The bare pomeron in quantum chromodynamics

L. N. Lipatov

B. P. Konstantinov Leningrad Institute of Nuclear Physics, Academy of Sciences of the USSR, Gatchina

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The Green's function is constructed in the leading logarithmic approximation for the vacuum Regge singularity in quantum chromodynamics, and is used to calculate the pomeron trajectory in the limit of high momentum transfer. A lower bound is found for its intercept.

I. INTRODUCTION

It is well known that the behavior of the hadron scattering amplitudes at high energies $s^{1/2} \gg m$ and fixed momentum transfer $|q| = (-t)^{1/2} \sim m$ is completely determined by singularities in the t-channel partial waves $f_j(t)$ on the complex plane of the angular momentum j (Ref. 1). There is particular interest in the nature of the Pomeranchuk singularity in the channel with vacuum quantum numbers. This singularity is located near the point $\omega \equiv j - 1 = 0$, in accordance with the very small rise in the observed total cross sections.

In the leading logarithmic approximation (LLA) $[g^2 \ln(s/m^2) \sim 1, g \ll 1$, where g is the Yang-Mills coupling constant] of the $SU(N)$ Yang-Mills theory, the Pomeranchuk singularity (or, briefly, the pomeron) appears as a bound state of two Reggeized gluons. The corresponding partial waves have a fixed root singularity at $\omega = g^2 N(\ln 2/\pi^2)$ (Ref. 2). In quantum chromodynamics (QCD) corresponding to $SU(3)$, asymptotic freedom ensures that the singularity transforms into a set of Regge poles which accumulate to the right of the point $\omega = 0$ (Refs. 2 and 3). Within the framework of the Reggeon diagram technique,[4] these poles must be looked upon as bare. We shall calculate the trajectories of the bare pomerons for large q and give the lower bounds for their intercepts, using the properties of the conformal invariance of the pomeron Green's function in LLA[5] and the renormalization group.

We now recall the main results of LLA. The scattering amplitudes of colorless objects in QCD can be written in the following form in this approximation:[3]

$$A(s,t) = is \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d\omega}{2\pi i} s^\omega f_\omega(q^2), \qquad t = -q^2,$$
$$f_\omega(q^2) = \int d^2k\, d^2k'\, \Phi^1(k,q)\Phi^2(k',q) f_\omega(k,k',q), \tag{1}$$

where k', k are the two-dimensional transverse components of the momenta of the exchanged gluons. The function $f_\omega(k,k',q)$ can be interpreted as the t-channel partial wave for the gluon-gluon scattering amplitude with virtualities $-k^2$, $-k'^2$, $-(q-k)^2$, $-(q-k')^2$, where the gluon propagators are included in $f_\omega(k,k',q)$. The functions $\Phi^{1,2}(k,q)$ characterize the internal structure of the colliding particles 1 and 2 and can be calculated from perturbation theory in some cases.[3] By virtue of gauge invariance, we have[3]

$$\Phi^{1,2}(k,q)|_{k=0} = \Phi^{1,2}(k,q)|_{k=q} = 0. \tag{2}$$

It will be convenient to use the representation of impact parameters ρ_i:

$$\delta^2(q-q') f_\omega(k,k',q) = (2\pi)^{-8} \int \prod_{r=1}^{2} d^2\rho_r \prod_{r=1}^{2} d^2\rho_{r'}$$
$$\times f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'}) \exp[ik\rho_1 + i(q-k)\rho_2 - ik'\rho_{1'} - i(q'-k')\rho_{2'}]. \tag{3}$$

The expression for $f_\omega(q^2)$ in (1) can then be written in the form

$$\delta^2(q-q') f_\omega(q^2) = \int \prod_{r=1}^{2} \frac{d^2\rho_r\, d^2\rho_{r'}}{(2\pi)^2(2\pi)^2}$$
$$\times \Phi^1(\rho_1,\rho_2,q)\Phi^2(\rho_{1'},\rho_{2'},q) f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'}), \tag{4}$$

where

$$\Phi^{1,2}(\rho_1,\rho_2,q) = \int d^2k\, \Phi^{1,2}(k,q) e^{ik\rho_1} e^{i(q-k)\rho_2}. \tag{5}$$

In this representation, (2) assumes the form

$$\int d^2\rho_1\, \Phi^{1,2}(\rho_1,\rho_2,q) = \int d^2\rho_2\, \Phi^{1,2}(\rho_1,\rho_2,q) = 0. \tag{6}$$

The function $f_\omega(\rho_1,\rho_2) \equiv f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$ satisfies the following Bethe-Salpeter equation[1,3,5] in LLA:

$$\omega \nabla_1^2 \nabla_2^2 f_\omega(\rho_1,\rho_2) = (2\pi)^4 \delta^2(\rho_{11'})\delta^2(\rho_{22'})$$
$$+ \frac{g^2 N}{(2\pi)^3} \Big\{ (2\pi)^2 \delta^2(\rho_{12}) (\nabla_1 + \nabla_2)^2 f_\omega(\rho_1,\rho_2)$$
$$+ \nabla_1^2 \int \frac{d^2\rho_0}{|\rho_{01}|^2} \Big[\nabla_2^2 f_\omega(\rho_0,\rho_2) - \frac{|\rho_{12}|^2}{|\rho_{01}|^2 + |\rho_{02}|^2} \nabla_2^2 f_\omega(\rho_1,\rho_2) \Big]$$
$$+ \nabla_2^2 \int \frac{d^2\rho_0}{|\rho_{02}|^2} \Big[\nabla_1^2 f_\omega(\rho_1,\rho_0) - \frac{|\rho_{12}|^2}{|\rho_{01}|^2 + |\rho_{02}|^2} \nabla_1^2 f_\omega(\rho_1,\rho_2) \Big] \Big\}, \tag{7}$$

where $\nabla_{1,2} = \partial/\partial\rho_{1,2}$, $\rho_{ij} = \rho_i - \rho_j$. When (7) is iterated, the infrared and ultraviolet divergences in $f_\omega(q^2)$ (4) cancel out in each order of perturbation theory,[3] so that the characteristic transverse momenta $k_\perp$ (the reciprocals of the impact parameters) of the virtual gluons in LLA are found to be of the order of the transverse momenta of quarks inside the colliding particles. In particular, for virtual photon scattering, we have $|k_\perp| \sim (-p_i^2)^{1/2}$, where p_i are the photon momenta. Hence, when LLA is valid, the running coupling constant $g(k_\perp)$ can be looked upon as a constant number

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$g[(-p^2)^{1/2}]$ which, in principle, can be made as small as desired.

In the next section, we shall examine in greater detail than in Ref. 5 the conformal invariance properties of the four-point Green's function $f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$.

2. Conformal properties of partial waves in LLA

It is convenient to introduce the complex notation for the transverse coordinates $(x_r,y_r), r=1,2,1',2'$:

$$\rho_r=x_r+iy_r,\quad \rho_r^*=x_r-iy_r,\quad d^2\rho_r=dx_r dy_r. \tag{8}$$

The six-parameter group of conformal transformations then corresponds to the bilinear transformation[6]

$$\rho_r=\frac{a_{11}\rho_r'+a_{12}}{a_{21}\rho_r'+a_{22}},\quad \rho_r^*=\frac{a_{11}^*\rho_r'^*+a_{12}^*}{a_{21}^*\rho_r'^*+a_{22}^*}. \tag{9}$$

We shall show that the Green's function $f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$ can be represented in LLA by a conformally invariant form. To begin, let us consider this function in the Born approximation $g=0$. The solution of (7) is then the product of free-gluon Green's functions

$$f_\omega{}^0(\rho_1,\rho_2,\rho_{1'},\rho_{2'})=(2\pi)^2\omega^{-1}\ln|\rho_{11'}|\ln|\rho_{22'}|. \tag{10}$$

This solution is not invariant under the conformal transformations. We note, however, that, instead of (10), we can use the following conformally invariant expression:

$$f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})|_{g=0}=\frac{2\pi^2}{\omega}\ln\left|\frac{\rho_{11'}\rho_{22'}}{\rho_{12'}\rho_{1'2}}\right|\ln\left|\frac{\rho_{11'}\rho_{22'}}{\rho_{12}\rho_{1'2'}}\right|, \tag{11}$$

because this differs from (10) by a term independent of one of the coordinates ρ_1, ρ_2, $\rho_{1'}$, or $\rho_{2'}$ and, by virtue of (6), provides a zero contribution to $f_\omega(q^2)$ in (4).

Moreover, the expression in (11) can be chosen as the bare term in the solution of (7) because the integral kernel of this equation gives zero when acting on a function independent of ρ_1 or ρ_2.

To prove the possibility that f_ω can be written in the conformally invariant form for arbitrary $g\neq 0$, it is sufficient to verify the invariance of (7) under the inversion transformation

$$\rho_i\to\rho_i/|\rho_i|^2,\quad \rho_i^*\to\rho_i^*/|\rho_i|^2. \tag{12}$$

We now introduce the intermediate ultraviolet regularization in the dimensionality of space: $d^2\rho_0\to d^{2+\varepsilon}\rho_0$ and set $\varepsilon=0$ after the transformation $\rho\to 1/\rho$. For example, we consider the following terms on the right-hand side of (7):

$$K_\varepsilon f_\omega(\rho_1,\rho_2)\equiv\nabla_1^2\int\frac{d^{2+\varepsilon}\rho_0}{|\rho_{01}|^2}\Big[\nabla_2^2 f_\omega(\rho_0,\rho_2)-\frac{|\rho_{12}|^2}{|\rho_{01}|^2+|\rho_{02}|^2}\nabla_2^2 f_\omega(\rho_1,\rho_2)\Big]. \tag{13}$$

The transformation $\rho_i\to 1/\rho_i$ gives

$$\lim_{\varepsilon\to 0}K_\varepsilon f_\omega(\rho_1,\rho_2)|_{\rho_i\to 1/\rho_i}=|\rho_1|^4|\rho_2|^4K_0f_\omega\left(\frac{1}{\rho_1},\frac{1}{\rho_2}\right)+|\rho_1|^4|\rho_2|^4\nabla_1^2\left\{\int\frac{d^2\rho_0 2(\rho_0,\rho_{10})}{|\rho_0|^2|\rho_{01}|^2}\nabla_2^2 f_\omega(\rho_0^{-1},\rho_2^{-1})+\pi\ln\left|\frac{\rho_2}{\rho_1}\right|^2\nabla_2^2 f_\omega(\rho_1^{-1},\rho_2^{-1})\right\}=|\rho_1|^4|\rho_2|^4K_0f_\omega(\rho_1^{-1},\rho_2^{-1}), \tag{14}$$

because the last two terms in the intermediate formula cancel out. The separation of the factor $|\rho_1|^4|\rho_2|^4$ from the other terms in (7) under inversion can be verified in a similar way.

Thus, the solution of (7) for a particular modification of the inhomogeneous terms [see (11)] can be taken to be conformally invariant. In particular, the solution of the corresponding homogeneous equation for $f(\rho_1,\rho_2)$ for each eigenvalue ω should generate an irreducible representation of the conformal group, normalized as follows:

$$\|f\|^2=\int\frac{d^2\rho_1\,d^2\rho_2}{|\rho_{12}|^4}|f(\rho_1,\rho_2)|^2. \tag{15}$$

The function $f(\rho_1,\rho_2)$ within the representation can be numbered by introducing the auxiliary function ρ_0. Because of translational invariance, f depends on ρ_{10} and ρ_{20} alone. Since the representation is irreducible with respect to extension and rotation, f must be a homogeneous polynomial in ρ_{10}, ρ_{20} of degree d and in ρ_{10}^*, ρ_{20}^* of degree $\bar d$. Since the solution must have the same value after rotation through 2π, we have $d-\bar d=n$, where n is an integer equal to the conformal spin. Finally, the invariance of the set of eigenfunctions under the inversions $\rho_{1,2}\to 1/\rho_{1,2}$

$$f(\rho_{10},\rho_{20})\to f'(\rho_{10},\rho_{20})=\rho_0^{-2d}\rho_0^{*-2\bar d}f(\rho_{10'},\rho_{20'}),\qquad \rho_{0'}=1/\rho_0 \tag{16}$$

establishes the form of these functions:[7]

$$f(\rho_{10},\rho_{20})\propto(\rho_{10}\rho_{20}/\rho_{12})^d(\rho_{10}^*\rho_{20}^*/\rho_{12}^*)^{\bar d},\quad d-\bar d=n. \tag{17}$$

It is convenient to parametrize the quantum numbers d and $\bar d$ of the irreducible representation as follows:

$$d=n/2-{}^1/_2-i\nu,\qquad \bar d=-n/2-{}^1/_2-i\nu, \tag{18}$$

since the set of functions (17) is complete for real ν and integral n. Henceforth, we shall use the following notation for these functions:

$$E^{n,\nu}(\rho_{10},\rho_{20})\equiv e^n(\rho_{10},\rho_{20})|\rho_{12}/\rho_{10}\rho_{20}|^{1+2i\nu}, \tag{19}$$

where e is the complex unit vector:

$$e(\rho_{10},\rho_{20})=\left(\frac{\rho_{10}}{|\rho_{10}|^2}-\frac{\rho_{20}}{|\rho_{20}|^2}\right)\left|\frac{\rho_{10}\rho_{20}}{\rho_{12}}\right|=-\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\left|\frac{\rho_{10}\rho_{20}}{\rho_{12}}\right|. \tag{20}$$

The functions (19) are the eigenfunctions of the two Casimir operators of conformal algebra:

$$(\rho_{12})^2\partial_1\partial_2E^{n,\nu}=\lambda_{n,\nu}E^{n,\nu},\quad (\rho_{12}^*)^2\partial_1^*\partial_2^*E^{n,\nu}=\lambda_{n,-\nu}E^{n,\nu}, \tag{21}$$

$$\lambda_{n,\nu}={}^1/_2-(i\nu-n/2)^2,\qquad \lambda_{n,-\nu}=\lambda_{n,\nu}^*,$$

and are also unaffected by conformal transformations of the form

$$\rho_i'=\rho_0+1/(\alpha+\rho_{i0}^{-1}),\quad i=1,\,2, \tag{22}$$

where α is an arbitrary complex number.

The eigenvalues of the Casimir operators coincide on the functions $E^{n,\nu}$ and $E^{-n,-\nu}$ [see (21)]. Moreover, these functions are not linearly independent. In fact, we have [see Eq. (A.12) in the Appendix]

$$E^{-n,-\nu}(\rho_{10},\rho_{20})=\frac{b_{n,-\nu}}{a_{n,\nu}}\int d^2\rho_{0'}E^{n,\nu}(\rho_{10'},\rho_{20'})\,|\rho_{00'}|^{-2+4i\nu}\left(\frac{\rho_{0'0}^*}{\rho_{0'0}}\right)^n, \tag{23}$$

where the constants $b_{n,\nu}$ and $a_{n,\nu}$ are given by (A.3) and (A.13).

Substituting the eigenfunction $E^{n,\nu}$ (19) into the right-hand side of the homogeneous equation (7), we can readily calculate the corresponding eigenvalue (see Ref. 3):

$$\omega(\nu,n)=\frac{Ng^2}{2\pi^2}\int_0^1\frac{dx}{1-x}[x^{(|n|-1)/2}\cos(\nu\ln x)-1]. \tag{24}$$

The completeness condition for the functions $E^{n,\nu}$ has the following form:

$$(2\pi)^4\delta^2(\rho_{11'})\delta^2(\rho_{22'})=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\nu\times\int d^2\rho_0\frac{16(\nu^2+n^2/4)}{|\rho_{12}|^2|\rho_{1'2'}|^2}E^{n,\nu^*}(\rho_{1'0},\rho_{2'0})E^{n,\nu}(\rho_{10},\rho_{20}), \tag{25}$$

which can be verified by multiplying (25) by $E^{m,\mu*}(\rho_{\tilde{1}0},\rho_{\tilde{2}0})$ and then integrating with respect to ρ_1,ρ_2 using the orthonormalization relations [see formula (A.16) in the Appendix]. We note that (25) is valid in a certain generalized sense. This can be used only when the left- and right-hand sides can be integrated with sufficiently well-behaved functions [see (6)].

Expanding the solution of (7) in terms of the complete set of functions (19) and using (24) and (25), we obtain the following explicit expression:[5]

$$f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})=\sum_{n=-\infty}^{+\infty}\fint_{-\infty}^{+\infty}d\nu\int d^2\rho_0(\nu^2+n^2/4)E^{n,\nu^*}(\rho_{1'0},\rho_{2'0})E^{n,\nu}(\rho_{10},\rho_{20})\times\left\{\left[\nu^2+\left(\frac{n-1}{2}\right)^2\right]\left[\nu^2+\left(\frac{n+1}{2}\right)^2\right][\omega-\omega(\nu,n)]\right\}^{-1}, \tag{26}$$

where, by definition, the crossed integral sign represents the ordinary integral for $n\neq\pm1$ and the integral in the sense of the "principal value",

$$\fint_{-\infty}^{+\infty}d\nu\frac{\varphi(\nu)}{\nu^2}\equiv\lim_{\varepsilon\to0}\left[\int_{-\infty}^{+\infty}d\nu\frac{\theta(\nu^2-\varepsilon^2)}{\nu^2}\varphi(\nu)-2\frac{\varphi(0)}{|\varepsilon|}\right] \tag{27}$$

for $n=\pm1$. The substracted term $2\varphi(0)/|\varepsilon|$ appears because the corresponding contribution to f_ω is proportional to the quantities $E^{\pm1,0}(\rho_{10},\rho_{20})$, and the solution of (7) is defined to within these terms [see (21) and (24)], where, because of (6), this ambiguity does not affect $f_\omega(q^2)$ in (4).

The function given by (26) is conformally invariant and therefore depends on the two anharmonic ratios [see (11)]:

$$a=|\rho_{11'}\rho_{22'}/\rho_{12'}\rho_{1'2}|,\quad b=|\rho_{11'}\rho_{22'}/\rho_{12}\rho_{1'2'}|. \tag{28}$$

To investigate the region of small momentum transfer, it is convenient to rewrite (26) in the mixed representation:

$$f_\omega^q(\rho,\rho')=\frac{1}{(2\pi)^2}\int d^2\left(\frac{\rho_1+\rho_2}{2}\right)\times\exp\left(iq,\frac{\rho_{11'}+\rho_{22'}}{2}\right)f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$$

$$=\frac{|\rho\rho'|}{16}\sum_{n=-\infty}^{+\infty}\fint_{-\infty}^{+\infty}d\nu\,E_q^{n,\nu^*}(\rho')E_q^{n,\nu}(\rho)\times\left\{\left[\nu^2+\left(\frac{n-1}{2}\right)^2\right]\left[\nu^2+\left(\frac{n+1}{2}\right)^2\right][\omega-\omega(\nu,n)]\right\}^{-1}, \tag{29}$$

where

$$E_q^{n,\nu}(\rho)=\frac{2\pi^2}{b_{n,\nu}}\int\frac{d^2R}{|\rho|}e^{iqR}E^{n,\nu}\left(R+\frac{\rho}{2},\,R-\frac{\rho}{2}\right). \tag{30}$$

The constant $b_{n,\nu}$ is given in the Appendix [see (A.3)]. Using (A.1), we can consider the region of small q:

$$f_\omega^q(\rho,\rho')|_{q\ll1/\rho\sim1/\rho'}=\frac{|\rho\rho'|}{4}\sum_{n=-\infty}^{+\infty}\int d\nu\times\left\{\left[\nu^2+\left(\frac{n-1}{2}\right)^2\right]\left[\nu^2+\left(\frac{n+1}{2}\right)^2\right][\omega-\omega(\nu,n)]\right\}^{-1}$$
$$\times\sin\left[2\nu\ln|q\rho|+n(\arg q-\arg\rho)+\frac{\tilde\delta(n,\nu)}{2}\right]\times\sin\left[2\nu\ln|q\rho'|+n(\arg q-\arg\rho')+\frac{\tilde\delta(n,\nu)}{2}\right],$$
$$\tilde\delta(n,\nu)=\delta(n,\nu)-\pi. \tag{31}$$

Thus, when $q^2\to0$, the function f_ω^q has singularities of the form $(q/q^*)^n\cdot(q^2)^{2i\nu}$, where $\nu=\nu(\omega,n)$ is determined from the solution of (24). Nevertheless, (31) allows the passage to the limit $q=0$ (Ref. 3):

$$f_\omega^q(\rho,\rho')|_{q=0}=\frac{|\rho\rho'|}{8}\sum_{n=-\infty}^{+\infty}\int d\nu|\rho/\rho'|^{2i\nu}\cdot\times(\rho^*\rho'/\rho\rho'^*)^{n/2}\left\{\left[\nu^2+\left(\frac{n-1}{2}\right)^2\right]\left[\nu^2+\left(\frac{n+1}{2}\right)^2\right]\times[\omega-\omega(\nu,n)]\right\}^{-1}. \tag{32}$$

To conclude this section, let us discuss the singularities of the partial wave (29) in the ω plane, which determine the asymptotic behavior of the scattering amplitudes. These singularities arise because the moving poles $\nu=(\nu,n)$ in the upper and lower half-planes clamp the contour of integration with respect to ν for some $\omega=\omega_n$ when $\nu=0$. The position of these root singularities can be found from (24):

$$\omega_n=\frac{Ng^2}{2\pi^2}\int_0^1\frac{dx}{1-x}[x^{\frac{|n|-1}{2}}-1]. \quad (33)$$

In particular, the leading singularity lies at $\omega=\omega_0=(g^2/\pi^2)N\ln^2$ (Ref. 2) and give rise to a power-type rise in the total cross sections in the range in which LLA is valid. The other singularities lie at the point $\omega_1=0$, $\omega_2=-(g^2/\pi^2)N(1-\ln 2)$.... The singular part of the partial wave (29) behaves in the following way near the point $\omega=\omega_0$:

$$f_\omega^q(\rho,\rho')|_{\omega\to\omega_0}$$

$$=\frac{1}{16}\varphi_q^*(\rho)\varphi_q(\rho')(\omega-\omega_0)^{1/2}(-16\pi)\left(\frac{Ng^2}{\pi^2}\frac{7}{2}\zeta(3)\right)^{-1/2}, \quad (34)$$

where

$$\varphi_q(\rho)=\lim_{\nu\to0}\frac{\rho}{\nu}E_q^{0\nu}(\rho),\qquad \zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}.$$

Formula (34) contains a logarithmic divergence for $q\to0$. This is essentially a manifestation of the fact that, when $q=0$, the nature of the singularity is different for $\omega\to\omega_0$, i.e., $f_\omega^0|_{\omega\to\omega_0}\sim(\omega-\omega_0)^{-1/2}$ (Ref. 3). The behavior of the partial wave in the intermediate region, $(\omega-\omega_0)\ln^2|q\rho|\sim1$, can be found from (31):

$$f_\omega^q(\rho,\rho')|_{\substack{q\to0\\ \omega\to\omega_0}}$$

$$=|\rho\rho'|\frac{\pi c}{(\omega-\omega_0)^{1/2}}\left(\left|\frac{\rho}{\rho'}\right|^{2c(\omega-\omega_0)^{1/2}}+\left|\frac{\rho'}{\rho}\right|^{2c(\omega-\omega_0)^{1/2}}\right.$$

$$\left.-2|q^2\rho\rho'|^{2c(\omega-\omega_0)^{1/2}}\right),\quad c=[2\pi^2/7\zeta(3)Ng^2]^{1/2}. \quad (35)$$

The result given by (26) will be examined in the next section from the point of view of the Wilson operator expansion.[8]

3. Anomalous dimensions of composite operators for $j\to1$

Let us consider the asymptotic behavior of (26) in the range

$$|\rho_{1'2'}|\ll|\rho_{1'2}|\sim|\rho_{1'1}|\sim|\rho_{12}|. \quad (36)$$

Two nonoverlapping regions are then significant in the integrals with respect to ρ_0, namely, $|\rho_{01'}|\sim|\rho_{02'}|\sim|\rho_{1'2'}|$ and $|\rho_{01}|\sim|\rho_{12}|$, which provide the same contribution [see (A.1), (A.12), (A.13)]:

$$f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})|_{\rho_{1'2'}\to0}=\sum_{n=-\infty}^{+\infty}\oint d\nu E^{n,\nu}(\rho_{10'},\rho_{20'})$$

$$\times\pi^4\left(\frac{\rho^*_{1'2'}}{\rho_{1'2'}}\right)^{n/2}|\rho_{1'2'}|^{1+2i\nu}\left\{b_{n,\nu}\left[\nu^2+\left(\frac{n-1}{2}\right)^2\right]\right.$$

$$\left.\times\left[\nu^2+\left(\frac{n+1}{2}\right)^2\right][\omega-\omega(\nu,n)]\right\}^{-1}, \quad (37)$$

where $\rho_{0'}=(\rho_{1'}+\rho_{2'})/2$. If we close the integration contour on the poles in the lower half-plane, we obtain the representation of f_ω in the form of a sum. If we now introduce the auxiliary field $\varphi(\rho)$ and write the Green's function (26) in the form

$$f_\omega(\rho_1,\rho_2,\rho_1',\rho_2')=\langle0|\varphi(\rho_1)\varphi(\rho_2)\varphi(\rho_1')\varphi(\rho_2')|0\rangle, \quad (38)$$

we can interpret the resulting sum as the representation of the four-point Green's function

$$f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'})|_{\rho_{1'2'}\to0}$$

$$=\sum_{n=-\infty}^{+\infty}\sum_r\langle0|\varphi(\rho_1)\varphi(\rho_2)O_{n,r}(\rho_{0'})|0\rangle C_{n,r}$$

$$\times\left(\frac{\rho^*_{1'2'}}{\rho_{1'2'}}\right)^{n/2}|\rho_{1'2'}|^{2d_{r,n}}, \quad (39)$$

which arises when the Wilson operator expansion is used.[8] The dimension of the field $\varphi(\rho)$ is then the same as the canonical dimension ($d_\varphi=0$). This follows from a comparison of the general conformally invariant expression[7] for the three-point Green's function with the result for our case [see (19)]:

$$\langle0|\varphi(\rho_1)\varphi(\rho_2)O_{n,r}(\rho_{0'})|0\rangle\infty E^{n,\nu_r(n)}, \quad (40)$$

where $\nu_r(\omega,n)$ is a solution of (24).

Let us now elucidate the connection between the $O_{n,r}(\rho_{0'})$ in (39) and the gauge-invariant operators $O(x)$ constructed bilinearly from the strengths $G_{\mu\nu}$ and their covariant derivatives D_ρ.

It is precisely these operators that provide the principal contribution to the asymptotic behavior of the scattering amplitude in LLA.

At high energies, the colorless objects with momenta p_1, p_2, which consist of quarks, interact with one another through their gluon fields A_μ. Integration over the quark degrees of freedom leads to the appearance of two gauge-invariant factors in the functional integral over A_μ. For simplicity, let us consider an Abelian group. In LLA, these two factors are then constructed from the product of the following Lorentz components: $p_1^\mu F^\perp_{\sigma_1\mu}(x_1)$, $p_1^\mu F_{\sigma_2\mu}(x_2)$ and $p_2^\mu F^\perp_{\sigma_1\mu}(x_{1'})$, $p_2^\mu F_{\sigma_{2'}\mu}(x_{2'})$, respectively, and the scattering amplitude is expressed in terms of the four-point Green's function

$$G_{\sigma_1\sigma_2\sigma_{1'}\sigma_{2'}}=\langle0|T\prod_{i=1}^2p_1^\mu F^\perp_{\sigma_i\mu}(x_i)\prod_{i=1}^2p_2^\mu F^\perp_{\sigma_i'\mu}(x_{i'})|0\rangle. \quad (41)$$

The Wilson expansion can now be used to write the function G in factorized form, involving integration over the spins $j=1+\omega$ of the intermediate composite operators. The following operator of twist 2 is an example:

$$O_{\mu_1\ldots\mu_j}(x)=\underset{\mu_1\ldots\mu_j}{\mathrm{S}}\sum_{n_1}C_{n_1}\left(\prod_{i=2}^{n_1}\partial_{\mu_i}F_{\mu_1\sigma}(x)\right)\prod_{i=n_1+1}^{j-1}\partial_{\mu_i}F_{\mu_j\sigma}(x), \quad (42)$$

where the symbol $\underset{\mu_1\ldots\mu_j}{\mathrm{S}}$ stands for symmetrization in the Lorentz indices $\mu_1\ldots\mu_j$ and with the trace subtracted off. The coefficients C_{n1} are fixed by the condition for the irreducibility of the tensor (42) under conformal transformations.[9]

The expansion for the operator product $p_2^\mu F^\perp_{\sigma'_i\mu}(x_{i'})$

($i=1,2$) includes different Lorentz components of the tensor (42). The simplest is the contraction of (42) with the vector p_2 in each index:

$$\underbrace{O\ldots}_{j}(x)\equiv\prod_{i=1}^{j}p_2^{\mu_i}O_{\mu_1\ldots\mu_j}(x)\sim F_{\cdot\sigma}^{\perp}\partial_{\cdot}^{j-2}F_{\cdot\sigma}^{\perp}, \qquad (43)$$

where we have used the fact that, in LLA, each term in (42) provides the same contribution apart from the sign.

When the Mellin transform is applied to (41), the dependence on the longitudinal components of the coordinates x_i disappears and we have

$$G^{\omega}_{\sigma_1\sigma_2\sigma_1'\sigma_2'}\sim\partial_{\sigma_1}^{\perp}\partial_{\sigma_2}^{\perp}\partial_{\sigma_1'}^{\perp}\partial_{\sigma_2'}^{\perp}f_\omega(\rho_1,\rho_2,\rho_{1'},\rho_{2'}), \qquad (44)$$

where the derivatives act on the corresponding arguments. We note that, from the point of view of the representation (4), the appearance of the differentiations on the right-hand side of (44) is due to the fact that the functions Φ^1 and Φ^2 are proportional to the derivatives with respect to each of the impact parameters because of gauge invariance. This ensures that (6) is satisfied.

Comparison of (38), (41), and (44) leads to the conclusion that the following connection between the fields of the four- and two-dimensional theories can be made after the Mellin transformation:

$$p_{1,2}^{\mu}G_{\sigma\mu}^{\perp}(x_i)\xrightarrow{\omega}\partial_\sigma^{\perp}\varphi(\rho_i). \qquad (45)$$

Here and in what follows, we are considering a non-Abelian model.

A comparison analogous to (45) in the case of the operator (43) yields

$$\underbrace{O\ldots}_{j}(x)=\mathrm{Sp}\,G_{\cdot\sigma}(x)\,\partial_{\cdot}^{j-2}G_{\cdot\sigma}(x)\xrightarrow{\omega}\partial_\sigma^{\perp}\varphi(\rho)\,\partial_\sigma^{\perp}\varphi(\rho)$$
$$\equiv O_{0,0}(x), \qquad (46)$$

where we use the notation introduced above [see (39) and (40)].

The matrix element of (40) can readily be calculated in the free theory:

$$\langle 0|\varphi(\rho_1)\varphi(\rho_2)\partial_\sigma^{\perp}\varphi(\rho_0)\partial_\sigma^{\perp}\varphi(\rho_0)|0\rangle\sim\left|\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right|^2, \qquad (47)$$

where we have neglected terms that provide no contribution because of (6). When the interaction is turned on, the anomalous dimension of the operator (42) is, in our approximation [see (19), (24), (39), (40)]

$$\gamma_{00}=2-d_{00}=1-2i\nu_{00}=Ng^2/\pi^2\omega+(Ng^2/4\pi^2\omega)^4 2\zeta(3)+\ldots. \qquad (48)$$

This results agrees with exact two-loop calculations of γ_{00} in which the correction $\sim g^4$ contains only a first-order pole at the point $\omega=0$.

In the general case, the following operators have a nonzero anomalous dimension in LLA:

$$O_r^{\rho\varphi\mu_1\ldots\mu_m}(x)=\underset{\rho\varphi}{\mathrm{S}}\ \underset{\mu_1\ldots\mu_m}{\mathrm{S}}\ \mathrm{Sp}\sum_{n_1n_2}C_{n_1n_2}\Big((D^2)^{n_2}\prod_{i=2}^{n_1-1}D_{\mu_i}D_\sigma G_{\rho\mu_1}\Big)$$
$$\times(D^2)^{r-n_2}\prod_{i=n_1}^{m-1}D_{\mu_i}D_\sigma G_{\mu_m\varphi}, \qquad (49)$$

where $C_{n_1n_2}$ are determined by considering the conformal invariance.

The two-dimensional operator $O_{rn}(\rho)$ on the right-hand side of (39) and (40) is associated with a particular projection of the operator (49) after passing to the Mellin transformation:

$$\prod_{i=1}^{n}e_{\sigma_i}\underbrace{O_r^{\sigma_1\ldots\sigma_n}}_{j}\xrightarrow{\omega}O_{rn}(\rho)$$
$$=\sum_{n_1n_2}\tilde{C}_{n_1n_2}(\nabla^{2n_2}\partial^{n_1-2}\nabla\varphi)\,\nabla^{2(r-n_2)}\partial^{n-n_1}\nabla\varphi, \qquad (50)$$

where e is the two-dimensional light-like vector $e=(e^1+ie^2)/\sqrt{2}$.

Conformal and scale invariance then yield

$$\langle 0|\varphi(\rho_1)\varphi(\rho_2)O_{rn}(\rho_0)|0\rangle=E_{n,\nu_r(n)}(\rho_{10},\rho_{20}), \qquad (51)$$

where

$$\nu_r(n)=-i(^1/_2(|n|+1)+r-{}^1/_2\gamma_{n,r}), \qquad (52)$$

and $\gamma_{n,r}$ is the anomalous dimension of the operator (49) which can be found from (24) in the form of a series expansion:

$$\gamma_{n,r}=\frac{Ng^2}{\pi^2\omega}-\left(\frac{Ng^2}{2\pi^2\omega}\right)^2(s_{|n|+r}+s_r)+\ldots,\qquad s_r=\sum_{k=1}^{r}\frac{1}{k}. \qquad (53)$$

The solutions (52) of (24) for different r can be looked upon as the values of a single analytic function on different sheets of a Riemann surface. It is therefore natural to combine the quantities in (49) into families containing operators of different twist. The reason for this unification can be the approximate invariance of the theory under generally conformal two-dimensional transformations.[10] It is clear from (52) and (53) that this property is satisfied only in the single-loop approximation $\gamma\sim g^2$. The breaking of the general conformal invariance is apparently due to the fact that the energy-momentum tensor is not traceless in transverse space.

4. Trajectory of the bare pomeron in QCD

We shall now use the above results to find the trajectories $j(q^2)$ of the bare pomeron in QED for large q^2, and the lower bound for its intercepts. The derivation is based on the idea that if, in accordance with experiment, we suppose that the trajectory of the bare pomeron is close to unity, i.e.,

$$\omega(q^2)\ll 1, \qquad (54)$$

the effective coupling constant

$$\alpha_s(k)\equiv g^2(k)/4\pi=4\pi/\beta_2\ln\frac{k^2}{\Lambda^2},\qquad \beta^2=11-\frac{2}{3}n_f \qquad (55)$$

(n_f is the number of flavors) turns out to be small:

$$\alpha_s\sim\omega(q^2)\ll 1, \qquad (56)$$

so that the LLA equations are valid (with one modification) for the evaluation of the pomeron trajectory.

Let us begin with small values of q^2:

$$q^2 \lesssim \Lambda^2. \quad (57)$$

It will be convenient to use the mixed representation [see (30)]. For sufficiently small ρ such that

$$|\rho| \ll 1/|q| \quad (58)$$

we can seek the solution of (7) in the form [see (A.1)]

$$E^\nu(\rho) = E_0^{0,\nu}(\rho) = |\rho|^{2i\nu}. \quad (59)$$

We shall confine our attention to the $n = 0$ case, since it is only for zero conformal spin that the singularities of f_ω appear for $\omega > 0$ [see (33)].

The function (59) is a solution of the Bethe-Salpeter equation (7) in the mixed representation, but only ν satisfies (24) which, in our case ($N = 3, n = 0$), we write in the form

$$\omega = \alpha_s \chi(\nu), \quad (60)$$

where $\alpha_s = g^2/4\pi$, and

$$\chi(\nu) = \frac{6}{\pi} \int_0^1 \frac{dx}{1-x} [x^{-1/2} \cos(\nu \ln x) - 1]. \quad (61)$$

Equation (24) for the function (59) has the form

$$\omega \mathscr{E}(\rho) = \alpha_s \chi\left(-\frac{i}{2}\frac{d}{d\ln|\rho|}\right) \mathscr{E}(\rho). \quad (62)$$

In accordance with (55), to take asymptotic freedom into account, we can replace the constant α_s in (62) with a variable guantity:

$$\alpha_s \to 4\pi/\beta_2 \ln \frac{1}{|\rho|^2 \Lambda^2}. \quad (63)$$

Another method is to use the renormalization group equations. Equation (60) is then solved for ν:

$$\nu = \chi^{-1}(\omega/\alpha_s), \quad (64)$$

where each solution corresponds to the anomalous dimension (52) of some local operator. The renormalization group equation for the matrix element $\mathscr{E}(\rho)$ is constructed from (64) in a standard manner:

$$-\frac{i}{2}\frac{d}{d\ln|\rho|} \mathscr{E}(\rho) = \chi^{-1}\left(\frac{\omega}{\alpha_s}\right) \mathscr{E}(\rho), \quad (65)$$

where the constant α_s must be replaced with the running constant (63). Let us rewrite (65) in the form:

$$\omega = \alpha_s(\rho) \chi\left(-\frac{i}{2}\frac{d\ln \mathscr{E}(\rho)}{d\ln|\rho|}\right) \quad (66)$$

and compare it with (62). The Bethe-Salpeter equation (62) may be looked upon as an analog of the Schrödinger equation of a mechanical system in which the stationary Hamilton-Jacobi equation is identical with (66). Clearly, the "quantization" of the renormalization group equation (66) is not single-valued because $d/d\ln|\rho|$ and $\alpha_s^{-1} \sim \ln|\rho|$ do not commute. Equation (62) is one variant of this quantization. To determine the order in which the noncommuting operators appear on the right-hand side of (62), we must calculate the corrections in LLA. However, in the region defined by (56), $\alpha_s(\rho)$ as given by (63) is quasiclassical, so that this order is unimportant. Equations (62) and (65) are then equivalent except at the "turning points."

The solution of (65) for small enough ρ is

$$\mathscr{E}(\rho) \propto \exp\left[2i \int^r dr' \chi^{-1}\left(\frac{\omega}{\alpha_s(r')}\right)\right],$$
$$r = \ln(|\rho|\Lambda), \quad \alpha_s(r) = -4\pi/2\beta_2 r, \quad (67)$$

where it is assumed [see (48)] that

$$\chi^{-1}\left(\frac{\omega}{\alpha_s(r')}\right)\Big|_{r'\to-\infty} \to \frac{1}{2i} + O(\alpha_s), \quad (68)$$

i.e., we select the solution that decreases as $\rho \to 0$.

As r increases, the imaginary part of $\chi^{-1}[\omega/\alpha_s(r)]$ decreases and, when $r > r_0$, where [see (33)]

$$-\beta_2(\omega r_0/2\pi) = 12 \ln 2/\pi, \quad (69)$$

the function becomes real, which means that we have reached the cut of anomalous dimension (48). Equation (62) becomes simpler near the turning point (69):

$$-\frac{\beta_2}{2\pi}(r - r_0)\omega \mathscr{E}(r) = -\frac{42}{\pi}\zeta(3)\left(-\frac{i}{2}\frac{d}{dr}\right)^2 \mathscr{E}(r) \quad (70)$$

and can be solved in terms of Airy functions:

$$\mathscr{E}(r) \propto \int dp \exp\left\{i\left[p(r - r_0) - \frac{7\zeta(3)}{\beta_2\omega} p^3\right]\right\}, \quad (71)$$

which enables us to find the phase $\varphi = \pi/4$ in the quasiclassical solution below the turning point:

$$\mathscr{E}(r)|_{r>r_0} \propto \cos\left[2\int_{r_0}^r dr' \chi^{-1}\left(\frac{\omega}{\alpha_s(r')}\right) - \frac{\pi}{4}\right]. \quad (72)$$

It is important to note that this phase does not depend on ω. It is convenient to transform (72) to the new variable of integration ν':

$$\mathscr{E}(r) \propto \cos\left[2r\nu(r) + \frac{4\pi}{\beta_2\omega}\int_0^{\nu(r)} \chi(\nu') d\nu' - \frac{\pi}{4}\right], \quad (73)$$

where $\nu(r)$ is given by (64). When

$$\alpha_s(r)/\omega \gg 1, \quad \alpha_s(r) \ll 1 \quad (74)$$

we have

$$\nu(r) \to \nu_0 \approx 0.637, \quad (75)$$

$$\int_0^{\nu(r)} \chi(\nu') d\nu' \to a \approx 0.919. \quad (76)$$

We note that $\nu = \nu_0$ corresponds to a solution of the equation $\chi(\nu) = 0$, i.e., when (74) is satisfied, the term proportional to ω can be neglected (62). We shall assume that (62) remains independent of ω for $\omega \ll 1$ if $\alpha_s(r) \sim 1$, which implies that perturbation theory is invalid. The asymptotic behavior of the solution as we leave this region should then be

$$\mathscr{E}(r)\propto\cos(2r\nu_0+\pi\eta(q^2)), \tag{77}$$

where the phase $\pi\eta$ is determined by the interaction at large distances and, by definition, tends to a finite limit when $\omega\to 0$.

The mixing of the logarithmic derivatives of the functions (73) and (77) in the region (74) leads to the quantization of the spectrum of Regge poles on the ω plane:

$$\omega=\omega_k(q^2)=\frac{4a}{\beta_2}\frac{1}{k+\eta(q^2)+{}^1/_4}, \quad k=0,\ \pm1,\ \pm2\ldots, \tag{78}$$

where the quantity a is given by (76). Without loss of generality, the parameter η can be regarded as lying in the region $-1/4\leqslant\eta\leqslant 3/4$ for sufficiently small q^2. We then obtain the following inequalities for the intercepts of the poles (78), which condense from the right to the point $\omega=0$:

$$\frac{4a}{\beta_2 k}\geqslant\omega_k(q^2)\,|_{q^2=0}\geqslant\frac{4a}{\beta_2(k+1)}\approx\frac{0{,}4}{k+1}, \quad k=0,1,\ldots\ . \tag{79}$$

We recall that (78) and (79) were derived on the assumption that the ω_k were so small that the following two inequalities became compatible for some range of values of r:

$$\omega_k\ll\chi(0)\alpha_s(r)=12\pi^{-1}\ln 2\alpha_s(r), \quad \alpha_s(r)\ll 1. \tag{80}$$

It is clear from (79) that the inequalities (80) can readily be satisfied for parametrically large k. The conditions in (80) can be satisfied only approximately $[0.4<3\alpha_s(r)<3]$ for the leading pole corresponding to $k=0$.

The numerical size of the nonlogarithmic terms $\sim\alpha_s^2$ in (62), which at least in principle can be calculated, would enable us to find its range of validity, i.e., the upper bound $\alpha_s(r_0)$ for the effective coupling constant:

$$\alpha_s(r)<\alpha_s(r_0). \tag{81}$$

The matching of (77) and (73) at $r=r_0$ gives us expressions more accurate than (78):

$$\omega=\omega_k(q^2)=\frac{4a(r_0)}{\beta_2}\frac{1}{k+\eta(q^2)+{}^1/_4},\qquad a(r_0)=\int\limits_0^{\nu(r_0)}\chi(\nu')\,d\nu', \tag{82}$$

where $\nu(r_0)$ is determined from (64):

$$-4a(r_0)r_0=2\pi\chi(\nu(r_0))\,(k+\eta(q^2)+{}^1/_4). \tag{83}$$

Equations (82) and (83) are valid if the following inequalities can be simultaneously satisfied for certain r [see (80)]

$$\omega_k(q^2)\leqslant 12\pi^{-1}\ln 2\alpha_s(r), \quad \alpha_s(r)<\alpha_s(r_0)\ll 1. \tag{84}$$

The numerical value of the lower bound for the intercept of the bare pomerons obtained from (82) and (83) is somewhat lower than that in (79) [for $\alpha_s(r_0)=0.2$, we have min $\omega_0\simeq 0.3$].

We must now consider large momentum transfers $(-q^2)^{1/2}\gg\Lambda$. In the region $\rho\sim 1/q$, where the running coupling constant is fixed by $\alpha_s=\alpha_s[-\ln(q/\Lambda)^{1/2}]$, the solution of the homogeneous equation in the mixed representation is given by (30). Its asymptotic behavior for $\rho\ll 1/q$ and $n=0$ can be obtained from (A.1) in the Appendix:

$$\mathscr{E}(r)\propto E_q^{0,\nu^k}(\rho)\,|_{\rho\ll 1/q}=2\,|q|^{2i\nu^k}\cos[2\nu^k r+\pi\eta^k(q^2)],$$
$$\eta^k(q^2)=\frac{1}{\pi}\left(2\nu^k\ln\frac{|q|}{\Lambda}+\frac{\delta(0,\nu^k)}{2}\right), \tag{85}$$

where the phase $\delta(0,\nu)$ is given in (A.2) and the spectrum of the values of ν^k is obtained from the set of equations (82) and (83), which arises from matching the functions (73) and (85) for $\rho\sim 1/q$:

$$\int\limits_0^{\nu^k}\chi(\nu')\,d\nu'/\chi(\nu^k)-\nu^k=\alpha_s(q^2)\,\frac{\beta_2}{8}\left(2k+\frac{1}{2}+\frac{\delta(0,\nu^k)}{\pi}\right),\quad k=0,1,2,\ldots \tag{86}$$

The corresponding frequency spectrum is given by (83):

$$\omega_k(q^2)=\alpha_s(q^2)\chi(\nu^k). \tag{87}$$

For $\ln(|q|/\Lambda)\to\infty$, we can neglect the phase $\delta(0,\nu)$ in (85), and (86) then yields $\nu^k\to 0$, i.e., the set of Regge poles is approximately simulated by a moving cut with the trajectory

$$\omega_k(q^2)\,|_{\ln(q/\Lambda)\gg 1}=\frac{12}{\pi}\ln 2\alpha_s(q^2). \tag{88}$$

The table lists the trajectories $\omega=\omega_k[\alpha(q^2)]$ (87) for $\alpha(q^2)<1$ when $\beta_2=11-(2/3)n_f=9$.

TABLE I.

$k=0$			$k=1$		
ν	α_s	ω	ν	α_s	ω
0.2	0.028	0.058	0.3	0.040	0.062
0.25	0.062	0.114	0.4	0.123	0.128
0.3	0.126	0.198	0.45	0.215	0.169
0.34	0.216	0.291	0.48	0.304	0.197
0.37	0.319	0.380	0.5	0.389	0.216
0.4	0.473	0.490	0.52	0.507	0.237
0.42	0.618	0.578	0.54	0.677	0.258
0.44	0.812	0.679	0.55	0.793	0.269
0.45	0.934	0.736	–	–	–

Thus, for large values of q^2 and $\rho \sim \rho' \sim 1/q$, inclusion of asymptotic freedom will modify the pomeron Green's function (29) as follows:

$$f_\omega{}^q(\rho,\rho') = \frac{|\rho\rho'|}{16} \sum_{n=-\infty}^{+\infty} \sum_k \frac{d\nu^k}{dk} (E_q^{n,\nu^k}(\rho'))^* E_q^{n,\nu^k}(\rho) \times \Big\{ \Big[(\nu^k)^2 + \Big(\frac{n-1}{2}\Big)^2 \Big] \Big[(\nu^k)^2 + \Big(\frac{n+1}{2}\Big)^2 \Big] \times [\omega - \omega_k{}^n(q^2)] \Big\}^{-1}, \quad (89)$$

where the trajectories $\omega_k^n(q^2)$ are given by (87), generalized (for $n \neq 0$) in accordance with (24), which leads to a dependence of ν^k on n.

We have thus calculated the Green's function (89) of the bare pomeron within the framework of QCD. The next step would be to construct the Reggeon diagram technique[4] and hence find the parameters of the renormalized Pomeranchuk singularity. Conformal invariance would enable us to reduce the solution of the set of equations for the vertex function and the pomeron Green's function to a purely algebraic problem of finding the anomalous dimensions of the composite operators for $\omega \to 0$ (see Section 3 above), since the three-pomeron vertex is fixed by this invariant to within a constant factor.

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APPENDIX

We shall now derive the mathematical formulas used in the main text. First, we show that, for small q, the function $E_q^{n,\nu}(\rho)$ (35) is given by the following asymptotic formula:

$$E_q^{n,\nu}(\rho)\big|_{q\ll 1/\rho} = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2R}{|\rho|} e^{iqR} E^{n,\nu}\Big(R+\frac{\rho}{2}, R-\frac{\rho}{2}\Big)\Big|_{q\ll 1/\rho} = \Big(\frac{\rho}{\rho^*}\Big)^{n/2} |\rho|^{-2i\nu} \Big[1 + \Big(\frac{q\rho^*}{q^*\rho}\Big)^n |q\rho|^{4i\nu} e^{i\delta(n,\nu)}\Big], \quad (A.1)$$

where

$$e^{i\delta(n,\nu)} = 2^{-8i\nu} \frac{\Gamma(-2i\nu+|n|+1)}{\Gamma(2i\nu+|n|+1)} \frac{\Gamma(i\nu+(|n|+1)/2)}{\Gamma(-i\nu+(|n|+1)/2)} \times \frac{\Gamma(-i\nu+|n|/2)}{\Gamma(i\nu+|n|/2)}, \quad (A.2)$$

$$b_{n,\nu} = \pi^3 \frac{1}{-i\nu+|n|/2} 2^{4i\nu} \times \frac{\Gamma(-i\nu+(1+|n|)/2)}{\Gamma(i\nu+(1+|n|)/2)} \frac{\Gamma(i\nu+|n|/2)}{\Gamma(-i\nu+|n|/2)}. \quad (A.3)$$

Let us subdivide the integral in (A.1) into two terms corresponding to the regions of integration $R \sim \rho$ and $R \sim 1/q$:

$$|\rho| \frac{b_{n,\nu}}{2\pi^2} E_q^{n,\nu}(\rho)\Big|_{q\ll 1/\rho} = J_1(\rho) + J_2{}^q(\rho), \quad (A.4)$$

where

$$J_1(\rho) = \int d^2R E^{n,\nu}\Big(R+\frac{\rho}{2}, R-\frac{\rho}{2}\Big) = (-1)^n \int d^2R \Big[\frac{(R+\rho/2)(R-\rho/2)\rho^*}{(R^*+\rho^*/2)(R^*-\rho^*/2)\rho}\Big]^{n/2} \times \Big[\frac{|\rho|}{|R+\rho/2||R-\rho/2|}\Big]^{1+2i\nu} \quad (A.5)$$

$$J_2{}^q(\rho) = (-1)^n \int d^2R\, e^{iqR} \Big(\frac{R^2\rho^*}{R^{*2}\rho}\Big)^{n/2} \Big|\frac{\rho}{R^2}\Big|^{1+2i\nu}. \quad (A.6)$$

It is implied in (A.5) and (A.6) that the parameter ν has infinitesimal imaginary components that ensure that the integrals converge.

We now transform to a new integration variable z such that $R = \rho z/2$. We then have

$$J_1(\rho) = \Big(\frac{\rho}{\rho^*}\Big)^{n/2} |\rho|^{1-2i\nu} (-1)^n 2^{4i\nu} C, \quad (A.7)$$

where

$$C = \int d^2z \Big(\frac{z^2-1}{z^{2*}-1}\Big)^{n/2} \Big|\frac{1}{z^2-1}\Big|^{1+2i\nu} \quad (A.8)$$

depends only on the modulus of n. The constant C is readily calculated by using the Wick development of the contour of integration with respect to y ($z = x - t \equiv \alpha, z^* = x + t \equiv \beta$):

$$C = \frac{i}{2} \int d\alpha\, d\beta \times \frac{(1-\alpha^2)^{|n|}(-1)^n}{\{[(1-\alpha)(1-\beta)+i\varepsilon][(1+\alpha)(1+\beta)+i\varepsilon]\}^{(1+|n|)/2+i\nu}} = (-1)^n \int_{-1}^{1} d\alpha (1-\alpha^2)^{(|n|-1)/2} \sin\Big(\frac{|n|+1}{2} + i\pi\Big) \times \int_1^\infty \frac{d\beta}{(\beta^2-1)^{(|n|+1)/2+i\nu}}. \quad (A.9)$$

The last integrals can be expressed in terms of Γ-functions, which yields

$$J_1(\rho) = |\rho| \frac{b_{n,\nu}}{2\pi^2} \Big(\frac{\rho}{\rho^*}\Big)^{n/2} |\rho|^{-2i\nu}, \quad (A.10)$$

where $b_{n,\nu}$ is given by (A.3). By transforming to radial variables, we can readily evaluate (A.6):

$$J_2{}^q(\rho) = |\rho| \frac{b_{n,\nu}}{2\pi^2} \Big(\frac{\rho^*}{\rho}\Big)^{n/2} \Big(\frac{q}{q^*}\Big)^n |\rho q^2|^{2i\nu} e^{i\delta(n,\nu)}, \quad (A.11)$$

where the last factor is given above [see (A.3)]. Using (A.4), (A.10), and (A.11), we finally arrive at (A.1).

We must now prove that

$$E^{n,\nu^*}(\rho_{10},\rho_{20}) = \frac{b^*_{n,\nu}}{a_{n,\nu}} \int d^2\rho_{0'} E^{n,\nu}(\rho_{10'},\rho_{20'}) |\rho_{00'}|^{-2+4i\nu} \Big(\frac{\rho^*_{0'0}}{\rho_{0'0}}\Big)^n, \quad (A.12)$$

where

$$a_{n,\nu} = \frac{\pi^4/2}{\nu^2+n^2/4} = \frac{|b_{n,\nu}|^2}{2\pi^2}. \quad (A.13)$$

The fact the right- and left-hand sides of (A.12) are proportional follows from considerations of conformal invariance. Hence, all that we have to do is to find the proportionality constant. Integrating (A.12) with respect to the variable $(\rho_1+\rho_2)/2$ with the weight $\exp\{-i^{1/2}(\rho_{10}+\rho_{20})q$ and using [see (A.6) and (A.11)]

$$\int d^2\rho e^{iq\rho}|\rho|^{-2+4i\nu}\left(\frac{\rho^*}{\rho}\right)^n$$
$$=(-1)^n\frac{b^*_{n,\nu}}{2\pi^2}|q|^{-4i\nu}\left(\frac{q^*}{q}\right)^n e^{-i\delta(n,\nu)} \qquad (A.14)$$

we can rewrite (A.12) in the mixed representation:

$$E_q^{n,\nu^*}(\rho)=e^{i\delta(-n,-\nu)}E_q^{n,\nu}(\rho)|q|^{-4i\nu}(q^*/q)^n. \qquad (A.15)$$

It is readily verified that, when (A.1) is used, (A.15) is actually valid for $q\rho\ll 1$, which ensures its validity for any q and ρ.

Next, we show that the orthonormalization condition for the functions $E^{n,\nu}(\rho_{10},\rho_{20})$ is

$$\int\frac{d^2\rho_1\,d^2\rho_2}{|\rho_{12}|^4}E^{n,\nu}(\rho_{10},\rho_{20})E^{m,\mu^*}(\rho_{10'}\rho_{20'})=a_{n,\nu}\delta_{n,m}\delta(\nu-\mu)\delta^2(\rho_{00'})$$
$$+b_{n,\nu}|\rho_{00'}|^{-2-4i\nu}(\rho_{00'}/\rho^*_{00'})^n\delta_{n,-m}\delta(\nu+\mu). \quad (A.16)$$

The functional dependence on $\rho_{00'}$, $\rho^*_{00'}$ and m, μ on the right-hand side of (A.16) follows from considerations of conformal invariance. It is sufficient for our purposes to verify the constant factors $a_{n,\nu}b_{n,\nu}$ (A.3) and (A.13). Let $\rho_0=1,\rho_{0'}=0$ and let us introduce the new integration variables ρ and λ, defined by

$$\rho_1=\frac{\rho}{\lambda+(1-\lambda)\rho},\quad \rho_2=-\frac{\rho}{\lambda-(1-\lambda)\rho},$$
$$\frac{d^2\rho_1\,d^2\rho_2}{|\rho_{12}|^4}=\frac{d^2\rho\,d^2\lambda}{4|\lambda|^2}. \qquad (A.17)$$

Since the dependence on λ,λ^* in the expressions for $E^{n,\nu}$ and $E^{m,\mu}$ can be factored:

$$E^{n,\nu}(\rho_{10},\rho_{20})=(-1)^n\left(\frac{\lambda}{\lambda^*}\right)^{n/2}\left|\frac{1}{\lambda}\right|^{1+2i\lambda}$$
$$\times\left[\frac{(1-\rho)(1+\rho)\rho^*}{(1-\rho^*)(1+\rho^*)\rho}\right]^{n/2}\left|\frac{2\rho}{(1-\rho)(1+\rho)}\right|^{1+2i\nu},$$
$$E^{m,\mu}(\rho_{10'},\rho_{20'})=\left(\frac{\lambda}{\lambda^*}\right)^{m/2}|\lambda|^{1-2i\mu}\left(\frac{\rho^*}{\rho}\right)^{m/2}\left|\frac{2}{\rho}\right|^{1-2i\mu}, \qquad (A.18)$$

we can integrate with respect to λ and ρ:

$$\int\frac{d^2\rho_1\,d^2\rho_2}{|\rho_{12}|^4}E^{n,\nu}(\rho_1-1,\rho_2-1)E^{m,\mu^*}(\rho_1,\rho_2)$$
$$=2\pi^2\delta_{n,-m}\delta(\nu+\mu)(-1)^n 2^{4i\nu}\int d^2\rho$$
$$\times\left(\frac{1-\rho^2}{1-\rho^{*2}}\right)^{n/2}\left|\frac{1}{1-\rho^2}\right|^{1+2i\nu}=b_{n,\nu}\delta_{n,-m}\delta(\nu+\mu), \qquad (A.19)$$

where, in the last transformation, we use (A.7), (A.8), and (A.10). Thus, the constant factor in the second term on the right-hand side of (A.16) is, in fact, equal to $b_{n,\nu}$. To verify the constant factor in the first term, we integrate both sides of (A.16) with respect to $\rho_{00'}$. From the left-hand side, we obtain [see (A.5) and (A.10)]

$$\frac{b_{n,\nu}b^*_{m\mu}}{(2\pi^2)^2}\int\frac{d^2\rho}{|\rho|^2}\left(\frac{\rho}{\rho^*}\right)^{(n-m)/2}|\rho|^{2i(\mu-\nu)}$$
$$=\frac{\pi^6}{2\pi^2(\nu^2+n^2/4)}\delta_{nm}\delta(\nu-\mu). \qquad (A.20)$$

On the right-hand side of (A.16), only the first term provides a contribution and, by virtue of (A.13), the results of integration are the same, i.e., (A.16) is proved.

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