) leads to an equation for $\Psi_{\rm int}(\mathbf{r},t)$:

$$i\hbar\frac{\partial\Psi_{int}(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi_{int}(\mathbf{r},t)-\frac{i\hbar}{m}\left\{\mathbf{p}-\mathbf{k}_1\frac{dS_1(\varphi_1)}{d\varphi_1}-\mathbf{k}_2\frac{dS_2(\varphi_2)}{d\varphi_2}-\frac{e}{c}[\mathbf{A}_1(\varphi_1)+\mathbf{A}_2(\varphi_2)]\right\}\nabla\Psi_{int}(\mathbf{r},t)+\left[\frac{e^2}{mc^2}\mathbf{A}_1(\varphi_1)\mathbf{A}_2(\varphi_2)+\frac{1}{m}\mathbf{k}_1\mathbf{k}_2\frac{dS_1(\varphi_1)}{d\varphi_1}\frac{dS_2(\varphi_2)}{d\varphi_2}+\frac{e}{mc}\mathbf{A}_1\mathbf{k}_2\frac{dS_2(\varphi_2)}{d\varphi_2}+\frac{e}{mc}\mathbf{A}_2\mathbf{k}_1\frac{dS_1(\varphi_1)}{d\varphi_1}\right]\Psi_{int}(\mathbf{r},t). \quad (5)$$

We can leave out of (5) the terms

$$\mathbf{k}_1[dS_1(\varphi_1)/d\varphi_1],\quad \mathbf{k}_2[dS_2(\varphi_2)/d\varphi_2],$$
$$e(\mathbf{A}_1+\mathbf{A}_2)/c,\quad m^{-1}\mathbf{k}_1\mathbf{k}_2[dS_1(\varphi_1)/d\varphi_1]dS_2(\varphi_2)/d\varphi_2,$$
$$(e\mathbf{A}_1\mathbf{k}_2/mc)[dS_2(\varphi_2)/d\varphi_2],\quad (e\mathbf{A}_2\mathbf{k}_1/mc)dS_1(\varphi_1)/d\varphi_1,$$

if the inequalities[1)] $cp\cos\theta \gg e\mathscr{E}_i\lambda\!\!\!^{-}_i$ $(i=1,2)$ and $v/c \ll 1$.

Equation (5) takes then the form

$$i\hbar\frac{\partial\Psi_{int}(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi_{int}(\mathbf{r},t)-\frac{i\hbar}{m}\mathbf{p}\nabla\Psi_{int}(\mathbf{r},t)+\frac{e^2}{mc^2}\mathbf{A}_1(\varphi_1)\mathbf{A}_2(\varphi_2)\Psi_{int}(\mathbf{r},t). \quad (6)$$

We consider for the sake of argument linearly polarized electromagnetic waves

$$\mathbf{A}_1(\varphi_1)=\mathbf{A}_1\sin\varphi_1, \quad (7)$$

$$\mathbf{A}_2(\varphi_2)=\mathbf{A}_2\sin\varphi_2. \quad (8)$$

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Substituting (7) and (8) in (6), we get

$$i\hbar\frac{\partial\Psi_{int}(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi_{int}(\mathbf{r},t)-\frac{i\hbar}{m}\mathbf{p}\nabla\Psi_{int}(\mathbf{r},t)+\frac{e^2}{2mc^2}\mathbf{A}_1\mathbf{A}_2[\cos(\varphi_1-\varphi_2)-\cos(\varphi_1+\varphi_2)]\Psi_{int}(\mathbf{r},t). \quad (9)$$

It is natural to seek the solution of (9) in the form

$$\Psi_{int}(\mathbf{r},t)=\Psi_{int}^{(0)}(\mathbf{r},t)\exp[iS_3(\mathbf{r},t)/\hbar], \quad (10)$$

where, if the inequalities $pc\gg e\mathscr{E}_i\lambda\!\!\!^{-}_i$ $(i=1,2)$, we have the following equations for $S_3(\mathbf{r},t)$ and $\Psi_{int}^{(0)}(\mathbf{r},t)$:

$$-\frac{\partial S_3(\mathbf{r},t)}{\partial t}=-\frac{i\hbar}{2m}\Delta S_3(\mathbf{r},t)+\frac{1}{2m}[\nabla S_3(\mathbf{r},t)]^2+\frac{\mathbf{p}}{m}\nabla S_3(\mathbf{r},t)-\frac{e^2\mathbf{A}_1\mathbf{A}_2}{2mc^2}\cos(\varphi_1+\varphi_2) \quad (11)$$

$$i\hbar\frac{\partial\Psi_{int}^{(0)}(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi_{int}^{(0)}(\mathbf{r},t)-i\hbar\frac{\mathbf{p}}{m}\nabla\Psi_{int}^{(0)}(\mathbf{r},t)+\frac{e^2\mathbf{A}_1\mathbf{A}_2}{2mc^2}\cos(\varphi_1-\varphi_2)\Psi_{int}^{(0)}(\mathbf{r},t). \quad (12)$$

It follows from the forms of (11) and (12) that the solutions of these equations should depend on $\varphi_1+\varphi_2$ and $\varphi_1-\varphi_2$, respectively. Neglecting the first, second, and third terms in the right-hand side of (11), we obtain the solution

$$S_3(\mathbf{r},t)=\frac{e^2A_1A_2}{2mc^2(\omega_1+\omega_2)}\sin(\varphi_1+\varphi_2). \quad (13)$$

Neglect of the first, second, and third terms in relation (11) is valid subject to the inequalities

$$\hbar(\omega_1+\omega_2)/2mc^2\ll1,\quad e^2\mathbf{A}_1\mathbf{A}_2/(2mc^2)^2\ll1,\quad v\ll c.$$

We seek the solution of (12) by introducing first a new variable $z=\varphi_1-\varphi_2$; we can then rewrite this equation in the form

$$\frac{\hbar^2(\mathbf{k}_2-\mathbf{k}_1)^2}{2m}\frac{d^2\Psi_{int}^{(0)}(z)}{dz^2}+i\hbar[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]\frac{d\Psi_{int}^{(0)}(z)}{dz}-\frac{e^2\mathbf{A}_1\mathbf{A}_2}{2mc^2}\cos z\,\Psi_{int}^{(0)}(z)=0 \quad (14)$$

or

$$\frac{d^2\Psi_{int}^{(0)}(z)}{dz^2}+i\frac{2m[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}\frac{d\Psi_{int}^{(0)}(z)}{dz}-\frac{e^2\mathbf{A}_1\mathbf{A}_2}{(\hbar c)^2(\mathbf{k}_2-\mathbf{k}_1)^2}\cos z\,\Psi_{int}^{(0)}(z)=0, \quad (15)$$

where $\mathbf{v}$ is the initial velocity of the electron.

We eliminate from (15) the term with the first derivative with respect to z, by making the change of variables

$$\Psi_{int}^{(0)}(z)=P(z)u(z), \quad (16)$$

where

$$P(z)=\exp\left\{-i\frac{[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]m}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}z\right\}. \quad (17)$$

We have for $u(z)$ the equation

$$u''(z)+\left\{\left[\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}m\right]^2-\frac{e^2\mathbf{A}_1\mathbf{A}_2}{(\hbar c)^2(\mathbf{k}_2-\mathbf{k}_1)^2}\cos z\right\}u(z)=0. \quad (18)$$

We shall solve (18) by perturbation theory. We represent $u(z)$ in the form

$$u(z)=u_0(z)+u_1(z), \quad (19)$$

where $u_0(z)$ satisfies the equation

$$u_0''(z)+\left[\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}m\right]^2u_0(z)=0, \quad (19')$$

and $u_1(z)$ the equation

$$u_1''(z)+\left[\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}m\right]^2u_1(z)=\frac{e^2\mathbf{A}_1\mathbf{A}_2}{(\hbar c)^2(\mathbf{k}_2-\mathbf{k}_1)^2}\cos z\,u_0(z). \quad (19'')$$

The solutions of (19') and (19'') are

$$u_0(z)=\exp\left[i\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}mz\right],$$
$$u_1(z)=-\frac{e^2\mathbf{A}_1\mathbf{A}_2\sin z}{2mc^2[\hbar\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)+\hbar(\omega_1-\omega_2)]}\times\exp\left[i\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}mz\right]. \quad (20)$$

To obtain (20) it was necessary to meet the conditions

$$\left|\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{\hbar(\mathbf{k}_2-\mathbf{k}_1)^2}m\right|\gg1,$$
$$\left|\frac{\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)}{(\mathbf{k}_2-\mathbf{k}_1)^2}m\right|\gg\frac{|e^2\mathbf{A}_1\mathbf{A}_2|^{1/2}}{c|\mathbf{k}_2-\mathbf{k}_1|}, \quad (20')$$
$$\left|\frac{e^2\mathbf{A}_1\mathbf{A}_2}{2mc^2[\hbar\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)+\hbar(\omega_1-\omega_2)]}\right|\ll1.$$

The last condition of (20') is violated as $\hbar\to0$.

Taking (29) and (20) into account, we have

$$\Psi_{int}^{(0)}(z)=1-ie^2\mathbf{A}_1\mathbf{A}_2\{\hbar[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]2mc^2\}^{-1}\sin z. \quad (20'')$$

Substituting (10), (13), and (20'') in (2) and taking (3) and (4) into account, we obtain the wave function of the nonrelativistic electron in the field of two strong electromagnetic waves at an arbitrary arrangement of the vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{p}$. We shall, however, be interested hereafter only in a situation in which $\mathbf{k}_1$ and $\mathbf{k}_2$ are collinear and form a standing wave.

In this case the wave function takes the form

$$\Psi_i=\Psi_i^{(0)}\left\{1-ie^2\mathbf{A}_1\mathbf{A}_2\{\hbar[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]2mc^2\}^{-1}\sin z\right\}, \quad (21)$$

where Ψ_i^0 are basis functions defined by the relation

$$\Psi_i^{(0)}=\frac{1}{V^{1/2}}\exp\left[i\frac{\mathbf{pr}}{\hbar}-i\frac{Et}{\hbar}\right]$$

$$\times\exp\left[-i\frac{e\mathbf{A}_1\mathbf{p}}{mc\hbar\omega_1}\cos\varphi_1-i\frac{e^2A_1^2}{4mc^2\hbar\omega_1}\varphi_1\right.$$

$$+i\frac{e^2A_1^2}{8mc^2\hbar\omega_1}\sin 2\varphi_1-i\frac{e\mathbf{A}_2\mathbf{p}}{mc^2\hbar\omega_2}\cos\varphi_2-i\frac{e^2A_2^2}{4mc^2\hbar\omega_2}\varphi_2$$

$$\left.+i\frac{e^2A_2^2}{8mc^2\hbar\omega_2}\sin 2\varphi_2+i\frac{e^2\mathbf{A}_1\mathbf{A}_2}{2mc^2\hbar(\omega_1+\omega_2)}\sin(\varphi_1+\varphi_2)\right]. \tag{22}$$

It can be easily shown by direct calculation that the Ψ_i^0 satisfy the orthogonality relation (if $\mathbf{k}_1$ and $\mathbf{k}_2$ are collinear or form a standing wave)

$$\int\Psi_i^{(0)}\Psi_f^{(0)*}\,d\mathbf{r}=(2\pi)^3\delta(\mathbf{p}-\mathbf{p}'),$$

where Ψ_f^0 is obtained from Ψ_i^0 by replacing the exponents $\mathbf{p}$ by $\mathbf{p}'$. The expression (22) for the wave function of the nonrelativistic electron coincides in the case of two collinear electromagnetic waves with the wave function obtained in Ref. 1 in the nonrelativistic limit.

We define the field of the amplified electromagnetic wave by the relation

$$\mathbf{A}_3(\mathbf{r},t)=\mathbf{A}_3\sin(\omega_3t-\mathbf{k}_3\mathbf{r}), \tag{23}$$

where ω_3 is the frequency of the amplified wave and $\mathbf{k}_3$ is its wave vector. The processes investigated in this paper are charcterized by an element of an S matrix for which the expressions take, in first-order perturbation theory in the test-wave electromagnetic field, the form

$$S_{fi}=-i\int\Psi_f^*(\mathbf{r},t)\hat{V}\Psi_i(\mathbf{r},t)\,d\mathbf{r}\,dt, \tag{24}$$

$$\hat{V}=e\mathbf{A}_3(\mathbf{r},t)\hat{p}/mc. \tag{25}$$

The S-matrix element is, with allowance for relations (22) and (25),

$$S_{fi}=\frac{e\mathbf{A}_3\mathbf{p}}{2mcV}\int\exp\left[i\left(\frac{E'-E}{\hbar}\pm\omega_3\right)t+i\left(\frac{p-p'}{\hbar}\mp k_3\right)r\right]$$

$$\times\exp\left[i\frac{e\mathbf{A}_1(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_1}\cos(\omega_1t-\mathbf{k}_1\mathbf{r})\right.$$

$$\left.+i\frac{e\mathbf{A}_2(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_2}\cos(\omega_2t-\mathbf{k}_2\mathbf{r})\right]$$

$$\times\frac{e^2\mathbf{A}_1\mathbf{A}_2(\mathbf{v}-\mathbf{v}')(\mathbf{k}_2-\mathbf{k}_1)}{2mc^2\hbar\{[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]\}^2}\,d\mathbf{r}\,dt \tag{26}$$

(the + and − signs pertain to emission and absorption, respectively).

We use the known relation

$$\exp[iB\sin x]=\sum_{n=-\infty}^{+\infty}J_n(B)e^{inx}. \tag{27}$$

The expression for the S-matrix element is then

$$S_{fi}=-i\frac{e\mathbf{A}_3\mathbf{p}}{2mcV}\int\sum_{n_1,n_2}J_{n_1}\left(\frac{e\mathbf{A}_1(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_1}\right)J_{n_2}\left(\frac{e\mathbf{A}_2(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_2}\right)$$

$$\times\exp\left\{i\left[\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\mp\mathbf{k}_3+n_1\mathbf{k}_1+n_2\mathbf{k}_2+(\mathbf{k}_2-\mathbf{k}_1)\right]\mathbf{r}\right\}$$

$$\times\frac{e^2\mathbf{A}_1\mathbf{A}_2(\mathbf{p}-\mathbf{p}')(\mathbf{k}_2-\mathbf{k}_1)}{(2mc^2)^2\hbar(\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1))^2}$$

$$\times\exp\left[i\left(\frac{E'-E}{\hbar}\pm\omega_3+n_1\omega_1+n_2\omega_2+\omega_1-\omega_2\right)t\right]d\mathbf{r}\,dt. \tag{28}$$

Integration with respect to $\mathbf{r}$ and t in (28) leads to the following expression for the S-matrix element:

$$S_{fi}=-i\frac{e\mathbf{A}_3\mathbf{p}(2\pi)^4}{2mcV}\sum_{n_1,n_2}J_{n_1}\left(\frac{e\mathbf{A}_1(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_1}\right)J_{n_2}\left(\frac{e\mathbf{A}_2(\mathbf{p}'-\mathbf{p})}{mc\hbar\omega_2}\right)$$

$$\times\frac{e^2\mathbf{A}_1\mathbf{A}_2(\mathbf{p}-\mathbf{p}')(\mathbf{k}_2-\mathbf{k}_1)}{(2mc^2)^2\hbar[\omega_1-\omega_2+\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1)]^2}$$

$$\times\delta\left(\frac{E'-E}{\hbar}\pm\omega_3+n_1\omega_1+n_2\omega_2+\omega_1-\omega_2\right)$$

$$\times\delta\left[\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\mp\mathbf{k}_3+n_1\mathbf{k}_1+n_2\mathbf{k}_2+(\mathbf{k}_2-\mathbf{k}_1)\right]. \tag{29}$$

The situation optimal for the considered effect is that of two electromagnetic waves that form a standing electromagnetic wave, and $\mathbf{k}_3\perp\mathbf{A}_1$. It follows then, since the electromagnetic field is transverse, that the arguments of the Bessel functions vanish, so that a nonzero contribution is made by the Bessel functions with $n_1=n_2=0$. The S-matrix element takes the form

$$S_{fi}=-(2\pi)^4\,i\frac{e\mathbf{A}_3\mathbf{p}}{2mcV}\frac{e^2A_1^2(\mathbf{k}_2-\mathbf{k}_1)^2}{(2mc^2)^2\hbar(\mathbf{v}(\mathbf{k}_2-\mathbf{k}_1))^2}\delta\left(\frac{E'-E}{\hbar}\pm\omega_3\right)$$

$$\times\delta\left(\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\mp\mathbf{k}_3\pm2\mathbf{k}_1\right). \tag{30}$$

From the energy and momentum conservation laws inherent in the delta functions of (30), we obtain an expression for the test-wave frequency at which amplification is possible

$$\omega_3=2(v/c)\omega_1\cos\theta, \tag{31}$$

where θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{k}_1$ ($0\leqslant\theta\leqslant\pi/2$). Knowing the S-matrix element we can obtain the total probabilities of the induced emission and absorption processes. The total probabilities must be averaged over the initial energy distribution $f(E)$ of the electrons in the beam. We assume that the function $f(E)$ is normalized to unity, i.e., $\int f(E)dE=1$, and that the function $f(E)$ has a width $\Delta E\ll E$.

We write then the expression for the emission and absorption differential probabilities in the form

$$dW_{e,a}=|S_{fi}^{(e,a)}|^2\frac{dp'}{(2\pi)^3}Vf(E)dE. \tag{32}$$

Substituting (30) in (32) and integrating the result with respect to $d\mathbf{p}'$, we get

$$W_e^{(1)}=2\pi t\left(\frac{e\mathbf{A}_3\mathbf{p}}{2mc}\right)^2\left[\frac{e^2A_1^2}{(2mc^2)^2}\right]^2 \times\left(\frac{c}{v\cos\theta}\right)^4\int\delta\left(\frac{E'-E}{\hbar}+\omega_3\right)f(E)\,dE,$$

$$W_a^{(1)}=2\pi t\left(\frac{e\mathbf{A}_3\mathbf{p}}{2mc}\right)^2\left[\frac{e^2A_1^2}{(2mc^2)^2}\right]^2 \times\left(\frac{c}{v\cos\theta}\right)^4\int\delta\left(\frac{E'-E}{\hbar}-\omega_3\right)f(E)\,dE. \quad (33)$$

We assume next that **p** is directed along $\mathbf{A}_3$. The δ functions in (33) can be represented in the form

$$\delta(E'-E\pm\omega_3)=\frac{\delta(E-E_{e,a})}{|\partial E'/\partial E-1|}=\frac{\delta(E-E_{e,a})pc}{2\hbar\omega_1\cos\theta}, \quad (34)$$

where $E_{e,a}=E_0\pm\delta E$ is the energy of the electrons that emit or absorb a photon of a given frequency ω_3:

$$E_0=\frac{mc^2}{8\cos^2\theta}\left(\frac{\omega_3}{\omega_1}\right)^2,\quad \delta E=\frac{\hbar\omega_3}{2\cos^2\theta},\quad \delta E\ll\Delta E, \quad (35)$$

where ΔE is the width of the distribution function.

Integrating with respect to dE in (33) and using (34), we obtain for the difference between the total probabilities for emission and absorption of a photon of frequency ω_3:

$$\Delta W^{(1)}=4\pi t\frac{(eA_3)^2E^2}{mc^2\omega_3}\delta E\frac{df}{dE}\left[\frac{e^2A_1^2}{(2mc^2)^2}\right]^2\left(\frac{c}{v\cos\theta}\right)^4. \quad (36)$$

In the derivation of (36) we used the approximate equality

$$f(E_e)-f(E_a)=2\delta E\,df/dE, \quad (37)$$

the derivative df/dE is taken at the point $E=E_0$.

The gain of the electromagnetic wave is determined in the linear regime by the relation

$$G^{(1)}=8\pi(j/e)\Delta W^{(1)}\hbar\omega_3/c\mathscr{E}_3^2; \quad (38)$$

where $\mathscr{E}_3$ is the amplitude of the electromagnetic field intensity of the amplified wave and j is the electron-current density.

THE GAIN

Substituting in (38) the expression for $\Delta W^{(1)}$ from (36), we get

$$G^{(1)}=16\pi^2(j/e)r_0t\lambda\!\!\!^{-}_3E^2\left[\frac{e^2A_1^2}{(2mc^2)^2}\right]^2\left(\frac{c}{v\cos\theta}\right)^4\frac{1}{\cos^2\theta}\frac{df}{dE}, \quad (39)$$

where r_0 is the classical electron radius. Relation (39) was obtained by using (35). We make one remark. Expression (39) for the gain is valid if $\Delta E/E>1/\omega_1t$ (t is the interaction time). Further estimates show that this criterion is valid. In the case of the inverse condition $\Delta E/E<1/\omega_1t$, however, the spontaneous-emission line is determined by the homogeneous width connected with the finite region of the interaction between the electrons and the field. The formal transition from df/dE to homogeneous broadening is analogous to that in Ref. 5.

If the distribution function width is $\Delta E\gg\hbar\omega_3/2\cos^2\theta$, we have $df/dE\approx1/(\Delta E)^2$. Taking this fact into account, the expression for the gain can be written in the form

$$G^{(1)}=16\pi^2(j/e)r_0t\lambda\!\!\!^{-}_3\left(\frac{E}{\Delta E}\right)^2\left(\frac{e\mathscr{E}_1\lambda\!\!\!^{-}_1}{2mc^2}\right)^4\left(\frac{c}{v}\right)^4\frac{1}{\cos^6\theta}, \quad (40)$$

where $\mathscr{E}_1$ is the amplitude of the electric field strength.

The expression for the gain was obtained subject to validity of relations (20). These relations are violated in the classical limit $\hbar\omega_1\to0$.

CONCLUSION

The foregoing result can be interpreted as follows. Nonrelativisitc electrons interacting with a standing electromagnetic wave can absorb photons from one wave and transfer them to another. This rescattering changes the electron energy by an amount of the order of the recoil energy $2\hbar\mathbf{k}\cdot\mathbf{v}$, and the energy of the quantum emitted in the recoil process can be of the same order.

We present a numerical example. We estimate the gain in the case of an amplified wavelength $\lambda\!\!\!^{-}_3=50\ \mu$m. We choose the electron-beam parameters to be: $j=30$ A/cm^2, $(\Delta E/E)=10^{-3}$, $v/c=0.66$, and d (the beam diameter) $=1$ mm.

Let the standing pump wave be produced by a CO_2 laser with the following parameters: $\lambda^1=10.6\,\mu$m, E_0 (energy in the pulse) $=1$ kJ, and pulse duration $\Delta t=10$ ns. In this case $\cos\theta=0.15$ and the gain $G^{(1)}$ calculated from Eq. (40) is found to be unity.

It can be easily verified that the aid of the above parameters that in our case $\Delta E/E>1/\omega_1t$, therefore the decisive factor is the inhomogeneous broadening which takes into account the initial energy distribution of the electrons in the beam. One remark is in order.

The gain was obtained in the single-particle approximation. For this approximation to be valid it is necessary (see, e.g., Ref. 6) to satisfy the relation

$$t(4\pi N_ec^2/m)^{1/2}\ll1. \quad (41)$$

It is easily seen that the condition (41) with our parameters is met.

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[1)]These criteria follow from the form obtained for Ψ_{int}, see (10) and (20″).

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