Structure of vacuum in chiral supersymmetric quantum electrodynamics

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An effective Hamiltonian is found for supersymmetric quantum electrodynamics defined in a finite volume. When the right and left fields appear in the theory nonsymmetrically (but so that anomalies cancel out), this Hamiltonian turns out to be nontrivial and describes the motion of a particle in the "ionic crystal" consisting of magnetic charges of different sign with an additional scalar potential of the form $U = K^2/2$, where $\partial_i K = \mathscr{H}_i$. Despite the nontrivial form of the effective Hamiltonian, supersymmetry remains unbroken.

1. INTRODUCTION

One of the most acute problems in theoretical elementary-particle physics is the question of supersymmetry breaking, which is generally believed to occur in nature in one form or another at high enough energies. The most attractive mechanism of supersymmetry breaking is spontaneous breaking by dynamic effects, not manifest at the "tree" level. A well-known example of this type of breaking occurs in the Witten quantum mechanics with superpotential of the form $V(X) = \lambda(X^3/3 - a^2X)$ (Ref. 1). However, for a long time, searches for the dynamic breaking of supersymmetry in four-dimensional field theories remained unsuccessful. Thus, this type of breaking does not occur in supersymmetric Yang-Mills theories without matter.[2] The situation is no better in gauge theories with extended supersymmetry, or in supersymmetric QCD with different numbers of right and left multiplets of matter in the fundamental representation.

However, nontrivial field theories in which supersymmetry is dynamically broken have recently been found.[3,4] The simplest example of this type of theory is the SU(5) supersymmetric Yang-Mills theory that includes chiral matter fields in the 5_L and 10_R representations. The triangular anomaly cancels out in this theory, but right and left fields appear in different representations, so that matter remains massless. It is precisely these "theories with irremovable chirality" that were not fully examined by Witten in his well-known paper, in which he analyzed vacuum states for a wide class of theories.[2]

The arguments advanced in Refs. 3 and 4 in favor of supersymmetry breaking in this theory are indirect. Instanton calculations of particular correlators support the formation of a gluino condensate $\langle \lambda^a_\alpha \lambda^{a\alpha} \rangle_0$ suggesting spontaneous breaking of chiral symmetry. If supersymmetry is not broken, the pseudoscalar goldstone should be part of the chiral multiplet, which also contains a massless scalar particle. However, there is no place for this particle in the present case because the theory has no valleys, i.e., directions in the space of scalar fields for which the classical potential vanishes (a detailed discussion of this range of questions can be found in the review in Ref. 5).

Our ultimate aim is to examine directly the question of supersymmetry breaking in different chiral theories by constructing the effective Hamiltonian and then analyzing the spectrum of states of lowest energy by analogy with Witten's analysis of nonchiral theories. We shall investigate the simplest example of chiral supersymmetric QCD. Unfortunately, our original hopes have not been justified, and supersymmetry remains unbroken in this theory. The effective Hamiltonian has, however, turned out to be highly nontrivial and does not reduce to free motion, as was the case for nonchiral theories. The methods developed below can be used in the analysis of more complicated chiral theories, including those in which supersymmetry is definitely broken.

In Section 1, we examine a chiral supersymmetric QCD in the quantum-mechanical limit in which the fields are assumed to be independent of spatial coordinates. We obtain the effective Hamiltonian that turns out to describe the motion of a charged particle in the field of a Dirac monopole with an additional scalar potential of the form $\sim 1/r^2$. States of arbitrarily low energy are present in this problem, but there are no states with zero energy. In Section 3, we turn to field theory in a finite volume. The effective Hamiltonian for the zeroth harmonic of the gauge field, sought by precisely the same method, then describes motion in a lattice of monopoles. It is interesting that the final expression for H^{eff} can be written only for a theory free of anomalies (e.g., a theory with eight left fermions of unit charge and one right fermion with twice this charge). The lattice then contains monopoles of different sign, and the average magnetic space charge is zero. Analysis of the effective Hamiltonian for an anomaly-free theory in a finite volume shows that the Witten index is then nonzero and supersymmetry is not broken. A brief summary of all the results is given in Section 4.

2. QUANTUM MECHANICS OF SUPERSYMMETRIC QCD

Consider a theory describing the interaction of a photon A_i and a photino λ_α with one chiral left field (φ, ψ_α) in the limit where the fields do not depend on spatial coordinates [in one sense, a four-dimensional theory with one left field does not exist because of the anomaly, but in $(0+1)$-dimensional situations, this anomaly is not manifest]. The total hypercharges and the Hamiltonian for this theory are[6]

$$\begin{aligned}\hat{Q}_\alpha &= 2^{-1/2}[-i\hat{P}_k(\sigma_k)_\alpha{}^\gamma - e\varphi\bar{\varphi}\delta_\alpha{}^\gamma]\lambda_\gamma + [-\hat{\pi}_\varphi\delta_\alpha{}^\gamma \\ &\quad + ie\bar{\varphi}A_k(\sigma_k)_\alpha{}^\gamma]\psi_\gamma, \\ \hat{\bar{Q}}^\beta &= 2^{-1/2}\bar{\lambda}^\delta[i\hat{P}_k(\sigma_k)_\delta{}^\beta - e\varphi\bar{\varphi}\delta_\delta{}^\beta] + \bar{\psi}^\delta[-\hat{\bar{\pi}}_\varphi\delta_\delta{}^\beta \\ &\quad - ie\varphi A_k(\sigma_k)_\delta{}^\beta],\end{aligned} \tag{1}$$

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$$\hat{H}=\hat{P}_i\hat{P}_i/2+\hat{\pi}_\varphi\hat{\bar{\pi}}_\varphi+e^2(\varphi\bar{\varphi})^2/2+e^2\varphi\bar{\varphi}A_i^2 +i2^{1/2}e(\bar{\lambda}\psi\bar{\varphi}-\bar{\psi}\lambda\varphi)-eA_k\bar{\psi}\sigma_k\psi, \qquad (2)$$

where ψ_α and λ_α are two-component Weyl spinors and

$$\hat{P}_i=-i\partial/\partial A_i,\quad \hat{\pi}_\varphi=-i\partial/\partial\varphi,\quad \hat{\bar{\pi}}_\varphi=-i\partial/\partial\bar{\varphi},$$

$$\bar{\lambda}^\alpha=\partial/\partial\lambda_\alpha,\quad \bar{\psi}^\alpha=\partial/\partial\psi_\alpha.$$

The supersymmetry algebra has the form

$$\{\hat{Q}_\alpha,\hat{\bar{Q}}^\beta\}_+=\delta_\alpha{}^\beta\hat{H}-eA_k(\sigma_k)_\alpha{}^\beta\hat{G}, \qquad (3)$$

where

$$\hat{G}=i(\hat{\bar{\pi}}_\varphi\bar{\varphi}-\hat{\pi}_\varphi\varphi)+\bar{\psi}^\alpha\psi_\alpha-1 \qquad (4)$$

is the coupling that commutes with the hypercharge and the Hamiltonian. The significance of (3) is that the superposition of the transformed supersymmetry yields both a translation and a gauge transformation. On physical states $\hat{G}\Psi=0$, so that the second term on the right of (3) does not contribute to the physical matrix elements. We note that the $\hat{G}$ that follows from (3) is automatically "correctly ordered" in the sense of Ref. 6. It is also clear that all the remaining $\hat{G}$ that differ from (4) by a constant do not yield the correct supersymmetry algebra on the space $\hat{G}\Psi=0$.

It is clear that the classical potential following from the Hamiltonian (2) vanishes when $\varphi=\bar{\varphi}=0$ for any A_i, i.e., there is a valley along which the wave function tends to "spread out." In other words, in our problem, **A** is a "slow" variable and the lowest-energy states of the system are associated precisely with the excitation of this degree of freedom. The charged fields φ and $\bar{\varphi}$, on the other hand, are "fast" variables and their excitation energies are high. The Born-Oppenheimer approximation, discussed in detail in Ref. 2 in relation to field theories, is valid in this case.

It is therefore assumed that $|\mathbf{A}|\gg|\varphi|$. We shall classify the terms in the hypercharges and in the Hamiltonians in terms of the parameter $|\varphi|/|\mathbf{A}|$. We have $Q_\alpha=Q_\alpha^{(0)}+Q_\alpha^{(1)}$, where

$$\hat{Q}_\alpha^{(0)}=[-\hat{\pi}_\alpha\delta_\alpha{}^\gamma+ie\bar{\varphi}A_k(\sigma_k)_\alpha{}^\gamma]\psi_\gamma, \qquad (5a)$$

$$\hat{Q}_\alpha^{(1)}=2^{-1/2}[-i\hat{P}_k(\sigma_k)_\alpha{}^\gamma-e\varphi\bar{\varphi}\delta_\alpha{}^\gamma]\lambda_\gamma, \qquad (5b)$$

and, similarly, $\hat{H}=\hat{H}_0+\hat{H}_1+\hat{H}_2$, where

$$\hat{H}_0=\hat{\pi}_\varphi\hat{\bar{\pi}}_\varphi+e^2A_i^2\varphi\bar{\varphi}-eA_i\bar{\psi}\sigma_i\psi, \qquad (6a)$$

$$\hat{H}_1=ie2^{1/2}(\bar{\lambda}\psi\bar{\varphi}-\bar{\psi}\lambda\varphi), \qquad (6b)$$

$$\hat{H}_2=\hat{P}_i\hat{P}_i/2+e^2(\varphi\bar{\varphi})^2/2. \qquad (6c)$$

The Hamiltonian $\hat{H}_0$ (in which A_i appears as a parameter) is quadratic in the charged fields φ and ψ. It is not difficult to find the explicit form of the vacuum wave function of this Hamiltonian (its energy is zero):

$$\Phi_\mathbf{A}^{\rm vac}(\varphi,\psi)=CA^{1/2}\exp(-eA\varphi\bar{\varphi})\psi^\alpha w_\alpha(\mathbf{A}/A), \qquad (7)$$

where $A=|\mathbf{A}|$ and w_α is a spinor depending on the direction of **A** and is defined (to within a phase factor) by the relation $(A_k/A)(\sigma_k)_\alpha^\beta w_\beta=w_\alpha$. We shall take it in the form

$$w_\alpha=\begin{pmatrix} e^{-i\varphi}\cos\theta/2\\ \sin\theta/2\end{pmatrix}, \qquad (8)$$

where θ and φ are angles that parametrize the direction of **A**. The freedom in choosing the common phase will be explained below. The factor $A^{1/2}$ appears in (7) for convenience, in order to ensure that the normalization integral of $|\Phi_A|^2$ over the charged field does not depend on A. The characteristic angles φ in the wave function (7) are given by $|\varphi_{\rm char}|\sim(eA)^{-1/2}$. It is clear that the approximation $A\gg|\varphi|$ is self-consistent for $\gamma=1/eA^3\ll1$. The quantity γ is, in fact, the true expansion parameter in the effective Hamiltonian.

The complete wave function of the system is

$$\psi^{\rm compl}=f_0(\mathbf{A},\lambda_\alpha)\Phi_\mathbf{A}^{\rm vac}(\varphi,\psi)+\sum_n f_n(\mathbf{A},\lambda_\alpha)\Phi_\mathbf{A}{}^n(\varphi,\psi), \qquad (9)$$

where Φ_A^n are the wave functions of excited states of $\hat{H}_0$. The effective Hamiltonian acting on $f_0(\mathbf{A},\lambda_\alpha)$ is given by

$$\hat{H}^{\rm eff}=\langle\hat{H}_2\rangle_{00}-\sum_n(\langle\hat{H}_1\rangle_{0n}\langle\hat{H}_1\rangle_{n0}/E_n)+\dots \qquad (10)$$

In our case, in which the problem is supersymmetric, we can also define an effective hypercharge $\hat{Q}_\alpha^{\rm eff}=\langle\hat{Q}^{(1)}\rangle_{00}+\dots$, so that

$$\{\hat{Q}_\alpha^{\rm eff},\hat{\bar{Q}}_{\rm eff}^\beta\}=\delta_\alpha{}^\beta\hat{H}^{\rm eff} \qquad (11)$$

[We note that (11) does not contain a component due to $\hat{G}$ because, in the quantum-mechanical limit, the gauge transformation rotates only the charged fields.] At the end of the present paper, we shall produce arguments showing that higher-order perturbation-theory terms are absent from $\hat{Q}^{\rm eff}$ and $\hat{H}^{\rm eff}$. Direct calculation yields

$$\hat{Q}_\alpha^{\rm eff}=-2^{-1/2}\lambda_\gamma[i(\sigma_k)_\alpha{}^\gamma(\hat{P}_k-\mathscr{A}_k)+\delta_\alpha{}^\gamma/2A],$$
$$\hat{\bar{Q}}_{\rm eff}^\beta=2^{-1/2}\bar{\lambda}^\delta[i(\sigma_k)_\delta{}^\beta(\hat{P}_k-\mathscr{A}_k)-\delta_\delta{}^\beta/2A], \qquad (12)$$

$$\hat{H}^{\rm eff}=\frac{1}{2}(\hat{\mathbf{P}}-\vec{\mathscr{A}})^2+\frac{1}{8A^2}+\frac{\mathbf{A}}{8A^3}\cdot\bar{\lambda}\sigma\lambda, \qquad (13)$$

where $\mathscr{A}$ is a vector function of the fields **A**, which is identical with the vector potential of a Dirac monopole placed at the origin $\mathbf{A}=0$, with a filament running along the positive direction of the z axis, chosen in accordance with (8). Other choices of the phase of the spinor w_α correspond to other gauges for the vector potential $\mathscr{A}$! The last term in the Hamiltonian will work for wave functions in the sector with fermion charge $F=1$, and describes the interaction between the magnetic field $\mathscr{H}=-\mathbf{A}/2A^3$ and a spin $\frac{1}{2}$ particle, the gyromagnetic ratio being greater by a factor of two than the Dirac value. (We note that, in the $F=0$ and $F=2$ sectors, the Hamiltonian corresponds to the scattering of a scalar particle.) The scalar potential $1/8A^2$ is necessary to ensure the supersymmetry of the Hamiltonian. The Hamiltonian (13) belongs to the $N=2$ class of supersymmetric Hamiltonians constructed by De Grombrugghe and Rittenberg:[7]

$$\hat{H}_{\rm GR}={}^1/_2(\hat{\mathbf{P}}-\vec{\mathscr{A}})^2+{}^1/_2K^2-\vec{\mathscr{H}}\bar{\lambda}\sigma\lambda, \qquad (14)$$

where $\mathscr{H}=\nabla K$. In the present case $K=1/2A$; another example of a Hamiltonian belonging to this class was examined in Ref. 8.

The Hamiltonian (13) was among those analyzed in Ref. 9. Its spectrum is continuous and includes states with arbitrarily low energy, but there are no zero-energy states. This is particularly clear in the case of the boson spectrum, for which the Hamiltonian is positive-definite and, as the state energy is reduced, the wave function becomes localized at larger distances in the space of **A**.

Consequently, supersymmetry is spontaneously broken in this problem, but only in a certain arbitrary sense, since states of arbitrarily low energy are present.

Consider now the quantum mechanics obtained by reduction to the $(0+1)$-space of the anomaly-free chiral supersymmetric QCD. As an example, we take the theory with eight left fields of matter of charge 1 and one right field of charge 2. The sum of the cubes of the charges of the left fields is now equal to the cube of the charge of the right field, and the anomaly cancels out. The effective Hamiltonian in this problem is also given by (14), where $\mathscr{A}$ is the vector potential of the monopole of charge $-N_L + N_R = -7$ and $K = 7/2A$.

However, it is important to note that, in this problem, there is not only the valley along **A** but also a valley in the space of the scalar fields, defined by

$$\sum_{f=1}^{8} \varphi_f \bar{\varphi}_f = 2\chi\bar{\chi}, \tag{15}$$

where φ_j are scalar components of the left multiplets S_f and $\bar{\chi}$ is the scalar component of the right multiplet T. This valley can be disposed of by blocking it with the Yukawa term in the superpotential

$$V_{Yuk} = h \sum_f S_f S_f \bar{T}.$$

If the Yukawa constant h is of the order of or greater than the gauge constant, the dynamics of the system is wholly determined by the "gauge valley" along **A**, and is described by the effective Hamiltonian (14).

In the usual supersymmetric QCD, with one right and one left multiplet, the charge of the monopole, $N_R - N_L$, is zero and the effective Hamiltonian describes free three-dimensional motion in the space of **A** (so that supersymmetry is not broken in any sense). The valley $\varphi\bar{\varphi} = \chi\bar{\chi}$ can then be blocked by the mass term $V_m mS\bar{T}$. In the massless case, the wave function of low-lying states will spread out along the scalar valley. It is possible to find the corresponding effective Hamiltonian which, in this case, describes the free two-dimensional motion in the space $(\varphi, \bar{\varphi})$ (the modulus of χ is obtained from the condition $\varphi\bar{\varphi} = \chi\bar{\chi}$, and the phase of χ is the gauge degree of freedom which disappears once the coupling is settled).

3. CHIRAL SUPERSYMMETRY OF QCD IN A FINITE VOLUME

We now turn to the effective Hamiltonian in field theory. We employ our theory in a box of size L and subject the field to the periodic boundary conditions $(A_i, \varphi_f, ...) \times (x+L, y, z) = (A_i, \varphi_f, ...)(x, y, z)$ and, similarly, along the y and z directions. Each of the fields can be expanded into a Fourier series of the form

$$A_i(x) = \sum_{\mathbf{n}} A_i^{(\mathbf{n})} \exp(2\pi i \mathbf{n}\cdot\mathbf{x}/L),$$

$$\varphi(x) = \sum_{\mathbf{n}} \varphi^{(\mathbf{n})} \exp(2\pi i \mathbf{n}\cdot\mathbf{x}/L)$$

and so on, where $\mathbf{n}$ are three-dimensional integer vectors. We shall take the Hamiltonian in the quadratic approximation in the charged fields and in the excited harmonics A_i:

$$\begin{aligned}
\hat{H}_0 = \frac{1}{2V} \sum_{\mathbf{n}} \hat{P}_i^{(\mathbf{n})} \hat{P}_i^{(\mathbf{n})} + 2\pi^2 L \sum_{\mathbf{n}} \mathbf{n}^2 A_i^{(\mathbf{n})} \bar{A}_i^{(\mathbf{n})} \\
+ \frac{2\pi}{L} \sum_{\mathbf{n}} n_k \bar{\lambda}^{(\mathbf{n})} \sigma_k \lambda^{(\mathbf{n})} \\
+ \frac{1}{V} \sum_{\mathbf{n}} \Big[\sum_{f=1}^{8} \hat{\pi}_f^{(\mathbf{n})} \hat{\bar{\pi}}_f^{(\mathbf{n})} + \pi_\chi^{(\mathbf{n})} \bar{\pi}_\chi^{(\mathbf{n})} \Big] \\
+ V \sum_{\mathbf{n}} \Big[\Big(\frac{2\pi n_k}{L} + 2eA_k^{(0)} \Big)^2 \chi^{(\mathbf{n})} \bar{\chi}^{(\mathbf{n})} \\
+ \sum_{f=1}^{8} \Big(\frac{2\pi n_k}{L} - eA_k^{(0)} \Big)^2 \varphi_f^{(\mathbf{n})} \bar{\varphi}_f^{(\mathbf{n})} \Big] \\
+ \sum_{\mathbf{n}} \Big[\bar{\xi}^{(\mathbf{n})} \sigma_h \xi^{(\mathbf{n})} \Big(\frac{2\pi n_k}{L} + 2eA_k^{(0)} \Big) \\
+ \sum_{f=1}^{8} \bar{\psi}_f^{(\mathbf{n})} \sigma_k \psi_f^{(\mathbf{n})} \Big(\frac{2\pi n_k}{I} - eA_k^{(0)} \Big) \Big]
\end{aligned} \tag{16}$$

[cf. (6a)]. When $\mathbf{n} \neq 0$, the sum over i in the first two terms on the right of (16) is evaluated only over the transverse polarizations, i.e., for $n_i A_i^{(\mathbf{n})} = 0$. The longitudinal components $A_i^{(\mathbf{n})}$ are the gauge degrees of freedom, eliminated after the coupling is settled, and ξ_α is the fermion component of the multiplet $\bar{T}$. The Hamiltonian (16) is invariant under the transformations

$$\begin{aligned}
A_i^{(0)\prime} = A_i^{(0)} + 2\pi m_i/eL, \\
(\varphi_f^{(\mathbf{n})}, \psi_{\alpha f}^{(\mathbf{n})})' = (\varphi_f^{(\mathbf{n}_1)}, \varphi_{\alpha f}^{(\mathbf{n}_1)}), \\
(\chi^{(\mathbf{n})}, \xi_\alpha^{(\mathbf{n})})' = (\chi^{(\mathbf{n}_2)}, \xi_\alpha^{(\mathbf{n}_2)}),
\end{aligned} \tag{17}$$

where $\mathbf{m}$ is an integer vector and $\mathbf{n}_1 = \mathbf{m} - \mathbf{n}$, $\mathbf{n}_2 = \mathbf{n} + 2\mathbf{m}$. The transformation given by (17) is none other than the gauge transformation of the original field theory

$$\begin{aligned}
A_i'(\mathbf{x}) = A_i(\mathbf{x}) + \partial_i \alpha(\mathbf{x}), \\
(\varphi_f, \psi_{\alpha f})'(\mathbf{x}) = e^{ie\alpha(\mathbf{x})} (\varphi_f, \psi_{\alpha f})(\mathbf{x}), \\
(\chi, \xi_\alpha)'(\mathbf{x}) = e^{-2ie\alpha(\mathbf{x})} (\chi, \xi_\alpha)(\mathbf{x}).
\end{aligned} \tag{18}$$

In a finite volume, transformation (18) should not violate the periodic conditions, and this leads to the following restriction on $\alpha(x)$: $\alpha(\mathbf{x}) = 2\pi \mathbf{m}\cdot\mathbf{x}/L$. If we expand the field in terms of the harmonics, we are led to[1)] (17). The wave functions of physical states should be invariant under these gauge transformations. This means that the wave functions of the effective Hamiltonian corresponding to the zeroth harmonic $A_i^{(0)}$, which we aim to find, must also be periodic in the reciprocal-lattice shifts, i.e., $A_i^{(0)\prime} = A_i^{(0)} + 2\pi m_i/eL$. The valley along $A_i^{(0)}$ thus ceases to be unbounded (as was the case in quantum mechanics), the motion becomes finite, and a discrete spectrum appears (we have given a brief summary of the logic of Witten's discussion[2)]).

The vacuum eigenfunction of the Hamiltonian $\hat{H}_0$ is

$$\Phi^{vac} \sim \prod_{\mathbf{n}} \Big(|C^{(\mathbf{n})}|^4 |D^{(\mathbf{n})}|^{1/2} \exp\Big\{-eV\Big[2|D^{(\mathbf{n})}|\chi^{(\mathbf{n})}\bar{\chi}^{(\mathbf{n})} + |C^{(\mathbf{n})}| \sum_{f=1}^{8} \varphi_f^{(\mathbf{n})} \bar{\varphi}_f^{(\mathbf{n})} \Big]\Big\} \bar{w}^{\beta}\Big(\frac{D^{(\mathbf{n})}}{|D^{(\mathbf{n})}|}\Big)$$

$$\times\, \xi_\beta^{(\mathbf{n})} \prod_f \psi_f^{\alpha(\mathbf{n})} w_\alpha \Big(\frac{C^{(\mathbf{n})}}{|C^{(\mathbf{n})}|}\Big)\Big)$$

$$\times \prod_{\mathbf{n}\neq 0} \Big(\exp\{-\pi L^2 |\mathbf{n}| A_i^{(\mathbf{n})} \bar{A}_i^{(\mathbf{n})}\} \lambda^{\alpha(\mathbf{n})} w_\alpha \Big(\frac{\mathbf{n}}{|\mathbf{n}|}\Big)\Big),$$

where

$$C^{(\mathbf{n})} = 2\pi\mathbf{n}/L - e\mathbf{A}^{(0)}, \qquad D^{(\mathbf{n})} = 2\pi\mathbf{n}/L + 2e\mathbf{A}^{(0)},$$

and w_α is given by (8) and $n_i A_i^{(\mathbf{n})} = 0$.

The effective Hamiltonian is most simply found as the anticommutator of the effective hypercharges, and Q_α^{eff} is given by the matrix element of the hypercharge:

$$Q_\alpha = \int dx \Big\{ 2^{-1/2}\lambda_\gamma \Big[(H_k - iE_k)(\sigma_k)_\alpha{}^\gamma + e\Big(2\chi\bar{\chi} - \sum_f \varphi_f \bar{\varphi}_f\Big)\delta_\alpha{}^\gamma\Big] + \sum_f [-\hat{\pi}_{\varphi_f}\delta_\alpha{}^\gamma + ie\bar{\varphi}_f A_k(\sigma_k)_\alpha{}^\gamma]\psi_{f\gamma} - [\pi_\chi \delta_\alpha{}^\gamma + 2ie\bar{\chi}A_k(\sigma_k)_\alpha{}^\gamma]\xi_\gamma \Big\} \tag{20}$$

(where we recall that we must substitute the harmonic expansion of the fields, eliminate longitudinal components $n_i E_i^{(\mathbf{n})}$ from the coupling condition, and impose the gauge conditions $n_i A_i^{(\mathbf{n})} = 0$). In the lowest nontrivial order, the problem reduces to finding the average of

$$\hat{Q}_\alpha^{(1)} = (2V)^{-1/2} \lambda_\gamma^{(0)} \Big[-iE_k^{(0)} (\sigma_k)_\alpha{}^\gamma + eV \sum_{\mathbf{n}} \Big(2\chi^{(\mathbf{n})}\bar{\chi}^{(\mathbf{n})} - \sum_{f=1}^{8} \varphi_f^{(\mathbf{n})} \bar{\varphi}_f^{(\mathbf{n})} \Big)\delta^\gamma \Big]. \tag{21}$$

The higher harmonics $A_i^{(\mathbf{n})}$, and the terms arising after the coupling is settled, do not contribute to Q_α^{eff}. The result is

$$\hat{Q}_\alpha^{\text{eff}} = -\lambda_\gamma [i(\sigma_k)_\alpha{}^\gamma (\hat{P}_k^{(0)} - \mathscr{A}_k) + K\delta_\alpha{}^\gamma]/(2V)^{1/2}, \tag{22}$$

where $\mathscr{A}_k$ is the sum of the vector potentials of the set of monopoles of charge $+1$, located at the sites of the cubic lattice of edge length $\pi/|e|L$, and monopoles of charge -8, located at the sites of a lattice of edge length $2\pi/|3|L$, which is less dense by a factor of two; K is the Coulomb potential of an analogous lattice consisting of electric charges, so that $\mathscr{H}_i = \partial_i K$. The effective Hamiltonian is again given by (14) (to within the factor $1/V$).

We note that the average space charge of the lattice is then zero, i.e., the sum over the lattice sites, which determines the vector potential $\mathscr{A}$, the magnetic field $\mathscr{H}$, and the scalar potential $U = K^2/2$, is found to converge. For the anomalous theory with one left field, we would obtain a lattice consisting of monopoles of the same sign, but the effective Hamiltonian could not be defined:

The wave functions of the gauge-invariant physical states have zero quasimomentum. The spectrum of such states is discrete, with characteristic level separation $\sim e^2(L)/L$ (Ref. 2). The question is: does the spectrum of the Hamiltonian (14) contain states of zero energy? In other words: is supersymmetry broken in this theory?

Direct solution of the Schrödinger equation with a Hamiltonian (14) in which K is given by

$$K = {}^1/_2 \sum_{\mathbf{n}} \Big| \mathbf{A}^{(0)} - \frac{\pi\mathbf{n}}{eL}\Big|^{-1} - 4\sum_{\mathbf{n}} \Big| \mathbf{A}^{(0)} - \frac{2\pi\mathbf{n}}{eL}\Big|^{-1}, \tag{23}$$

is difficult. There is, however, a method that enables us to obtain information about zero modes of the Hamiltonian without solving the Schrödinger equation. It is based on the representation of the Witten index by an integral over the phase space:[10,11]

$$I_W = \lim_{\beta\to 0} \int \prod_n \frac{dx_n\, dp_n}{2\pi} \prod_a d\psi^a\, d\bar{\psi}^a \exp\{-\beta H(x_n, p_n, \psi^a, \bar{\psi}^a)\}. \tag{24}$$

Let us substitute the effective Hamiltonian (14) into this expression. Integration over the fermion variables reduces to the evaluation of the determinant

$$\text{Det}\, \|\beta(\sigma_i)_\alpha{}^\beta \mathscr{H}_i\| = -\beta^2 \vec{\mathscr{H}}^2.$$

If we also integrate with respect to the momenta, we obtain

$$I_W = \lim_{\beta\to 0} \Big[-\Big(\frac{\beta}{8\pi^3}\Big)^{1/2} \int d\mathbf{A}\, (\nabla K)^2 e^{-\beta K^2/2} \Big], \tag{25}$$

where the integral is evaluated over the unit cell of the lattice. For small β, the integral is largely determined by the singularities of K, i.e., the region of integration near the lattice sites. The contribution of each site is $-|q|/2$, where q is the charge of the monopole resident on the site. In our case, the unit cell contains a site with charge -7 and seven sites with charge $+1$. The final result is

$$I_W = -{}^7/_2 - {}^7/_2 = -7 \tag{26}$$

(it can be verified that, for any anomaly-free chiral QCD, the Witten index must be a nonzero negative integer).

The result given by (26) can also be obtained in another way. Let us modify the effective Hamiltonian (14) by adding an arbitrary constant C to the function K in (23). This modification does not change the meaning of the index. We shall make C large and negative: $-C \gg 1/eL$. The wave functions of low-lying states are then localized in the region of small $\tilde{K} = K + C$, i.e., near the sites of the lattice with unit-charge monopoles. The effect of the other sites can be neglected, and the problem reduces to the solution of the Schrödinger equation with the Hamiltonian

$$\hat{H}_n = \frac{1}{2}(\hat{\mathbf{P}}-\vec{\mathscr{A}})^2 + \frac{1}{2}\left(\frac{1}{2\tilde{A}}+C\right)^2 + \frac{\tilde{\mathbf{A}}}{8\tilde{A}^3}\cdot\bar{\lambda}\boldsymbol{\sigma}\lambda, \qquad (27)$$

where $\tilde{\mathbf{A}} = \mathbf{A} - \pi\mathbf{n}/eL$. Zero-energy states are possible only in the $F = 1$ sector, where the problem reduces to the motion of a fermion with gyromagnetic ratio $\gamma = 2$ in the field of a monopole and an additional spherically-symmetric potential. The angular variables can then be separated (see Ref. 12), and the lowest harmonic has total angular momentum $j = |g/2| - \frac{1}{2} = 0$. The equation for the corresponding radial function is

$$\left\{-\frac{1}{2}\frac{\partial^2}{\partial\tilde{A}^2} + \frac{1}{2}\left[\left(\frac{1}{2\tilde{A}}+C\right)^2 - \frac{1}{2\tilde{A}^2}\right]\right\}\chi(\tilde{A}) = E\chi(\tilde{A}). \qquad (28)$$

The zero-energy solution of (28) is readily found:

$$\chi_0(\tilde{A}) \sim \tilde{A}^{1/2} e^{-|C|\tilde{A}}.$$

(We note that, generally, (28) can be formally satisfied by another normalized function with $E = 0$, and even a whole series of functions with $E<0$.) However, these solutions cease to be normalizable when they are operated on by the hypercharge operator, and must be excluded because the only physically meaningful are those on which the supersymmetry algebra is realized.

The unit cell contains seven sites with unit-charge monopoles, so that the spectrum contains seven zero-energy states.

When the constant C is large and positive, the wave function settles on the site with $q = -7$. The lowest angular harmonic then has $j = 3$, which produces a sevenfold angular degeneracy. The radial wave function, on the other hand, has only one zero-energy solution, as before.

Thus, the theory contains seven fermion states of zero energy, and supersymmetry is not broken. These states have fermion charge $S = 1$ for the effective Hamiltonian (14). The total fermion charge of the vacuum, on the other hand, determined by the function (19), is infinite.

In the case of the usual supersymmetric QCD, the lattice of monopoles of charge -1 is superimposed on a similar lattice of monopoles with charge $+1$; the effective Hamiltonian corresponds to free motion (with the additional condition that the wave functions are periodic), and there are four zero-energy states

$$\Psi^{vac}(A_i^{(0)}, \lambda_\alpha^{(0)}) = C; \quad C\lambda_\alpha^{(0)}; \quad C\lambda_\alpha^{(0)}\lambda^{\alpha(0)},$$

which corresponds to the Witten analysis.[2]

4. DISCUSSION AND CONCLUSIONS

We note first that, in the above formulation of the problem, the effective Hamiltonian can be constructed only in supersymmetric theories. In nonsupersymmetric theories, the valleys, even if they are present at the classical level, are broken up when quantum-mechanical effects are taken into account, even in the lowest perturbation-theory order. Consider, for example, ordinary electrodynamics interacting with a charged scalar field φ. The analog of the Hamiltonian (6a) for the corresponding quantum mechanics describes the ordinary (nonsupersymmetric) oscillator:

$$\hat{H}_0 = \hat{\pi}_\varphi\hat{\bar{\pi}}_\varphi + e^2 A_i^2 \varphi\bar{\varphi}. \qquad (29)$$

The lowest state of (29) has the energy $E^0(A) = eA$. This ensures that the valley is blocked, the wave functions are concentrated in the region $eA^3 \lesssim 1$, and the Born-Oppenheimer approximation is not self-consistent (see, for example, Ref. 13, which investigates the spectrum of the quantum mechanics corresponding to the ordinary Yang-Mills theory. The valley blocking shows that the spectrum is discrete). In field theory, the situation is worse still: each harmonic $\varphi^{(\mathbf{n})}$ provides a contribution $|e\mathbf{A} - 2\pi\mathbf{n}/L|$ to $E^0(A)$, and the total $E^0(A)$ is infinite.

The physical reason for the fact that, in chiral supersymmetric theories, the effective Hamiltonian turns out to be nontrivial is very clear and reduces to the spontaneous generation of the D-term by quantum-mechanical effects. In fact, the equations of motion give

$$D = e\left(\sum_{f=1}^{8}\varphi_f\bar{\varphi}_f - 2\chi\bar{\chi}\right). \qquad (30)$$

If we substitute into this expression the harmonic expansions of the fields, and then average over the wave function (19), we find that $V\langle D\rangle_0$ becomes identical with the function K defined above and, in general, is nonzero. However, the resulting $\langle D\rangle_0$ is not sign-definite and does not lead, in this case, to supersymmetry breaking. The $A_i^{(0)}$ configuration space contains surfaces on which the Coulomb potential of the lattice, given by (23), vanishes. The vacuum wave functions are largely concentrated on these surfaces (the Born–Oppenheimer approximation for these new valleys does not, however, work, and the width of the distribution of the wave functions is comparable with the edge length of the lattice).

Our conclusions about the conditions for the generation of the D-term do not agree with the conclusions of Ref. 13, where it is reported that the D-term does not arise when the sum of the charges of the chiral multiplets is $\Sigma_f Q_f = 0$. The logic of Ref. 14 is simple: when the expression $D = \Sigma_f Q_f \varphi_f \bar{\varphi}_f$ is averaged, each term produces a quadratic divergence at the single-loop level. When all the integrals are equally cut off for all the flavors, the divergence cancels out if $\Sigma_f Q_f = 0$. (We note that the authors of Ref. 14 have also analyzed higher-order loops, but this did not change the conclusions of the single-loop analysis.) However, the quadratic divergences require a consistent treatment. It is well-known that a simple momentum cutoff will upset the gauge invariance. Our arguments, which do not explicitly involve a specific ultraviolet regularization, show that the D-term, averaged over the lattice and corresponding to the field theory D-term, is zero in the limit of infinite volume, subject to the cancellation of the anomalies $\Sigma_f Q_f^3 = 0$. The condition $\Sigma_f Q_f = 0$ is not essential. A "fine structure" appears in the finite volume, and the D-term is nonzero practically everywhere.

We must now consider the higher-order perturbation-theory corrections to the $\hat{Q}_{\alpha}^{\rm eff}$ and $\hat{H}^{\rm eff}$ that we have found. We have some arguments that are not rigorous but are, nevertheless, convincing, that such corrections are absent. The first is the aesthetic argument: the Hamiltonian (14), in which K given by (23) is the Coulomb potential of the lattice, has a definite internal beauty and completeness. Higher-order corrections in the Born-Oppenheimer parameter will give rise to unpleasant singularities in the D-term near the lattice sites, of the form $\sim 1/eA^4$, $\sim 1/e^2A^7$, and so on. We have carried out an explicit evaluation of the corrections $\sim 1/eA^4$ to the D-term and have shown that they do, in fact, cancel out. The superfield analysis given in Ref. 14, in which it is shown that there are no higher-order corrections in the D-term, is a further argument. Unfortunately, this analysis does not directly apply to our case because it merely shows that the D-term averaged over the unit cell of the lattice is zero, even when higher orders are taken into account. In general, it does not follow that the corrections are zero at a given point in the $A_i^{(0)}$ configuration space. The absence of corrections to the effective Hamiltonian acting on f_0 in the expansion (9) does not, of course, signify the absence of corrections in the Born-Oppenheimer parameter to the total vacuum wave function, which are due to the contribution of the excited states of H_0 in (9).

To conclude, we must briefly examine the SU(2) supersymmetric theory without matter. In the quantum-mechanical limit, as in the QCD case examined above, we now have valleys of Abelian directions in the space of the fields A_i^a [strictly speaking, there is only one valley if we factor $\{A_i^a\}$ with respect to the action of the gauge group]. If we direct the valley along the third isotopic axis, and integrate over the fast variables $A_i^{\pm}$ ($A_i^{\pm}A_i^3 = 0$) and $\lambda_{\alpha}^{\pm}$, we can construct the effective Hamiltonian for motion along the valley coordinates A_i^3 and λ_{α}^3. Without going into details, we merely note that the resulting effective Hamiltonian corresponds to free motion in a way very similar to the situation in nonchiral supersymmetric QCD. A continuous spectrum is therefore found to appear in the problem.

This explains the reason why the evaluation of the Witten index undertaken in Ref. 15 led to a result different from that given in Ref. 2. In Ref. 15, we used a method based on the reduction of field theory to the $(0+1)$-dimensional space, followed by the evaluation of the integral (24) for I_W.

However, in quantum mechanics, the spectrum is continuous and the very concept of the index is not well established. It is therefore not surprising that the evaluation of the integral for I_W led to a result that was different from the value of I_W obtained in field theory defined in a finite volume in which the motion was finite and the spectrum discrete.

The phenomenon of spontaneous generation of the D-term, found above, is quite universal and occurs in a wide class of chiral supersymmetric gauge fields. We shall examine this question in greater detail elsewhere.

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