Kinetic equations in the theory of localization

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Integration with respect to anticommuting variables and the method of replicas are applied to derive the effective Lagrangian in the Keldysh representation for a gas of electrons interacting with phonons. The technique developed makes it possible to obtain in a natural manner the kinetic equations for the cooperons and diffusons, and expressions for the current and other kinetic quantities. As applications, the problem of the temperature cutoff of the quantum corrections to the conductivity in the unitary limit and the problem of the high-frequency conductivity of a long thin wire are considered.

I. INTRODUCTION

The electronic properties of metals and semiconductors with a high impurity concentration are determined to a considerable extent by Anderson localization of electrons. A complete theory of this phenomenon has still not been constructed, but there has recently been considerable progress in our understanding of the phenomenon as a result of the discovery of weak-localization effects (see, e.g., the review in Ref. 1). The theory of these effects (the theory of small corrections contributed to the kinetic coefficients by the interaction of diffusion modes—cooperons and diffusons has acquired, after the introduction of an effective Lagrangian[2–5] that makes it possible to describe the interaction of the diffusion modes in a regular manner, a form that is both convenient to use and adequate for the formulated problem.

In conditions of localization an important role is played by inelastic processes, e.g., abosorption or emission of a phonon. In the strong-localization limit the conductivity is determined entirely by these processes, while in the weak-localization limit these processes determine the magnitude of the quantum corrections. The basic aim of the present paper is to reformulate the theory of Refs. 2–5 in a form that makes it possible to take account of the interaction of electrons with phonons and to calculate various kinetic effects. For this it turns out to be convenient to use the method of Keldysh[6,7] and to formulate the problem in the time representation (Sec. 2). In this case the correlators in the diffusion and Cooper channels in conditions of weak localization[1)] are found to satisfy kinetic equations.[8,9]

As applications of the method, two problems are solved. In Sec. 3 we consider the problem of the temporary cutoff of the quantum corrections to the conductivity in the unitary case (in a magnetic field or in the presence of magnetic impurities). In Sec. 4 we calculate the high-frequency conductivity of a metallic wire in conditions of strong localization. The results of Refs. 11 and 12 are generalized here to the case of a wire of macroscopic thickness.

2. THE EFFECTIVE LAGRANGIAN

2.1 The Keldysh technique

We shall derive the generating functional in the time representation. For this we consider first the usual causal Green's function

$$G(\mathbf{r},t,\mathbf{r}',t')=\sum_{\mathbf{k},\varepsilon}\frac{\varphi_{\mathbf{k}\varepsilon}(\mathbf{r},t)\varphi_{\mathbf{k}\varepsilon}^{*}(\mathbf{r}',t')}{\varepsilon-\varepsilon(\mathbf{k})},\tag{1}$$

where $\varepsilon = \varepsilon + i\delta\,\mathrm{sign}\,\varepsilon$ and $\varphi_{k\varepsilon}(\mathbf{r},t)$ satisfy the equation

$$\left(i\frac{\partial}{\partial t}-H\right)\varphi_{\mathbf{k}\varepsilon}(\mathbf{r},t)=[\varepsilon-\varepsilon(\mathbf{k})]\varphi_{\mathbf{k}\varepsilon}(\mathbf{r},t),\quad H=H_0+U(\mathbf{r}),\tag{2}$$

in which $U(\mathbf{r})$ is the random field of the impurities. Using the following properties of the anticommuting variables $\chi_{\mathbf{k}\varepsilon}$ and $\chi^{*}_{k\varepsilon}$ (Refs. 13, 10):

$$\{\chi_{\mathbf{k}\varepsilon},\chi_{\mathbf{k}'\varepsilon'}\}=\{\chi^{*}_{\mathbf{k}\varepsilon},\chi^{*}_{\mathbf{k}'\varepsilon'}\}=\{\chi^{*}_{\mathbf{k}\varepsilon},\chi_{\mathbf{k}'\varepsilon'}\}=0,$$
$$\int\chi_{\mathbf{k}\varepsilon}\,d\chi_{\mathbf{k}\varepsilon}=1,\qquad\int d\chi_{\mathbf{k}\varepsilon}=0,\tag{3}$$

we write the denominator in (1) in the form of a functional integral over anticommuting fields[3,4]:

$$\frac{1}{\varepsilon-\varepsilon(\mathbf{k})}=\frac{i}{Z}\int\chi_{\mathbf{k}\varepsilon}\chi^{*}_{\mathbf{k}\varepsilon}\exp\left\{-i\sum_{\mathbf{k}'\varepsilon'}\chi^{*}_{\mathbf{k}'\varepsilon'}[\varepsilon'-\varepsilon(\mathbf{k}')]\chi_{\mathbf{k}'\varepsilon'}\right\}D\chi^{*}D\chi,\tag{4}$$

where

$$Z=\int\exp\left\{-i\sum_{\mathbf{k}\varepsilon}\chi^{*}_{\mathbf{k}\varepsilon}[\varepsilon-\varepsilon(\mathbf{k})]\chi_{\mathbf{k}\varepsilon}\right\}D\chi^{*}D\chi.$$

Substituting (4) into (1), we obtain

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{i}{Z}\int\chi(\mathbf{r},t)\chi^{*}(\mathbf{r}',t')\exp\left[-\int\mathscr{L}\,d\mathbf{r}\,dt\right]D\chi^{*}D\chi,\tag{5}$$

where

$$\mathscr{L}=i\chi^{*}(\mathbf{r},t)(i\partial/\partial t-H)\chi(\mathbf{r},t),\tag{6}$$

$$\chi(\mathbf{r},t)=\sum_{\mathbf{k}\varepsilon}\chi_{\mathbf{k}\varepsilon}\varphi_{\mathbf{k}\varepsilon}(\mathbf{r},t).\tag{7}$$

We introduce the matrix notation

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi\\ \chi^{*}\end{pmatrix},\qquad\bar{\psi}=\frac{1}{\sqrt{2}}(\chi^{*},-\chi),\tag{8}$$

where ψ and $\bar{\psi}$ are related by the charge-conjugation matrix

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C:

$$\bar{\psi}=(C\psi)^T, \quad C=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

Using ψ and $\bar{\psi}$ we write the Green's function in the form of a matrix:

$$G(\mathbf{r},t;\mathbf{r}'t')=\frac{i}{Z}\int \psi(\mathbf{r},t)\bar{\psi}(\mathbf{r}',t')\exp\left[-i\int_{-\infty}^{+\infty} dt \int d\mathbf{r}\,\bar{\psi}(\mathbf{r},t)\right.$$
$$\left.\times\left(i\sigma_z\frac{\partial}{\partial t}-H\right)\psi(\mathbf{r},t)\right] D\bar{\psi}D\psi. \tag{10}$$

The causal character of the Green's function (10) is connected with the fact that the integral over t in (10) goes from $-\infty$ to $+\infty$, i.e., is time-ordered.

We now consider the Green's function ordered in t along the contour C (Fig. 1) (Ref. 6):

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{i}{Z}\int \psi(\mathbf{r},t)\bar{\psi}(\mathbf{r}',t')$$
$$\times\exp\left[-\int_C dt\int d\mathbf{r}\mathscr{L}\right] D\bar{\psi}D\psi. \tag{11}$$

It is obvious that when t and t' lie on the lower part of the contour the Green's function (11) is causal, while when t and t' lie on the upper part it is anticausal. In the other cases we obtain an average of the operators ψ and $\bar{\psi}$ that is not ordered in time.

We now turn to the Keldysh matrices; for this we denote the function ψ on the lower part of the contour C by ψ_-, and the function ψ on the upper part by ψ_+. Thus, dimensionality of ψ is doubled. Then, in place of (11), we can write

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{i}{Z}\int \psi(\mathbf{r},t)\bar{\psi}(\mathbf{r}',t')e^{-F}D\bar{\psi}D\psi, \tag{12}$$

$$F=\int_{-\infty}^{+\infty} dt\,dt'\int d\mathbf{r}\,d\mathbf{r}'\mathscr{F}, \tag{13}$$

$$\mathscr{F}=i\bar{\psi}(\mathbf{r},t)[G_0^{-1}(\mathbf{r},t;\mathbf{r}',t')$$
$$-U(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')]\psi(\mathbf{r}',t') \tag{14}$$

and $G_0(\mathbf{r},t;\mathbf{r}',t')$ satisfies the equation

$$[i\sigma_z\partial/\partial t-H_0]G_0(\mathbf{r},t;\mathbf{r}',t')=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{15}$$

We shall transform the basis in such a way that the function G_0 has the form

$$G_0=\begin{pmatrix} G_0^R & G_0^K \\ 0 & G_0^A \end{pmatrix}. \tag{16}$$

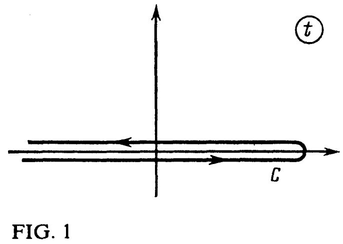

FIG. 1

The function G_0 is brought to the form (16) by a rotation (unitary transformation in the Keldysh space) and permutation of the elements of the basis.[6,7,14] Then

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix} \psi_-+\psi_+ \\ \psi_--\psi_+ \end{pmatrix}, \quad \bar{\psi}=\frac{1}{\sqrt{2}}(\bar{\psi}_--\bar{\psi}_+,\bar{\psi}_-+\bar{\psi}_+), \tag{17}$$

and ψ and $\bar{\psi}$ are connected by the relation

$$\bar{\psi}=(R\psi)^T, \tag{18}$$

where

$$R=\tau_x\otimes C, \quad R^TR=1 \tag{19}$$

in which τ_x is a Pauli matrix in the Keldysh space.

The final expresion for the Green's function G, unaveraged over the configurations of the random potential $U(\mathbf{r})$, is now written with allowance for the phonons and the external electromagnetic field:

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{i}{Z}\int \psi(\mathbf{r},t)\bar{\psi}(\mathbf{r}',t')e^{-\mathscr{F}}D\bar{\psi}D\psi D\varphi, \tag{20}$$

where

$$\mathscr{F}=i\psi(\mathbf{r},t)[G_0^{-1}(\mathbf{r},t;\mathbf{r}',t')-U(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')]\psi(\mathbf{r}',t')$$
$$+i\bar{\varphi}(\mathbf{r},t)D_0^{-1}(\mathbf{r},t;\mathbf{r}',t')\varphi(\mathbf{r}',t')$$
$$-ig\gamma_{ij}^{\alpha}\bar{\psi}_i\psi_j\varphi_\alpha\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{21}$$

$$\varphi(\mathbf{r},t)=\frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_-+\varphi_+ \\ \varphi_--\varphi_+ \end{pmatrix}, \quad \bar{\varphi}=(\tau_x\varphi)^T; \tag{22}$$

here $\varphi(\mathbf{r},t)$ is the phonon field, g is the electron-phonon coupling constant, the matrix γ has elements

$$\gamma_{ij}^1=(\tau_x/\sqrt{2})_{ij}, \quad \gamma_{ij}^2=\delta_{ij}/\sqrt{2}, \tag{23}$$

and the functions G_0 and D_0 satisfy the equations

$$\left[i\sigma_z\frac{\partial}{\partial t}-\frac{1}{2m}(-i\nabla-e\mathbf{A}\sigma_z)^2+\mu-e\Phi\right]G_0(\mathbf{r},t;\mathbf{r}',t')$$
$$=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{24}$$

$$\left[-\frac{\partial^2}{\partial t^2}+\omega^2(-i\nabla)\right]D_0(\mathbf{r},t;\mathbf{r}',t')=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{25}$$

where $\mathbf{A}$ and Φ are the vector and scalar potentials.

2.2 Averaging over the random potential

The next step is to average over the random potential $U(\mathbf{r})$. We shall assume that for scattering by impurities the Born approximation is applicable, and the random potential is Gaussian:

$$\langle U(\mathbf{r})\rangle=0, \quad \langle U(\mathbf{r})U(\mathbf{r}')\rangle=V(\mathbf{r}-\mathbf{r}'). \tag{26}$$

We shall make use of the method of replicas, as was done, e.g., in Ref. 3. As a result of the averaging of $\exp(-F)$ the function $\mathscr{F}$ acquires the term

$$\mathscr{F}_{imp}=\frac{1}{2}\sum_{nm}\bar{\psi}_n(\mathbf{r},t)\psi_n(\mathbf{r},t)V(\mathbf{r}-\mathbf{r}')\bar{\psi}_m(\mathbf{r}',t')\psi_m(\mathbf{r}',t'), \tag{27}$$

where n and m are replica indices. The expression (27) is transformed by the introduction of an integral over the matrix Q-fields,[3] and in the process we separate out the weakly nonuniform field

$$Q_{\mathbf{k}}^{nm}(\mathbf{q},t,t')\sim\langle\bar{\psi}_n(\mathbf{k}+\mathbf{q},t)\psi_m(\mathbf{k},t')\rangle. \tag{28}$$

Then

$$\exp\left\{-\int dt\,dt'\,d\mathbf{r}\,d\mathbf{r}'\,\mathcal{F}_{imp}\right\}=\int DQ\exp\left\{-\int dt\,dt'\sum_{\mathbf{q}\mathbf{k}_1\mathbf{k}_2}\left[\pi\nu V_{\mathbf{k}_1\mathbf{k}_2}\bar{\psi}\left(\mathbf{k}_1-\frac{\mathbf{q}}{2},t\right)Q_{\mathbf{k}_2}(\mathbf{q},t,t')\right.\right.$$
$$\left.\left.\times\psi\left(\mathbf{k}_1+\frac{\mathbf{q}}{2},t'\right)+\frac{\pi^2\nu^2}{2}V_{\mathbf{k}_1\mathbf{k}_2}\operatorname{Tr}Q_{\mathbf{k}_1}(\mathbf{q},t,t')Q_{\mathbf{k}_2}(-\mathbf{q},t',t)\right]\right\}\Big/\int DQ$$
$$\times\exp\left\{-\int dt\,dt'\sum_{\mathbf{q}\mathbf{k}_1\mathbf{k}_2}\frac{\pi^2\nu^2}{2}V_{\mathbf{k}_1\mathbf{k}_2}\operatorname{Tr}Q_{\mathbf{k}_1}(\mathbf{q},t,t')Q_{\mathbf{k}_2}(-\mathbf{q},t',t)\right\} \tag{29}$$

(here Q are matrices in the replica indices, and $V_{\mathbf{k}_1\mathbf{k}_2}=V(\mathbf{k}_1-\mathbf{k}_2)$). From the condition that the quantity $\bar{\psi}\tau_xQ\psi$ be real it follows that the matrix Q is hermitian:

$$Q=Q^+, \tag{30}$$

and from (18) follows the condition

$$Q=-RQ^TR. \tag{31}$$

The averaging over the phonon fields is performed analogously. After the integration over φ the free energy acquires a term quadratic in $\bar{\psi}\psi$ and this term is eliminated by integration over an auxiliary matrix Σ-field. As a result we obtain

$$F=\int d\mathbf{r}\,dt\,dt'\sum_{\mathbf{k}}\left\{\bar{\psi}(\mathbf{k},t)\left[iG_0^{-1}(\mathbf{k},t,t')-i\Sigma(\mathbf{k},t,t')\right.\right.$$
$$\left.+\pi\nu\sum_{\mathbf{k}_1}V_{\mathbf{k}\mathbf{k}_1}Q_{\mathbf{k}_1}(\mathbf{r},t,t')-ie\Phi(\mathbf{r},t)\right]$$
$$\psi(\mathbf{k},t')+\frac{\pi^2\nu^2}{2}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\operatorname{Tr}Q_{\mathbf{k}}(\mathbf{r},t,t')$$
$$\times Q_{\mathbf{k}'}(\mathbf{r},t',t)+\frac{i}{2g^2}\int dt_1\sum_{\mathbf{q}}\operatorname{Tr}\Sigma_{ij}(\mathbf{k},t,t_1)$$
$$[\mathbf{A}^{-1}(\mathbf{q},t_1,t')]_{ij,kl}\Sigma_{kl}(\mathbf{k}+\mathbf{q},t',t), \tag{32}$$

where

$$A_{ij,kl}(\mathbf{q},t,t')=\bar{\gamma}_{il}{}^{\alpha}D_0{}^{\alpha\beta}(\mathbf{q},t,t')\gamma_{kj}{}^{\beta},\quad\bar{\gamma}_{ij}=(\gamma_{ij}\tau_x)^T.$$

Now we can calculate the integrals over $\bar{\psi}$ and ψ:

$$\mathcal{L}=-\operatorname{Tr}\ln\left[-iG_0^{-1}(\mathbf{k})+i\Sigma(\mathbf{k})-\pi\nu\sum_{\mathbf{k}_1}V_{\mathbf{k}\mathbf{k}_1}Q_{\mathbf{k}_1}+ie\Phi(\mathbf{r})\right]$$
$$+\frac{\pi^2\nu^2}{2}\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\operatorname{Tr}Q_{\mathbf{k}}Q_{\mathbf{k}'}$$
$$+\frac{i}{2g^2}\sum_{\mathbf{k}\mathbf{q}}\operatorname{Tr}\Sigma_{ij}(\mathbf{k})[A^{-1}(\mathbf{q})]_{ij,kl}\Sigma_{kl}(\mathbf{k}+\mathbf{q}). \tag{33}$$

Here, for abbreviation, we have omitted t and t' in the arguments and regard the corresponding quantities as matrices in these variables; the symbol Tr incorporates integration over the time.

In the integral over the Σ-field we can confine ourselves to the saddle-point value (since $g\ll1$), which is obtained from the equation $\delta\mathcal{L}/\delta\Sigma=0$ and leads to

$$\Sigma(\mathbf{k})=ig^2\sum_{\mathbf{q}}\sum_{\alpha\beta}D_0{}^{\alpha\beta}(\mathbf{q})\gamma^{\alpha}G(\mathbf{k}-\mathbf{q})\gamma^{\beta}, \tag{34}$$

where

$$G(\mathbf{k})=\left[G_0^{-1}(\mathbf{k})-\Sigma(\mathbf{k})-i\pi\nu\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}Q_{\mathbf{k}'}-e\Phi(\mathbf{r})\right]^{-1}. \tag{35}$$

Actually, as follows from (33)–(35), a function $Q_{\mathbf{k}}$ over ξ appears in the equations. Without introducing new notation for this function, henceforth we assume that $Q_{\mathbf{k}}$ depends on the direction of the vector $\mathbf{k}$ but not on the modulus k. In the integration over the Q-fields it should be taken into account that $\varepsilon_F\tau\gg1$, and therefore it is also possible to use the method of steepest descent. The extremum of $\mathcal{L}$ corresponds to the function $Q_{\mathbf{k}}^s$ determined from

$$Q_{\mathbf{k}}^s=-\frac{i}{\pi}\int d\xi_{\mathbf{k}}G^s(\mathbf{k}) \tag{36}$$

in which G^s is given by (35) with the saddle-point function $Q_{\mathbf{K}}^s$.

By means of (35) and (36) we can obtain an equation satisfied by the function $Q_{\mathbf{k}}^s$. This is the kinetic equation (well known in the theory of dirty superconductors) for the quasiclassical Green's function[13]:

$$\sigma_z\frac{\partial Q_{\mathbf{k}}^s}{\partial t}+\frac{\partial Q_{\mathbf{k}}^s}{\partial t'}\sigma_z+\frac{k}{m}\left[\frac{\partial Q_{\mathbf{k}}^s}{\partial\mathbf{r}}-ie\mathbf{A}(t)\sigma_zQ_{\mathbf{k}}^s+ie\mathbf{A}(t')Q_{\mathbf{k}}^s\sigma_z\right]$$
$$+ie\Phi(\mathbf{r},t)Q_{\mathbf{k}}^s-ieQ_{\mathbf{k}}^s\Phi(\mathbf{r},t')+i\int dt_1[\Sigma(t,t_1)Q_{\mathbf{k}}^s(t,t')$$
$$-Q_{\mathbf{k}}^s(t,t_1)\Sigma(t_1,t)]-\pi\nu\int\frac{d\Omega_{\mathbf{k}'}}{2\pi}V_{\mathbf{k}\mathbf{k}'}(Q_{\mathbf{k}'}^sQ_{\mathbf{k}}^s-Q_{\mathbf{k}}^sQ_{\mathbf{k}'}^s)=0, \tag{37}$$

where

$$Q_{\mathbf{k}}^s=Q_{\mathbf{k}}^s(t,t').$$

2.3 The effective Lagrangian

We introduce the following notation (in the formulas containing averaging over the angles, for definiteness we take the total angle to be 2π, corresponding to dimensionality $d=2$):

$$P=2\pi\nu\tau\int\frac{d\Omega_{\mathbf{k}'}}{2\pi}V_{\mathbf{k}\mathbf{k}'}Q_{\mathbf{k}'}. \tag{38}$$

The saddle-point matrix P^s possesses the property

$(P^s)^2 = 1$. Expanding the free energy in small deviations of the matrix P from P^s, we can convince ourselves that matrices $P(\mathbf{r})$ which experience only "transverse" deviations from P^s that do not violate the condition

$$P^2(\mathbf{r})=1$$

fluctuate the most strongly. In fact, if external fields and phonons are absent, Eq. (37) is satisfied by any uniformly rotated matrix $Q_{\mathbf{k}} = T^{-1}Q^s_{\mathbf{k}}T$. Terms that break the symmetry under uniform rotations will be taken into account by perturbation theory.

We write P in the form

$$P(\mathbf{k},\mathbf{k}')=\sum_{\mathbf{k}_1\mathbf{k}_2}T^{-1}(\mathbf{k},\mathbf{k}_1)P^s(\mathbf{k}_1,\mathbf{k}_2)T(\mathbf{k}_2,\mathbf{k}'), \tag{39}$$

and assume that the matrices T are slowly fluctuating in time and weakly nonuniform (i.e., the difference $\mathbf{k}-\mathbf{k}'$ in the arguments of T is small). The Lagrangian can be written in the form

$$\mathscr{L}=-\operatorname{Tr}\ln\left\{-i\left[iT\sigma_z\,\partial_0T^{-1}-\frac{1}{2m}T(\mathbf{k}-e\mathbf{A}\sigma_z)^2T^{-1}-T\Sigma T^{-1}\right.\right.$$
$$\left.\left.-eT\Phi T^{-1}+\mu-\frac{i}{2\tau}P^s\right]\right\}$$
$$+\frac{\pi^2\nu^2}{2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}}V_{\mathbf{k}_1\mathbf{k}_2}\operatorname{Tr}Q(\mathbf{k}_1+\mathbf{q},\mathbf{k}_1)Q(\mathbf{k}_2-\mathbf{q},\mathbf{k}_2)$$
$$+\frac{i}{2g^2}\sum_{\mathbf{k}\mathbf{q}}\operatorname{Tr}\Sigma(\mathbf{k})\Lambda^{-1}(\mathbf{q})\Sigma(\mathbf{k}+\mathbf{q}), \tag{40}$$

where $\partial_0=\partial\delta(t-t')/\partial t$. Expanding to first order in the small quantities, we obtain

$$\mathscr{L}=\pi\nu\int\frac{d\Omega_{\mathbf{k}}}{2\pi}\operatorname{Tr}\left[\left(\sigma_z\frac{\partial}{\partial t}+\frac{ik^2}{2m}-\frac{ie\mathbf{kA}}{m}\sigma_z+i\Sigma+ie\Phi\right)\right.$$
$$\times Q(t,t',\mathbf{k},\mathbf{k})$$
$$\left.+\frac{\pi\nu}{2}\sum_{\mathbf{q}}\frac{d\Omega_{\mathbf{k}'}}{2\pi}Q(\mathbf{k}+\mathbf{q},\mathbf{k})V_{\mathbf{k}\mathbf{k}'}Q(\mathbf{k}'-\mathbf{q},\mathbf{k}')\right]. \tag{41}$$

The operator acting on Q in the expression (41) plays the role of the operator of the kinetic equation.

The condition for applicability of (41) is that Q vary slowly over distances $\hbar/p_F$. For distances greater than the mean free path $l_{tr}=v_F\tau_{tr}$, we can use the diffusion approximation

$$Q=Q_0+\delta Q-{}^1/_2Q_0\delta Q^2, \tag{42}$$

where

$$Q_0=Q_0(\mathbf{k}-\mathbf{k}'),\quad \delta Q(\mathbf{k},\mathbf{k}')=\mathbf{kw}(\mathbf{k}-\mathbf{k}'),\quad \delta QQ_0+Q_0\delta Q=0. \tag{43}$$

The Q chosen in this form satisfies the conditions (to within terms $\sim\delta Q^2$)

$$Q^2=1,\quad \operatorname{Tr}Q=0. \tag{44}$$

The quantity δQ is determined using the saddle-point equation (37):

$$\delta Q_{\mathbf{k}}(\mathbf{r})=(\tau_{tr}/m)Q_0(\mathbf{k}\nabla Q_0-ie\mathbf{k}\mathbf{A}[\sigma_z,Q_0]). \tag{45}$$

Using (42)–(45), we rewrite the effective Lagrangian in the diffusion approximation:

$$\mathscr{L}=\pi\nu\int d\mathbf{r}\operatorname{Tr}\left\{\left(\sigma_z\frac{\partial}{\partial t}+i\Sigma+ie\Phi\right)Q_0(t,t')+\frac{1}{4}D(\nabla Q_0\right.$$
$$\left.-ie\mathbf{A}[\sigma_z,Q_0])^2+\frac{1}{8}D\frac{eH\tau_{tr}}{m}\sigma_zQ_0[\nabla_xQ_0,\nabla_yQ_0]\right\}, \tag{46}$$

where D is the diffusion coefficient. The last term in formula (46) is written for the case when there is a magnetic field $\mathbf{H}=(0,0,H)$ (the existence of this term in the Lagrangian was first pointed out in Refs. 15 and 16). Formula (46) is a generalization of the results of Refs. 2–4 and 17, and is convenient for the description of nonstationary and inelastic processes.

2.4. The kinetic equations

We shall obtain the equations that are obeyed by the diffuson and cooperon propagators. For this we choose a parametrization of the matrix $Q(\mathbf{k},\mathbf{k}')$ in the form

$$Q=w+\tau_z(1-w^2)^{1/2} \tag{47}$$

and perform an expansion in the matrices w. The choice (47) corresponds to setting[2)] $Q^s=\tau_z$, and treating the deviation of Q^s from τ_z by perturbation theory. Since the matrix Q is related to Q^s by a unitary transformation, we have $\operatorname{Tr}Q=0$. Let $\mathbf{A}=\mathbf{A}(t)$. Then, expanding in (47) to w^2 and substituting into (41), we obtain

$$\mathscr{L}=\frac{\pi\nu}{2}\int\frac{d\Omega_{\mathbf{k}}}{2\pi}\sum_{\mathbf{q}}\operatorname{Tr}\left\{w_{\mathbf{k}}^{12}(\mathbf{q},t,t_1)\left[\frac{\partial}{\partial t}w_{\mathbf{k}}^{12}(-\mathbf{q},t_1,t)\right.\right.$$
$$+\sigma_z\frac{\partial}{\partial t_1}w_{\mathbf{k}}^{21}(-\mathbf{q},t_1,t)+\frac{ie\mathbf{kA}(t_1)}{m}w^{21}(-\mathbf{q})\sigma_z$$
$$\left.-\frac{ie\mathbf{kA}(t)}{m}\sigma_zw_{\mathbf{k}}^{21}(-\mathbf{q})\right]$$
$$+w_{\mathbf{k}}^{12}(\mathbf{q})\left[-\frac{i\mathbf{kq}}{m}+i\Sigma^A(\mathbf{k})-i\Sigma^R(\mathbf{k})\right.$$
$$\left.-ie\Phi(\mathbf{k})+\frac{1}{\tau}\right]w_{\mathbf{k}}^{21}(-\mathbf{q})$$
$$-2iw_{\mathbf{k}}^{12}(\mathbf{q})\Sigma^{21}(\mathbf{k}+\mathbf{q},\mathbf{k})$$
$$\left.-2\pi\nu w_{\mathbf{k}}^{12}(\mathbf{q})\int\frac{d\Omega_{\mathbf{k}'}}{2\pi}V_{\mathbf{k}\mathbf{k}'}w_{\mathbf{k}'}^{21}(-\mathbf{q})\right\}. \tag{48}$$

From this, after Fourier transformation with respect to $t-t'$, we obtain the equations that are obeyed by the diffusons (density correlators) $D_{\mathbf{k}\varepsilon,\mathbf{k}'\varepsilon'}(\mathbf{r},t,\mathbf{r}',t')$ and cooperons $C_{\mathbf{k}\eta,\mathbf{k}'\eta'}(\mathbf{r},t,\mathbf{r}',t')$:

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{k}}{m}\frac{\partial}{\partial\mathbf{r}}+i\Sigma^A(\varepsilon,\mathbf{k})-i\Sigma^R(\varepsilon,\mathbf{k})-ie\Phi(\mathbf{k})+\frac{1}{\tau}\right]$$
$$\times D_{\mathbf{k}\varepsilon,\mathbf{k}'\varepsilon'}(\mathbf{r},t,\mathbf{r}',t')$$
$$-2i\Sigma^K\{D\}-2\pi\nu\int\frac{d\Omega_{\mathbf{k}_1}}{2\pi}V_{\mathbf{k}\mathbf{k}_1}D_{\mathbf{k}_1\varepsilon,\mathbf{k}'\varepsilon'}(\mathbf{r},t,\mathbf{r}',t')$$
$$=\delta(\mathbf{k}-\mathbf{k}')\delta(\mathbf{r}-\mathbf{r}')\delta(\varepsilon-\varepsilon')\delta(t-t'). \tag{49}$$

$$\left[\frac{\partial}{\partial\eta}+\frac{\mathbf{k}}{m}\left(\frac{\partial}{\partial\mathbf{r}}-2ie\mathbf{A}(t)\right)+i\Sigma^{A}(\varepsilon,\mathbf{k})-i\Sigma^{R}(\varepsilon,\mathbf{k})-ie\Phi(\mathbf{k})+\frac{1}{\tau}\right]C_{\mathbf{k}\varepsilon,\mathbf{k}'\varepsilon'}(\mathbf{r},\eta,\mathbf{r}',\eta')-2i\Sigma^{K}\{C\}$$
$$-2\pi\nu\int\frac{d\Omega_{\mathbf{k}_1}}{2\pi}V_{\mathbf{k}\mathbf{k}_1}C_{\mathbf{k}_1\varepsilon,\mathbf{k}'\varepsilon'}(\mathbf{r},\eta,\mathbf{r}',\eta)$$
$$=\delta(\mathbf{k}-\mathbf{k}')\delta(\varepsilon-\varepsilon')\delta(\mathbf{r}-\mathbf{r}')\delta(\eta-\eta'),\qquad(50)$$

where $\eta=t-t'$. The kinetic equations (49) and (50) in the diffusion approximation and in the absence of phonons were obtained earlier in Ref. 9.

In an external magnetic field, field terms containing the Lorentz force appear in the kinetic equations (49) and (50). These terms can be obtained using (41), with **A** regarded as an operator in the **k**-representation, i.e., $A=\frac{1}{2}i\mathbf{H}\times\nabla_{\mathbf{k}}$ (in the axial gauge). Without writing out the equations for the diffusons and cooperons in this case, we note that the term with the Lorentz force appears only in the equation for the diffuson.

2.5. Formula for the conductivity

To derive the formula for the conductivity we shall make use of the following expression for the current in terms of a nondiagonal element of the Keldysh matrix:

$$\mathbf{j}(t)=\frac{\pi e\nu}{m}\int\frac{d\Omega_{\mathbf{k}}}{2\pi}\langle\mathrm{Tr}(\mathbf{k}-e\mathbf{A}\sigma_z)\tau_x\sigma_zQ_{\mathbf{k}}(t,t)\rangle,\qquad(51)$$

where the angular brackets denote averaging over the Q-fields with the effective Lagrangian (41). Here the integration is performed over all possible rotations of the saddle-point matrix $Q^s_{\mathbf{k}}$, i.e., $Q_{\mathbf{k}}=U^+Q^s_{\mathbf{k}}U$, where U is the matrix of the unitary transformation. We shall consider the linear response to a weak field $\delta\mathbf{A}(t)$. Solving the saddle-point equation (37) to terms linear in $\delta\mathbf{A}$:

$$Q_{\mathbf{k}}=Q_{\mathbf{k}}^{(0)}+\delta Q_{\mathbf{k}},\quad Q_{\mathbf{k}}^{(0)}\delta Q_{\mathbf{k}}+\delta Q_{\mathbf{k}}Q_{\mathbf{k}}^{(0)}=0,\qquad(52)$$

we obtain an equation for $\delta Q_{\mathbf{k}}(t,t')$:

$$-\frac{iek_\alpha}{m}[\delta A_\alpha\sigma_z,Q_{\mathbf{k}}]+\frac{1}{\tau_{tr}}Q_{\mathbf{k}}\delta Q_{\mathbf{k}}+i\left[\frac{\delta\Sigma}{\delta Q_{\mathbf{k}'}}\delta Q_{\mathbf{k}'},Q_{\mathbf{k}}\right]+i[\Sigma,\delta Q_{\mathbf{k}}]=0.\qquad(53)$$

The separation of the dependence on $\delta\mathbf{A}$ in (51) implies a transformation rotation $Q_{\mathbf{k}}\to Q_{\mathbf{k}}+\delta Q_{\mathbf{k}}\{\delta\mathbf{A}\}$. With allowance for the term linear in **A** in the effective Lagrangian (41) we arrive at the following expression for the current:

$$j_\alpha(\mathbf{r},t)=\frac{\pi e\nu}{m}\int\frac{d\Omega_{\mathbf{k}}}{2\pi}\Big\langle k_\alpha\mathrm{Tr}[\tau_x\sigma_z\delta Q_{\mathbf{k}}(t,t)]+\frac{ie\nu k_\alpha}{2m}$$
$$\times\int d\Omega_{\mathbf{k}'}\,dt'k_\beta'\delta A_\beta(t')\mathrm{Tr}[\tau_x\sigma_zQ_{\mathbf{k}}(t,t)]\mathrm{Tr}[\sigma_zQ_{\mathbf{k}'}(t',t')]$$
$$-e\delta A_\alpha(t)\mathrm{Tr}[\tau_xQ_{\mathbf{k}}(t,t)]\Big\rangle.\qquad(54)$$

The calculations are simplified if the effect of the phonons can be neglected. In this case, as follows from (53),

$$\delta Q_{\mathbf{k}}=\frac{ie\tau_{tr}k_\alpha}{m}Q_{\mathbf{k}}[\delta A_\alpha\sigma_z,Q_{\mathbf{k}}].\qquad(55)$$

Substituting this expression into (54) and taking into account that $\delta\mathbf{A}(t)=\delta\mathbf{A}_0e^{-i\omega t}$, we obtain the formula for the conductivity:

$$\sigma_{\alpha\beta}(\omega)$$
$$=\frac{e^2\nu}{2m^2\omega}\int d\Omega_{\mathbf{k}}\frac{d\varepsilon}{2\pi}\Big\langle k_\alpha k_\beta\tau_{tr}\mathrm{Tr}[\tau_x\sigma_zQ_{\mathbf{k}\varepsilon}(\mathbf{r},t)\sigma_zQ_{\mathbf{k},\varepsilon-\omega}(\mathbf{r},t)]$$
$$-\frac{\nu}{2}\int d\Omega_{\mathbf{k}'}\frac{d\varepsilon}{2\pi}d\mathbf{r}'\,dt'\,e^{-i\omega(t'-t)}k_\alpha k_\beta'\mathrm{Tr}[\tau_x\sigma_zQ_{\mathbf{k}\varepsilon}(\mathbf{r},t)]$$
$$\times\mathrm{Tr}[\sigma_zQ_{\mathbf{k}'\varepsilon}(\mathbf{r}',t')]-\frac{i}{m}\mathrm{Tr}[\tau_xQ_{\mathbf{k}\varepsilon}(\mathbf{r},t)]\Big\rangle.\qquad(56)$$

To go over the diffusion approximation we write $Q_{\mathbf{k}}$ in the form

$$Q_{\mathbf{k}}=Q+\mathbf{kw},\quad\{Q,\mathbf{kw}\}=0.\qquad(57)$$

Substitution into Eq. (37), with allowance for the fact that $\mathbf{k}\cdot\mathbf{w}$ is small in comparison with Q, gives ($A=\Sigma=0$)

$$\delta Q\equiv\mathbf{kw}=(k_\alpha\tau_{tr}/m)Q\nabla_\alpha Q,\qquad(58)$$

whence it follows that

$$\sigma_{\alpha\beta}(\omega)=\frac{e^2\nu\delta_{\alpha\beta}}{2\omega}\int\frac{d\varepsilon}{2\pi}\Big\langle D\,\mathrm{Tr}[\tau_x\sigma_zQ_\varepsilon(\mathbf{r},t)\sigma_zQ_{\varepsilon-\omega}(\mathbf{r},t)]$$
$$-\frac{\nu}{2}\int\frac{d\varepsilon'}{2\pi}d\mathbf{r}'\,dt'\exp[-i\omega(t'-t)]$$
$$\times D^2\,\mathrm{Tr}[\tau_x\sigma_zQ_\varepsilon(\mathbf{r},t)\nabla_\alpha Q_\varepsilon(\mathbf{r},t)]$$
$$\times\mathrm{Tr}[\sigma_zQ_{\varepsilon'}(\mathbf{r}',t')\nabla_\alpha Q_{\varepsilon'}(\mathbf{r}',t')]-\frac{i}{m}\mathrm{Tr}[\tau_xQ_\varepsilon(\mathbf{r},t)]\Big\rangle,\qquad(59)$$

which agrees with the result obtained in Ref. 18.

3. TEMPERATURE DEPENDENCE OF THE QUANTUM CORRECTION IN THE UNITARY CASE

In the case when the cooperons are suppressed by a magnetic field or by scattering by paramagnetic impurities (the unitary case), a quantum correction to the conductivity arises because of the interaction of diffusons and has relative order $(1/\varepsilon_F\tau)^2\ln\omega\tau$ for $d=2$. The corresponding Feynman diagrams are depicted in Fig. 2 (Ref. 4). The contributions of these diagrams are equal to $(1/\varepsilon_F\tau)^2\ln\omega\tau$ and have opposite signs. Nevertheless, the diagrams do not cancel each other, on account of the different character of their cutoffs at large momenta.[2,4,5] This result was obtained by means of a $(2+\varepsilon)$-expansion. The question of the incomplete cancellation of the graphs of Fig. 2 directly for $d=2$ dimensions is of independent interest, and, in principle, can be solved by means of the kinetic equation for the diffusons.

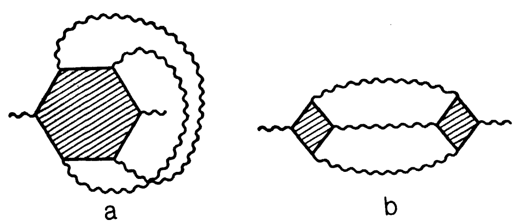

FIG. 2

Another topic that we consider in the present section is the need to cut off these corrections at small momenta. This cutoff is connected with inelastic processes and can also be obtained by means of the kinetic equation.

We shall find the eigenmodes of the operator of the kinetic equation in the diffusion region $q \ll 1/v\tau$. We shall assume that the phonons are equilibrium phonons with distribution function $N_{\mathbf{q}}$. The equation for the usual electron distribution function $f(\varepsilon_{\mathbf{p}})$ has the form

$$(\partial/\partial t+Dq^2-\hat{I}_{ph})f(\varepsilon_{\mathbf{p}})=0, \qquad (60)$$

where

$$\hat{I}_{ph}f(\varepsilon_{\mathbf{p}})=g^2\sum_{\mathbf{q}}\omega_{\mathbf{q}}\{N_{\mathbf{q}}f(\varepsilon_{\mathbf{p}}-\omega_{\mathbf{q}})[1-f(\varepsilon_{\mathbf{p}})]+(1+N_{\mathbf{q}})f(\varepsilon_{\mathbf{p}}+\omega_{\mathbf{q}})\times[1-f(\varepsilon_{\mathbf{p}})]-(1+N_{\mathbf{q}})[1-f(\varepsilon_{\mathbf{p}}-\omega_{\mathbf{q}})]f(\varepsilon_{\mathbf{p}})+N_{\mathbf{q}}[1-f(\varepsilon_{\mathbf{p}}+\omega_{\mathbf{q}})]f(\varepsilon_{\mathbf{p}})\}. \qquad (61)$$

In the case when a collision of an electron with a phonon occurs with small energy transfers, so that the electron experiences diffusion in energy, the collision integral (61) is simplified. We shall consider precisely the case, and then generalize the result obtained.

We write the equation linearized in the deviation $x(\varepsilon,t)\equiv\delta f(\varepsilon_{\mathbf{p}},t)$ of the distribution function from the equilibrium distribution function $f^{(0)}$:

$$\left(\frac{\partial}{\partial t}+Dq^2\right)x(\varepsilon,t)=B\left\{T\frac{\partial^2 x}{\partial\varepsilon^2}-[1-f^{(0)}(\varepsilon)]\frac{\partial x}{\partial\varepsilon}-2x\frac{\partial f^{(0)}}{\partial\varepsilon}\right\}, \qquad (62)$$

where

$$B=g^2\sum_{q\leqslant 2p_F}\omega_{\mathbf{q}}^2,\quad f^{(0)}(\varepsilon)=\frac{1}{2}\left(1-\text{th}\frac{\varepsilon-\mu}{2T}\right), \qquad (63)$$

(BT is the coefficient of diffusion in energy).

Let

$$x(\varepsilon,t)=\sum_n\xi_n(t)\varphi_n(\varepsilon). \qquad (64)$$

Then it follows from (67) that

$$\xi_n(t)=\exp[-(B\lambda_n+Dq^2)t], \qquad (65)$$

where λ_n is found from the eigenvalue equation

$$\varphi''+2\varphi'\,\text{th}\,y+2(2T\lambda+\text{ch}^{-2}y)\varphi=0, \qquad (66)$$

in which $y=(\varepsilon-\mu)/2T$, and the derivatives are taken with respect to y. Equation (66) is reduced by the substitution $\varphi=\psi/\cosh y$ to the Schrödinger equation for a particle in the potential $1/\cosh^2 y$. For us, the only important point is the presence of one discrete level $\lambda_0=0$ and the corresponding zeroth mode

$$\psi_0=1/\text{ch}\,y. \qquad (67)$$

The continuous spectrum starts from $1/4T$:

$$\lambda_k=(k^2+1)/4T,\quad k^2=2E,\quad -\infty<k<\infty, \qquad (68)$$

and the corresponding eigenfunctions are expressed in terms of the hypergeometric function:

$$\psi_k=(\text{ch}\,y)^{-ik}F[ik-1,\ ik+2,\ ik+1;\ (1-\text{th}\,y)/2] \qquad (69)$$

(we do not write the normalization factors).

The equation for the diffuson $D_{\varepsilon\varepsilon'}(\mathbf{q},t,t')=\langle x(\varepsilon,t)x(\varepsilon',t')\rangle$ has the form

$$\left(\frac{\partial}{\partial t}+Dq^2-\hat{I}\right)D_{\varepsilon\varepsilon'}(\mathbf{q},t,t')=\delta(\varepsilon-\varepsilon')\delta(t-t'), \qquad (70)$$

and from this, using the expansion of (69), we obtain

$$D_{\varepsilon\varepsilon'}(q,\omega)=\sum_n\frac{\psi_0(\varepsilon)\psi_n(\varepsilon)\psi_n(\varepsilon')}{\psi_0(\varepsilon')}\frac{1}{-i\omega+Dq^2+B\lambda_n}. \qquad (71)$$

Using the orthogonality and normalization conditions for $\psi_n(\varepsilon)$ in (71), we can convince ourselves that the correlator of the total density satisfies the diffusion equation[3)]

$$\int d\varepsilon\, D_{\varepsilon\varepsilon'}(q,\omega)=\frac{1}{-i\omega+Dq^2}. \qquad (72)$$

Thus, electron-phonon collisions lead to the result that a diffuson with a definite energy is cut off at $B\lambda_k$, and only for the zeroth diffusion mode is this cutoff absent ($\lambda_0=0$). The scale of the quantity $B\lambda_k$ is equal to $1/\tau_\varepsilon$, which, in the case of quasielastic scattering by phonons, is of order $\tau_\varepsilon^{-1}\sim\Omega^2/\tau_{ph}T^2$, where Ω is the characteristic energy transfer in an electron-phonon collision ($\Omega\sim sp_F$, where s is the velocity of sound).

In contrast to the case of the cooperon, for which, in the quasielastic case, a new kinetic time τ_φ appeared,[9] the damping of diffusons is determined by the energy-relaxation time τ_ε.

According to (71), there is no need to know the exact functions $\psi_n(\varepsilon)$. The rather general statement that there exists a zeroth mode and an orthonormal set of functions $\psi_n(\varepsilon)$ is sufficient. The existence of the zeroth mode is connected with the invariance of Eq. (60) as the chemical potential μ changes; therefore, even without the assumption of quasielasticity, because of this invariance, for the correction

$$\varphi_0=\partial f(\varepsilon-\mu)/\partial\mu\sim\text{ch}^{-2}[(\varepsilon-\mu)/2T] \qquad (73)$$

to the distribution function the collision integral $\hat{I}_{ph}(\varphi_0)=0$. The possibility of choosing an orthonormal set of eigenfunctions of $\hat{I}_{ph}$ is connected with the hermiticity of this operator, which follows, e.g., from the condition of detailed balance.

We now elucidate the role played by the cutoff of diffusons of the form (71) in the calculation of the second-order graphs of Fig. 2. The expression corresponding to the graph of Fig. 2a is proportional to (for convenience we transpose the indices with the arguments of the D-function)

$$A=\int\frac{d\omega'\,d\omega''}{(2\pi)^2}\sum_{\mathbf{q}_1\mathbf{q}_2}D_{\mathbf{q}_1,\omega'}\left(\varepsilon-\frac{\omega'}{2},\varepsilon-\frac{\omega'}{2}-\omega''\right)D_{\mathbf{q}_2-\omega''}\left(\varepsilon-\frac{\omega''}{2},\varepsilon-\omega'-\frac{\omega''}{2}\right)$$

$$=\int\frac{d\omega'\,d\omega''}{(2\pi)^2}\sum_{\mathbf{q}_1\mathbf{q}_2}\sum_{n_1n_2}\frac{\psi_0(\varepsilon-\omega'/2)\psi_{n_1}(\varepsilon-\omega'/2)\psi_{n_1}(\varepsilon-\omega'/2-\omega'')}{\psi_0(\varepsilon-\omega'/2-\omega'')}$$

$$\times\frac{\psi_0(\varepsilon-\omega''/2)\psi_{n_2}(\varepsilon-\omega''/2)\psi_{n_2}(\varepsilon-\omega'-\omega''/2)}{\psi_0(\varepsilon-\omega'-\omega''/2)}\frac{1}{-i\omega'+Dq_1^2+B\lambda_{n_1}}\frac{1}{i\omega''+Dq_2^2+B\lambda_{n_2}}. \tag{74}$$

In this integral, values $|\omega',\omega''|<\Lambda\equiv B/4T\sim 1/\tau_\varepsilon$ are important. Therefore, in the cutoff at the lower limit an important role is played by Λ and

$$A\propto\ln^2[\tau z(\Lambda)], \tag{75}$$

where $z(\Lambda)$ is a certain function. To determine it we formally introduce two different Λ:

$$B\lambda_{k_1}=\Lambda_1(1+k_1^2),\quad B\lambda_{k_2}=\Lambda_2(1+k_2^2), \tag{76}$$

and differentiate (74) with respect to Λ_1 and Λ_2 (we neglect the zeroth mode, since for $n_1,n_2=0$ the singularity is weakened by the integration[4)] over ω_1 and ω_2). We then obtain

$$\frac{dA}{d\Lambda_1\,d\Lambda_2}=\frac{1}{\Lambda_1\Lambda_2},\quad\frac{d}{d\Lambda}\ln[\tau z(\Lambda)]=\frac{1}{\Lambda}, \tag{77}$$

whence, with logarithmic accuracy, we obtain

$$A=\ln^2(\tau_\varepsilon/\tau). \tag{78}$$

Thus, the cutoff of the graphs that diverge logarithmically in q occurs, at the lower limit, at the reciprocal of the diffusion length $L=(D\tau_\varepsilon)^{1/2}$ of the energy relaxation.

4. HIGH FREQUENCY CONDUCTIVITY OF METALLIC WIRES

We shall consider the high-frequency conductivity of thin wires at low temperatures, when

$$T\tau\ll 1,\quad \omega\tau_{ph}\gg 1,\quad \omega\tau\ll 1,\quad \tau_{ph}\gg\tau(p_F^2S)^2$$

(the latter condition follows from the inequality $L_{ph}=(D\tau_{ph})^{1/2}\gg L_c\sim l(p_F^2S)$, where L_c is the localization length in the wire, l is the mean free path, and S is the cross-sectional area). Under these conditions the electrons are localized and the conductivity at frequency ω is due to phonon-assisted hoppings. The condition $\omega\tau_{\rm ph}\gg 1$ permits us to confine ourselves to the lowest orders of perturbation theory in the interaction with the phonons. In this case the high-frequency conductivity can be expressed in terms of correlators of the Q-fields. Of course, we cannot calculate these correlators in the localization region by any perturbation theory. However, for estimates it is sufficient to know only that they all behave like $\exp(-|\mathbf{r}-\mathbf{r}'|/L_c)$. The problem of the conductivity of one-dimensional chains under the conditions mentioned above was solved in Refs. 11 and 12. The localization length in this case is of the order of the mean free path l. In wires, $L_c\sim lk_F^2S$. In the calculation of the conductivity we shall use the method given in Sec. 2, with allowance for electron-phonon interactions.

We shall calculate the phonon-related contribution to the conductivity. In conditions of strong localization it is convenient to go over to the representation of exact wavefunctions $\psi_i(x)$ in a given random potential. The saddle-point equation in this representation, after Fourier transformation with respect to the time, can be written in the form

$$-i(\varepsilon-\varepsilon'-\varepsilon_i+\varepsilon_j)Q_{ij}(\varepsilon,\varepsilon')-(ie\mathbf{A}_\omega/m)[\mathbf{k}_{il}\sigma_zQ_{lj}(\varepsilon-\omega,\varepsilon')$$
$$-Q_{il}(\varepsilon,\varepsilon'+\omega)\mathbf{k}_{lj}\sigma_z]+i[\Sigma,Q]_{ij}(\varepsilon,\varepsilon')=0, \tag{79}$$

where ω is the frequency of the external field and ε_i is the energy of the ith level. We shall treat the effect of the phonons by perturbation theory, and so we solve Eq. (79) by iterations:

$$\delta Q_{ij}^{(1)}(\varepsilon,\varepsilon')=-\frac{e\mathbf{A}_\omega}{m(\omega-\varepsilon_i+\varepsilon_j)}$$
$$\times[\mathbf{k}_{ij}\sigma_zQ_j^{(0)}(\varepsilon-\xi)-Q_i^{(0)}(\varepsilon)\mathbf{k}_{ij}\sigma_z]\delta(\varepsilon-\varepsilon'-\omega), \tag{80}$$

$$\delta Q_{ij}^{(2)}(\varepsilon,\varepsilon')=\frac{1}{\varepsilon_j-\varepsilon_i}[\Sigma_{ij}(\varepsilon)Q_j^{(0)}(\varepsilon)-Q_i^{(0)}(\varepsilon)\Sigma_{ij}(\varepsilon)]\delta(\varepsilon-\varepsilon'), \tag{81}$$

$$\delta Q_{ij}^{(3)}(\varepsilon,\varepsilon')=-\frac{1}{\varepsilon-\varepsilon'-\varepsilon_i+\varepsilon_j}\left\{\frac{e\mathbf{A}_\omega}{m}[\mathbf{k}\sigma_z,\delta Q^{(2)}]_{ij}(\varepsilon,\varepsilon')\right.$$
$$\left.-\left[\frac{\partial\Sigma}{\partial Q}\delta Q^{(1)},Q^{(0)}\right]_{ij}(\varepsilon,\varepsilon')-[\Sigma,\delta Q^{(1)}]_{ij}(\varepsilon,\varepsilon')\right\}. \tag{82}$$

Using (51) and (80)–(82) we obtain (the superscript (0) can be omitted)

$$\mathbf{j}_{ph}(t)=\frac{\pi e^2\mathbf{A}_\omega g^2}{4V}\sum_{ij}(d_{ii}-d_{jj})^2\int\frac{d^3\mathbf{q}\,d\eta}{(2\pi)^4}\frac{d\varepsilon}{2\pi}|e^{i\mathbf{q}\mathbf{r}}|_{il}^2D_{\mathbf{q}}^{\beta\alpha}(\eta)$$
$$\times\langle\mathrm{Tr}\,\tau_x[\gamma^\alpha Q_j(\varepsilon-\omega-\eta)\bar\gamma^\beta Q_i(\varepsilon-\omega)+Q_i(\varepsilon)\gamma^\alpha Q_j(\varepsilon-\eta)\bar\gamma^\beta$$
$$-\sigma_z\gamma^\alpha Q_j(\varepsilon-\eta)\bar\gamma^\beta\sigma_zQ_i(\varepsilon-\omega)$$
$$-\sigma_zQ_i(\varepsilon)\sigma_z\gamma^\alpha Q_j(\varepsilon-\eta-\omega)\bar\gamma^\beta]\rangle e^{-i\omega t}, \tag{83}$$

where the matrix element of the momentum has been represented in the form $k_{ij}=im(\varepsilon_i-\varepsilon_j)d_{ij}$, where d_{ij} is the matrix element of the coordinate. We treat the phonons in the momentum representation, and therefore the matrix elements of the electron-phonon interaction contain the factor $(e^{i\mathbf{q}\cdot\mathbf{r}})_{ij}$, where $\mathbf{q}$ is the phonon momentum.

The element Q_i^{k} of the matrix Q_i is a small perturbation, and its fluctuations can be neglected. For this element, therefore, we use the expression

$$Q_i^{\mathrm{k}}(\varepsilon)=-2\,\mathrm{th}\frac{\varepsilon-\mu}{2T}\delta(\varepsilon-\varepsilon_i). \tag{84}$$

Since the phonons are in equilibrium, for the D-matrices we take

$$D=\begin{pmatrix} D^R & D^K \\ 0 & D^A \end{pmatrix},$$

$$D_{\mathbf{q}}^R(\eta)=D_{\mathbf{q}}^{A*}(\eta)=-i\pi\omega_{\mathbf{q}}[\delta(\eta-\omega_{\mathbf{q}})-\delta(\eta+\omega_{\mathbf{q}})]/2, \quad (85)$$
$$D_{\mathbf{q}}^k(\eta)=\mathrm{cth}\,(\eta/2T)[D_{\mathbf{q}}^R(\eta)-D_{\mathbf{q}}^A(\eta)].$$

In order to estimate the expression (83), we shall omit the dependence on ε in the diagonal elements of the matrices Q. This corresponds to assuming that the Q-matrices are fluctuating equally over distances $\lesssim L_c$. With allowance for the property $\mathrm{Tr}Q = 0$, from (83) we obtain an expression for the high-frequency conductivity, which can be written in the form of an average over the distribution of levels:

$$\sigma_{ph}\sim\frac{e^2g^2}{V}\left\langle\sum_{ij}(d_{ii}-d_{jj})^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,|e^{i\mathbf{q}\mathbf{r}}|_{ij}^2\frac{\omega_{\mathbf{q}}^2}{T\,\mathrm{sh}^2(\omega_{\mathbf{q}}/2T)}\right.$$
$$\left.\times\delta(\varepsilon_F-\varepsilon_i)\,\delta(\varepsilon_i-\omega_{\mathbf{q}}-\varepsilon_j)\right\rangle.$$

The same formula for one-dimensional chains was obtained in Refs. 11 and 12. Using the method of Ref. 11, we adopt the following estimates (the contribution arises from hoppings over distances $r\gg l$ but $\ll L_c$):

$$|e^{i\mathbf{q}\mathbf{r}}|_{ij}^2\sim 1/q^2r_{ij}^2,\qquad |d_{ii}-d_{jj}|^2\sim L_c^2(\omega_{\mathbf{q}}\tau)^2.$$

Replacing the summation over i and j by the corresponding integration, we obtain

$$\sigma_{ph}\sim g^2\nu e^2T^4L_c^3/sv^3\hbar^5,$$

which coincides with the result of Refs. 11 and 12 when $L_c = l$.

5. CONCLUSION

The principal result of this paper is the derivation of the effective Lagrangian in the time representation for the Keldysh Green's function of a disordered system. The method developed makes it possible to solve various kinetic problems (in addition to the examples considered in this paper, the kinetic equations for cooperons have arisen in the problem of the magnetoresistance of a thin metallic film[19]). It is obvious that electron-electron interactions can be incorporated into the treatment, in the same way as was done in Ref. 15.

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[1)]For the determination of the matrix elements of the processes of emission and absorption of phonons it is necessary to take into account the Fermi statistics of the electrons. This cannot be done in the framework of the method of supersymmetry[10] using Bose and Fermi variables for the description of the electron. Therefore, in the article we use the replica method of Refs. 3 and 4.

[2)]It should be noted that the saddle-point matrix $Q^s=\tau_z$ does not satisfy the condition (31). This means that in the integral over Q it is necessary to displace the contour of integration in such a way that it passes through the saddle point. This question has been discussed in Refs. 18 and 10.

[3)]The expressions (71) and (72) are derived for a gas of noninteracting electrons. In considering the zeroth mode in (71) and the total density (72) it is necessary to take electron-electron interaction into account. As a result, in the denominator of the term with $n = 0$ a Maxwell relaxation time appears, and this denominator (and also the denominator in (72)) acquires the form $-i\omega + Dq^2 + D\varkappa q$, where $\varkappa = 4\pi e^2\nu$ is the inverse screening length.

[4)]See also footnote 3.

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