Excitation of transverse sound in a two-dimensional electron crystal

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An analysis is made of various methods for exciting shear (transverse) vibrations in a two-dimensional electron crystal. In one of these methods a coil of an oscillatory circuit is magnetically coupled to currents which appear in a crystal as a result of its vibration. In the other case the excitation of transverse modes of an electron crystal is by a parametric resonance in an external homogeneous magnetic field which oscillates with time. The two methods are suitable for studies of Wigner crystallization on the surface of helium and similar crystallization in inversion layers of semiconductors.

In spite of the discovery of Wigner crystallization in a two-dimensional system above liquid helium,[1–3] the problem of the existence of an electron crystal in other two-dimensional charged systems (for example, inversion layers) has not yet been solved. It is therefore desirable to consider alternative methods for exciting transverse sound in a $2D$ electron system which could be used more generally. Some of them are discussed below.

The existence of a shear rigidity in a two-dimensional electron disk of radius R should be manifested when, for example, attempts are made to rotate this disk at a variable velocity by means of external forces distributed nonlinearly along the disk radius. Such a distribution of forces is shown schematically in Fig. 1 and can be brought about by a planar coil of radius $R_1 > R$ connected to an oscillatory circuit of suitable frequency carrying an alternating current $I(t)$; such a coil surrounds the electron disk along its perimeter and may be used also as a guard ring of the kind usually employed in experiments on surface electrons above helium. An alternating magnetic field created by the current $I(t)$ then induces an alternating electric field with an azimuthal component $E_\varphi(r)$ dependent on the distance r from the center of the disk. The distribution of forces with a surface density $enE_\varphi(r)$, where e is the electron charge and n is the surface density of electrons, excites shear vibrations of the electron disk. We shall assume that the displacements due to such vibrations have only the azimuthal component $u_\varphi(r,t)$. In fact, the existence of an azimuthal velocity creates a Lorentz force in the radial direction, but because of the very low compressibility of a two-dimensional Coulomb system we can ignore the radial component of the displacement (which alters the charge density) and, consequently, we can ignore the Lorentz force.

A system comprising a coupled oscillatory circuit and electron disk can be described conveniently by the Lagrangian formalism using an electromechanical analogy.[4] The role of the kinetic energy is then played by the sum of the magnetic energy present in the system of currents and mechanical kinetic energy of electrons in the disk, and the effective potential energy is composed of the electrical energy of the capacitor in the oscillatory circuit and the elastic energy of the electron disk deformed in the process of vibrations. The charge Q on the capacitor in the oscillatory circuit can be used as a generalized coordinate and the disk can be described by the coefficients U_i $(i = 0,1,2,...)$ in the expansion of the azimuthal displacement

$$u_\varphi(r,t)=\sum_{i=0}^{\infty} U_i(t)\,u_\varphi^{(i)}(x), \qquad x=r/R \tag{1}$$

in a system of functions that appear in the solution of the problem of radially symmetric shear vibrations of an infinite cylinder made of an incompressible material considered in the conventional theory of elasticity:

$$u_\varphi^{(0)}(x)=x, \qquad u_\varphi^{(i)}(x)=J_1(\lambda_i x). \tag{2}$$

where $J_\nu(x)$ is a Bessel function; λ_i are positive roots of $J_2(x) = 0$ which represents a compact form of the condition for the absence of tangential stresses on the free surface of a cylinder. The calculations then yield the following results for the Lagrangians of the disk, circuit, and their interaction:

$$L_d=\sum_{i=0}^{\infty}\frac{M_i\dot U_i^2}{2}+\sum_{i,j=0}^{\infty} m_{ij}\dot U_i\dot U_j-\sum_{i=1}^{\infty}\frac{M_i\omega_i^2U_i^2}{2}, \tag{4}$$

$$L_c=L_0\dot Q^2/2c^2-Q^2/2C_0, \tag{5}$$

$$L_{ih}=\sum_{i=0}^{\infty} p_i\dot U_i\dot Q. \tag{6}$$

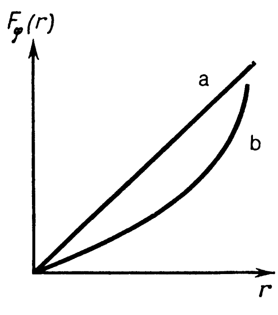

FIG. 1. a) Linear distribution of azimuthal torques resulting in rotation of a disk as a whole; b) nonlinear radial distribution of the forces along exciting shear vibrations in a disk.

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Here,

$$M_i=2mN\alpha_i,\quad \alpha_i=2\pi\int_0^1 [u_\varphi^{(i)}(x)]^2 x\,dx; \tag{7}$$

$$m_{ij}=\frac{2e^2N}{\pi Rc^2}\beta_{ij},$$

$$\beta_{ij}=2\pi\int_0^1 dx\int_0^1 dy\int_0^{2\pi} d\varphi\frac{u_\varphi^{(i)}(x)u_\varphi^{(j)}(y)xy\cos\varphi}{(x^2+y^2-2xy\cos\varphi)^{1/2}}; \tag{8}$$

$$p_i=\frac{2eN}{c^2}\gamma_i,\quad \gamma_i=2\pi\int_0^1 dx\int_0^{2\pi} d\varphi\frac{u_\varphi^{(i)}(x)x\cos\varphi}{(x^2+a^2-2ax\cos\varphi)^{1/2}}; \tag{9}$$

$a=R_1/R>1$; $\omega_i=\lambda_i c_t/R=k_i c_i$; $c_t^2=\mu/mn$ is the velocity of transverse sound in an electron crystal; μ is the shear modulus of this crystal; m is the electron mass; c is the velocity of light; $N=\pi nR^2$ is the total number of electrons in the disk; L_0 and C_0 are the inductance and capacitance of the components of the oscillatory circuit; in the case of a coil consisting of one turn of a thin (radius $b\ll R_1$) wire, we have

$$L_0\approx 4\pi R_1\left(\ln\frac{8R_1}{b}-\frac{7}{4}\right). \tag{10}$$

The nondiagonal terms in the expressions for the Lagrangian of the disk given by Eq. (4) describe the magnetic interaction of the currents created by vibrations of a disk consisting of charged particles. The sum in Eq. (4), corresponding to the potential energy, does not have a term with $i=0$, since the zeroth mode represents the rotation of the disk as a whole and therefore makes no contribution to the elastic deformation energy (i.e., $\omega_0=0$).

If we diagonalize L_d with the aid of a suitable matrix D, i.e., if we adopt new coordinates $V_i=D_i^k U_k$, we can write L in the form

$$L=\frac{M_0\dot V_0^2}{2}+\sum_{i=1}^{\infty}\left(\frac{M_i\dot V_i^2}{2}-\frac{M_i\omega_i'^2V_i^2}{2}\right)+\frac{L_0\dot Q^2}{2c^2}-\frac{Q^2}{2C_0}+\sum_{i=0}^{\infty}p_i'\dot V_i\dot Q, \tag{11}$$

where ω_i' are the eigenfrequencies of the vibrations of the disk calculated allowing for the magnetic interaction of the currents in the disk and $p_i'=(D^{-1})_i^k p_k$. Using the expression obtained in this way for L, we readily derive the following equation for the natural frequencies of the oscillatory circuit + disk system:

$$\frac{\omega^2}{\Omega^2}=1+\sum_{i=0}^{\infty}\frac{\omega^2}{\nu_i^2}\frac{\omega^2}{\omega^2-\omega_i'^2}, \tag{12}$$

where $\Omega^2=c^2/L_0C_0$ is the eigenfrequency of the circuit in the absence of the disk; $\nu_i^2=M_i/(C_0p_i'^2)$; it is assumed here that $\omega_0'=0$.

The solution of this equation is an infinite-valued function $\tilde\omega(\Omega)$, the first few branches of which are shown in Fig. 2. The branches are labeled so that for $p_i'\to 0 (i=0,1,2,...)$,

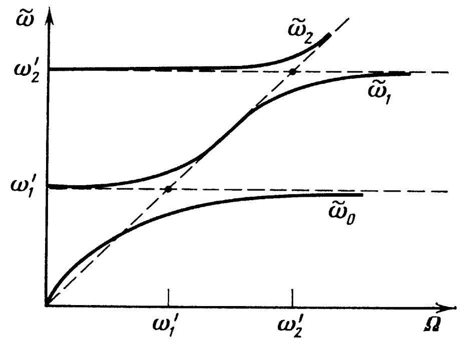

FIG. 2. Schematic representation of a multivalued function $\tilde\omega(\Omega)$ which is the solution of Eq. (12) and gives the eigenfrequencies of oscillations of the circuit + electron disk system as a function of the eigenfrequency of the circuit Ω.

the branch $\tilde\omega_0(\Omega)$ reduces the Ω and $\tilde\omega_i(\Omega)\to\omega_i'$. At low values Ω the inequality $\tilde\omega_0(\Omega)>\Omega$ is obeyed because the currents excited in a disk by the alternating magnetic field of the coil prevent the field from changing, which can be regarded as an effective reduction in the coil conductance.

In the vicinity of a point with coordinates (ω_i',ω_i') (Fig. 2) two resonance frequencies of the system are described by

$$\tilde\omega_i(\omega_i')=\omega_i'+\Delta_i,\quad \tilde\omega_{i-1}=\omega_i'+\Delta_{i-1}.$$

Assuming that $\Delta_i,\Delta_{i-1}\ll\omega_i'$ and retaining on this basis only the appropriate resonance term in the sum of Eq. (12), we can find approximately Δ_i and Δ_{i-1}, and also the total splitting δ_i:

$$\delta_i=\tilde\omega_i(\omega_i')-\tilde\omega_{i-1}(\omega_i')=\Delta_i-\Delta_{i-1},$$
$$\Delta_i\approx\omega_i'\frac{\omega_i'}{2\nu_i'},\quad \Delta_{i-1}\approx-\Delta_i, \tag{13}$$
$$\frac{\delta_i}{\omega_i'}\approx\frac{\omega_i'}{\nu_i}=cp_i'(L_0M_i)^{-1/2}=\frac{cp_i'}{(2mN\alpha_iL_0)^{1/2}}.$$

The quantity δ_i depends on p_i' which is a parameter governing the strength of the interaction of the oscillatory circuit with the ith shear vibration mode of the disk.

The quantities p_i' and ω_i' calculated explicitly require finding the matrix D which diagonalizes the Lagrangian of the disk. This is generally a difficult problem. The situation simplifies when the ratio of the coefficients m_{ij} and M_i, governed by the formulas (7) and (8), is small:

$$\frac{m_{ij}}{M_i}\ll\frac{r_eN}{\pi R}=\varepsilon\ll 1, \tag{14}$$

where $r_e=e^2/mc^2\approx 3\times 10^{-13}$ cm is the classical electron radius. The inequality (14) is obeyed in the case of the density of surface electrons $h\ll 10^{13}$ cm^{-2} for R of the order of several centimeters, which is always true in the case of electrons above helium and is readily realized in inversion layers of semiconductors.

If $\varepsilon\ll 1$, it is possible to ignore the nondiagonal terms in the Lagrangian (4) and substitute in Eq. (11) subject to Eq. (9)

$$p_i'\approx p_i=\frac{2eN}{c^2}\gamma_i,\quad \omega_i'\approx\omega_i.$$

Consequently, the splitting δ_i of Eq. (13) can be described by the approximate formula

$$\frac{\delta_i}{\omega_i} \simeq 2^{1/2}\gamma_i\alpha_i^{-1/2}(r_eN/L_0)^{1/2} \sim (r_eN/L_0)^{1/2} \qquad (15)$$

[in writing down this estimate we assumed in Eq. (15) that α_i and γ_i are numerical coefficients of the same order]. In the absence of an electron disk in the crystalline state the splitting causes the resonance curve $f(\omega)$ of an oscillatory circuit to have one peak at the frequency Ω; in the presence of an electron crystal when the frequency ω reaches the vicinity of one of the frequencies ω_i, the peak splits into two and they are separated by δ_i from one another. Moreover, the resonance singularities should appear near the remaining eigenfrequencies of the disk ω_j. When the temperature is increased to the melting point of the crystal, these effects disappear and they can therefore be used to detect the liquid-crystal phase transition in the electron system. Note that an allowance for the damping in transverse vibrations of a crystal by introducing a "dissipative" stress tensor[5] has the result that

$$\Gamma_i = \mathrm{Im}\,\omega_i/\mathrm{Re}\,\omega_i \approx \nu k_i/2c_t, \qquad (16)$$

where ν is the kinetic viscosity of the crystal. If we can estimate ν from Ref. 3 for $T = 70$ mK and assume that $c_t \sim 10^6$ cm/sec (which is consistent with typical electron densities above helium), we find that in the case of low values of i for R of the order of several centimeters we obtain $\Gamma \lesssim 10^{-3}$, i.e., the damping is weak.

It is worth considering particularly the case of a linear distribution $F_\varphi(r)$ of forces along the disk radius, represented by curve a in Fig. 1. In view of the electromagnetic origin of these forces, we may conclude that the situation described by curve a in Fig. 1 corresponds to the action on the electron system of a spatially homogeneous magnetic field $H(t)$ which oscillates in time. In this case the considerations mentioned above cease to be valid, because the distribution of the surface forces linear in respect of r simply results in solid-state rotation of the disk as a whole and does not excite shear modes. However, once again the shear degrees of freedom may be excited parametrically.

Let us assume that (x,y) is the plane of the crystal and

$$\mathbf{H}(t) = \mathbf{H}_0 + \mathbf{h}(t), \qquad h(t) \ll H_0$$

is the periodically varying magnetic field along the z axis direction. The electric field $\mathbf{E}$ induced by this alternating magnetic field depends on the geometry of the system: if $\mathbf{H}(t)$ is created inside a cylindrical solenoid, then $\mathbf{E}(\mathbf{r}) = [\mathbf{r},\dot{\mathbf{h}}]/2c$, where $\mathbf{r}$ is the radius vector in the (x,y) plane, but if the field is created by a coil of rectangular cross section with sides $X \ll Y$, then far from the smaller side the field has only one component $E_y = -\dot{h}x/c$.

A. We shall first consider the cylindrical geometry case. The equation of motion of an electron at the l th lattice site is

$$m\frac{d\mathbf{v}_l}{dt} = -\frac{\partial U}{\partial \mathbf{R}_l} + \frac{e}{c}[\mathbf{v}_l, \mathbf{H}] + \frac{e}{2c}[\mathbf{R}_l, \dot{\mathbf{h}}], \qquad (17)$$

where $\mathbf{R}_l$ and $\mathbf{v}_l$ are the radius vector and the electron velocity, and U is the Coulomb energy of the electron interaction. In the case of an electron disk which is completely rigid the last term in Eq. (17) would result in rotation of the crystal as a whole with an angular velocity $\Omega(t) = -eh(t)/2mc$. In view of the very low compressibility of the electron crystal, it remains to assume that Eq. (17) has a solution $\mathbf{R}_l(t)$ which in a system of coordinates rotating at an angular velocity $\Omega(t)$ takes the form of small displacements $\boldsymbol{\rho}_l(t)$ of an electron from the l th site, where these displacements are independent of l. Then, phonons correspond to motion of the $\boldsymbol{\rho}_l(t) + \mathbf{u}_l(t)$ type and if $\boldsymbol{\rho}_l(t)$ varies smoothly with l, then for phonons of moderate wavelength, less than the distance over which $\boldsymbol{\rho}_l(t)$ changes (which is practically equal to the size of the sample), the expansion of $\partial U/\partial \mathbf{R}_l$ in terms of $\mathbf{u}_l(t)$ is identical with the expansion for an unperturbed crystal at rest. Hence $\boldsymbol{\rho}_l(t)$ drops out from the equation of motion for $\mathbf{u}_l(t)$, which becomes

$$m\frac{d^2\mathbf{u}_l}{dt^2} = -\Phi_{ll'}\mathbf{u}_{l'} + \frac{e}{c}[\dot{\mathbf{u}}_l, \mathbf{H}] + m\mathbf{u}_l(\Omega(t)\omega_c - \Omega^2(t)). \qquad (18)$$

Here $\Phi_{ll'}$ is the dynamic matrix of a two-dimensional electron crystal and $\omega_c = eH_0/mc$ is the cylotron frequency in a field H_0.

Going over to normal coordinates Q_{kl} and Q_{kt} corresponding to longitudinal and transverse phonons with a wave vector in a crystal in the absence of a magnetic field, we can rewrite Eq. (18) as follows (we have introduced here the phonon damping γ_{kl} and γ_{kt}):

$$\begin{aligned}\ddot{Q}_{kl} &= -\omega_{kl}^2Q_{kl} + \omega_c\dot{Q}_{kt} - 2\gamma_{kl}\dot{Q}_{kl} + Q_{kl}[\Omega(t)\omega_c - \Omega^2(t)],\\ \ddot{Q}_{kt} &= -\omega_{kt}^2Q_{kt} - \omega_c\dot{Q}_{kl} - 2\gamma_{kt}\dot{Q}_{kt} + Q_{kt}[\Omega(t)\omega_c - \Omega^2(t)].\end{aligned} \qquad (19)$$

The last terms on the right-hand side in the system of equations (19) are time-dependent. Consequently, we may expect that for some amplitude of the oscillating field $h(t)$ expressed in the form $h(t) = 2h_0 \cos 2\omega t$ the vibrations with wave vector k become unstable if the frequency ω is close to one of the eigenfrequencies the crystal has with this wave vector when $h = 0$.

Proceeding by analogy with calculations of parametric resonance in the problem of the harmonic oscillator[6] and simplifying the system (19) by dropping the terms quadratic in $\Omega^2(t)$, we obtain the following value of the threshold for h_0 in the case of given k:

$$\frac{2h_0}{H_0} = \frac{2|(\omega - i\gamma_-)^2 - \omega_-^2||(\omega - i\gamma_+)^2 - \omega_+^2|}{\omega_c^2(\omega_l^2 - \omega_t^2)}. \qquad (20)$$

Here, the index k is omitted for brevity in the case of the frequencies $\gamma_\pm$, $\omega_\pm$, ω, and ω_t; $\pm\omega_- - i\gamma_-$ and $\pm\omega_+ - i\gamma_+$ are the frequencies of phonons in the presence of a magnetic field H_0 (the expressions for their real parts in terms of ω_l, ω_t, and ω_c are given in, for example, Ref. 7) and the damping $\gamma_\pm$ can be obtained from the damping of longitudinal and transverse phonons in the absence of a magnetic field using the formula

$$\gamma_\pm \approx \frac{\omega_l^2\gamma_t + \omega_t^2\gamma_l - \omega_\pm^2(\gamma_l + \gamma_t)}{\omega_\mp^2 - \omega_\pm^2}. \qquad (21)$$

Minimization of $2h_0/H_0$ with respect to k gives the instability threshold in terms of h_0 for the system as a whole. Assuming that the vibration with $k = k_0$ is the least stable, so that $\omega_-(k_0) = \omega$ [or $\omega_+(k_0) = \omega$, depending on the value of ω], we find that in the case of the threshold (the index k_0 is omitted for all the frequencies):

$$\left(\frac{2h_0}{H_0}\right)_\pm = \frac{4\gamma_\pm\omega}{\omega_c^2}\frac{\omega_+^2-\omega_-^2}{\omega_l^2-\omega_t^2}. \qquad (22)$$

In the case of frequencies ω corresponding to values of k_0 which are not too large (and located not too close to the boundary of the Brillouin zone), when the condition $\omega_l^2 \gg \omega_t^2$ is amply satisfied and we can assume that $\gamma_t \gg \omega_t^2\gamma_l/\omega_l^2$, we find from Eqs. (21) and (22) that

$$\left(\frac{2h_0}{H_0}\right)_- \approx \frac{4\omega\omega_t}{\omega_c^2}\frac{\gamma_t}{\omega_t} \approx \frac{4\omega^2}{\omega_c^2}\left(1+\frac{\omega_c^2}{\omega_l^2}\right)^{1/2}\frac{\gamma_t}{\omega_t}. \qquad (23)$$

For a typical electron density above helium amounting to $n \sim 5\times10^8$ cm^{-2} and $k \sim 10^3$ cm^{-1} in a field $H \sim 10^3$ Oe (which corresponds to $\omega_t \sim 10^9$ sec^{-1}, $\omega_l \sim \omega_c \sim 3\times10^{10}$ sec^{-1}, and $\omega_- = \omega \sim 7\times10^8$ sec^{-1}), we obtain

$$(2h_0/H_0)_- \approx 10^{-2}\gamma_t/\omega_t \ll 1.$$

One should mention that the value $\gamma_t/\omega_t = 1/4$ for $k \sim 520$ cm^{-1} is obtained in Ref. 3 for temperatures $T \sim 70$ mK.

We must also draw attention to the following important fact. Experiments on two-dimensional electron systems usually have a metal electrode located parallel to the electron system. Eddy currents created in the metal through the variation of $H(t)$ may distort considerably the field configuration assumed above. This stray effect can be eliminated by making a metal electrode in the form of a fan of strips which are in contact only at the center of the electrode. In this case the electrode remains an equipotential surface but it does not transmit eddy currents in the azimuthal direction.

B. In the rectangular geometry case the equation of motion of the l th electron is obtained by replacing the last term in Eq. (17) with $(-e\dot{h}x/c)\mathbf{e}_y$, where $\mathbf{e}_y$ is a unit vector in the direction of the y axis. Eliminating the part of the displacement $\boldsymbol{\rho}_l(t)$ which varies slowly with l and is due to this force (following a similar reasoning as that adopted above), we obtain the following system of equations

$$\ddot{Q}_{kl}+2\gamma_{kl}\dot{Q}_{kl}+\omega_{kl}^2Q_{kl}-\omega_c(t)\dot{Q}_{kt}=0,$$

$$\ddot{Q}_{kt}+2\gamma_{kt}\dot{Q}_{kt}+\omega_{kt}^2Q_{kt}+\frac{d}{dt}\{\omega_c(t)Q_{kl}(t)\}=0,$$

where $\omega_c(t) = e(H_0 + h(t))/mc$.

Calculations indicate that in this case the threshold of parametric excitation of the system is equal to the threshold in the cylindrical symmetry case, multiplied by the factor $(\omega_l^2-\omega_t^2)/\omega^2$.

It follows that changes in the threshold values of h_0/H_0 allow us to determine the damping of transverse phonons in an electron crystal; the existence of a parametric resonance at frequencies lying within the zone of values $\omega_-(k)$ can be regarded as an indication of crystallization of electrons.

The proposed method for the excitation of transverse waves in a two-dimensional electron system may be realized also in various two-dimensional charged systems in semiconductors. It is most natural for this to occur in periodic heterostructures when there are no auxiliary metal electrodes forming a homogeneous two-dimensional system, so that there are no problems with the special form of the metal gate.

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