

# Low-frequency fluctuation phenomena near the metal-dielectric transition point

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The  $1/f$  noise problem in the low-temperature limit and in the vicinity of the metal-dielectric transition point is reduced to the problem of the strong interaction between the long-range correlations of the diffusional and critical types. The physical causes of the enhancement of the frequency dependence of the current-current four-electron correlators as the metal-dielectric transition line is approached are elucidated. The critical-point exponent  $\alpha$  is computed by the  $\varepsilon$ -expansion method in  $d = 4 - \varepsilon$  dimensions.

The difficulties that must be overcome in order to find the general solution to the  $1/f$  noise problem is, for the present, even hard to imagine.<sup>1,2</sup> But the universality hypothesis suggests we look for the manifestations of similar phenomena in those temperature and frequency regions and in those models for which we have well-developed methods of theoretical investigations. In the present paper, as the simplest model, we choose the electron Fermi gas in which the particles are acted upon by a weak short-range repulsive force, and undergo scattering on fixed spinless impurities. In the low-temperature limit this system has two relaxation times: the momentum relaxation time  $\tau_p$  and the energy relaxation time  $\tau_\varepsilon$ . The problem is to find a way of computing the spectral composition of the noise in the frequency region

$$1/\tau_\varepsilon \ll \omega \ll \omega^* = \min(1/\tau_p, T), \quad (1)$$

where  $1/\tau_\varepsilon \sim T^2/\varepsilon_f$  ( $\varepsilon_f$  is the Fermi energy).

We shall show that, within the framework of the model in question and in the broad frequency range (1), the spectral composition of the noise both in the case of a prescribed constant current and in the case of a prescribed constant voltage potential has the intrinsically singular character

$$S_\omega \sim a\omega^{-\alpha}. \quad (2)$$

Here  $a$  is a dimensionless and nonuniversal quantity, called the Hooze constant. The critical exponent  $\alpha$  will be computed in  $(d = 4 - \varepsilon)$ -dimensional space both in the far-metallic phase and in the high-frequency region in the vicinity of the metal-dielectric transition point. The low-frequency region  $\omega \ll 1/\tau_\varphi$ , where  $\tau_\varphi$  is the electron-phase breaking time,<sup>3</sup> remains outside the purview of our analysis, since the model under investigation has  $\tau_\varphi \sim \tau_\varepsilon$ . But in practice, in the low-temperature limit  $1/\tau_\varphi \gg 1/\tau_\varepsilon$ , which drastically narrows the region of applicability of our results.

The critical exponent  $\alpha$ , by the scheme of its computation and by its physical meaning, resembles the corresponding exponent for the specific heat of a material in the vicinity of a second-order phase transition point. It is on this analogy that the above-mentioned universality hypothesis is based.

In Sec. 1 of the present paper we derive the first, experimentally measurable terms of the expansion of the spectral correlator in powers of the constant current and in powers of the constant voltage potential. It turns out in this case that the corresponding expansion coefficients can be expressed in terms of four-current irreducible correlators differing in the signs of the frequencies of the single-particle electron Green functions.

In Sec. 2 we show that the four-current correlators can be expressed in terms of two-current four-electron correlators, which in turn can be expressed in terms of scalar four-electron irreducible vertex parts. In this section we also write out for the scalar vertices a system of nonlinear equations that is valid in four-dimensional space, and is determined by the long-range interaction between the electronlike excitations. Section 3 is devoted to the computation of the exponent  $\alpha$  in  $(4 - \varepsilon)$ -dimensional space. To do this, we find the asymptotic solutions to the system of coupled (renormalization-group) equations obtained in Sec. 2. An  $\varepsilon \rightarrow 1$  extrapolation furnishes the results pertaining to three-dimensional space. In the far-metallic phase  $\alpha = 5/8$ , and the singular character of the spectral density is governed by the interaction between the diffusional and Cooper<sup>4</sup> excitations. As the transition point is approached in the direction of the dielectric phase, the  $\alpha$  increases, and attains a value of unity, this being made possible by additional (critical) fluctuations of the metal order parameter,<sup>5</sup> which are important at fairly high frequencies in the vicinity of the transition point:

$$1 \gg \omega/\varepsilon_0 \gg |\tau|^{1/2} \gg 1/\tau_\varphi \varepsilon_0. \quad (3)$$

Here  $\tau$  is a dimensionless quantity characterizing the proximity to the metal-dielectric transition point (hereafter: *M-D* transition);  $\varepsilon_0$  is a characteristic energy not exceeding the reciprocal momentum relaxation time. In the region of lower frequencies

$$1 \gg |\tau|^{1/2} \gg \omega/\varepsilon_0 \gg 1/\tau_\varphi \varepsilon_0 \quad (4)$$

the critical vibrations are unimportant, so that we again find that  $\alpha = 5/8$ —as before, in the  $\varepsilon$  approximation linear in  $\varepsilon$ .

To conclude the paper, we present various estimates for the time  $\tau_\varphi$ , and discuss the question of the extrapolation of the results obtained into the region of extremely low frequencies and finite temperatures, where the  $1/f$ -type noise is usually observed.

## 1. FORMULATION OF THE PROBLEM AND GENERAL RELATIONS

The spectral density (2) is the first nonvanishing term (attached to  $j^2$ ) of the expansion of the spectral function of the mean-square current-current correlator  $\langle \hat{j}_\alpha \hat{j}_\beta \rangle_{\omega q}$ . For the purpose of using the thermodynamic diagrammatic technique, it is convenient to express the quantum current-current correlator in terms of the retarded  $Q_{\alpha\beta}^R$  and advanced  $Q_{\alpha\beta}^A$  correlators with the aid of the dissipation-fluctuation

theorem (see Ref. 6):

$$\langle j_{\alpha\beta} \rangle_{\omega, \mathbf{q}} = \frac{i}{2} [Q_{\alpha\beta}^A(\omega, \mathbf{q}) - Q_{\alpha\beta}^B(\omega, \mathbf{q})] \text{cth} \left( \frac{\omega}{2T} \right). \quad (5)$$

Using the principle of analytic continuation, we express the correlator difference in terms of the product of the retarded and advanced electron Green functions, which has to be averaged over the random disposition of the impurity centers. In the limit  $|\omega| \ll T$  the integration over the internal frequency is replaced by the product of  $\omega$  and the mean of the product of the Matsubara Green functions, one of which is taken at  $\omega_n = (2n + 1)\pi T > 0$  and the other at  $\omega_n < 0$ . The proof of this assertion is entirely equivalent to the proof that is given in the computation of static conductivity (see, for example, Ref. 7).

In a system with a prescribed total current the Hamiltonian has an additional term,  $\hat{\mathbf{j}}\mathbf{v}$ , where  $\mathbf{v}$  is a Lagrange multiplier proportional to the mean current and  $\hat{\mathbf{j}}$  is the current-density operator in second quantization. Expanding each thermodynamic Green function in powers of  $\hat{\mathbf{j}}\mathbf{v}$  up to second order, we obtain the required coefficient  $S_\omega$ . Thus, the problem reduces to that of computing the four-current and four-electron Green functions that do not break up into irreducible two-electron correlators (Fig. 1). At each vertex in the diagrams in Figs. 1a, 1b, and 1c, in which the sign (+ or -) of the frequency does not change during the motion along an electron line, the operator  $\hat{\mathbf{j}}$ , represented by a cross, is scalar-multiplied by the average velocity  $\mathbf{v}$ . The spectral density, (1), of interest to us is proportional to the sum of the four-current correlators without the factors  $v_\alpha v_\beta$ , i.e.,

$$S_j \propto \Gamma_j(-++-) + \Gamma_j(-+++)+\Gamma_j(+---). \quad (6)$$

Here and below we do not write the vector indices  $\alpha, \beta, \gamma$ , and  $\mu$ ;  $\Gamma_j(\omega_1, \omega_2, \omega_3, \omega_4) \equiv \Gamma_j(1, 2, 3, 4)$  is the four-current vertex, which is invariant under cyclic permutation of the indices:

$$\begin{aligned} \Gamma_j(1, 2, 3, 4) &= \Gamma_j(2, 3, 4, 1) \\ &= \Gamma_j(3, 4, 1, 2) = \Gamma_j(4, 1, 2, 3). \end{aligned} \quad (7)$$

The decisive role played by the four-current correlators and their scale invariance in the determination of  $1/f$  noise is pointed out in the review in Ref. 8. Usually, in experiment, it is not the mean current that is prescribed, but a constant potential difference. In this case the two-current spectral

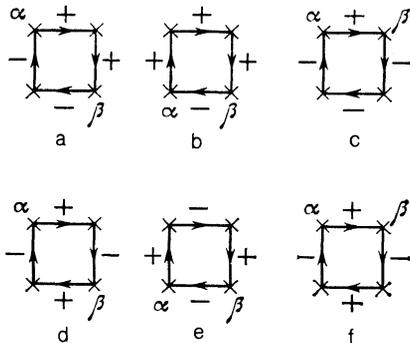


FIG. 1. The various types of current vertices, written in the zeroth-order approximation.

density (5) should be expanded in powers of the constant electric field  $\mathbf{E}$ . Slightly more complicated, but entirely similar arguments lead to the conclusion that we must, in the low-frequency limit, average the sum of the three four-current correlators with the alternating-sign thermodynamic Green functions, as shown in Figs. 1d, 1e, and 1f. Separating out the factor  $E_\alpha E_\beta$ , and using the cyclic-permutation invariance, we obtain

$$S_E \propto \Gamma_j(+--+). \quad (8)$$

In the present paper we do not propose to compute the Hooke constant, i.e., the constant of proportionality between  $S_E, S_j$  and  $\Gamma_j$ ; the problem is to find the exponent  $\alpha$  only. For this same reason, it is not necessary to compute the vertex parts  $\Gamma_j$  in the zeroth and first approximations in the number of critical and diffusion propagators (see below). Let us only note that in the zeroth approximation the current-current vertices are constants; in the first approximation they contain the factor  $1/\sqrt{\omega}$ , but these singularities are completely canceled out in the final formula (6).

## 2. RENORMALIZATION-GROUP EQUATIONS FOR THE FOUR-VERTEX CORRELATORS

In spite of the generality of the relations obtained below, all the specific intermediate calculations are carried out within the framework of the impurity model of the excitonic dielectric with a half-filled band. In doing this, we put in correspondence with each electron Green function a  $2 \times 2$  inverse matrix, i.e., we set

$$G_\omega^{-1}(\mathbf{p}) = \begin{vmatrix} i\tilde{\omega} - \xi_{\mathbf{p}}, & \tilde{\Delta} \\ \tilde{\Delta}, & i\tilde{\omega} + \xi_{\mathbf{p}} \end{vmatrix}. \quad (9)$$

Here  $\xi_{\mathbf{p}}$  is the electron energy, measured from the Fermi level, and satisfying two conditions:  $\xi_{\mathbf{p}} = \xi_{-\mathbf{p}}$ —parity;  $\xi_{\mathbf{p}+\mathbf{Q}} = -\xi_{\mathbf{p}}$ —full nesting. The functions  $\tilde{\omega}$  and  $\tilde{\Delta}$  satisfy the self-consistency conditions<sup>9</sup>

$$\begin{aligned} i\tilde{\omega} &= i\omega_n + \frac{i\tilde{\omega}}{2\tau_p(\tilde{\omega}^2 + \tilde{\Delta}^2)^{1/2}}, \\ \tilde{\Delta} &= \Delta - \frac{\tilde{\Delta}}{2\tau_p(\tilde{\omega}^2 + \tilde{\Delta}^2)^{1/2}}, \end{aligned} \quad (10)$$

where  $\tau_p$  is the relaxation time on the impurity centers,  $\Delta$  is a constant quantity proportional to the antiferromagnetic order parameter, and  $\omega_n = (2n + 1)\pi T$ . In this case the parameter  $\tau$  characterizing the proximity to the  $M$ - $D$  transition point and entering into the relations (3) and (4) is given by the difference  $\tau = 1 - 1/\tau_p \Delta$ . The quantity  $\tilde{\omega}$ , as a function of  $\omega$  in the vicinity of the  $M$ - $D$  transition point, satisfies the same equation that the spontaneous moment as a function of the magnetic field satisfies:

$$\tau\tilde{\omega} + \tilde{\omega}^3 = \omega. \quad (11)$$

Equation (11), which is in dimensionless variables, is the general self-consistency equation characterizing the  $M$ - $D$  transition in the zeroth-order approximation of the self-consistent field. In this approximation the four-current correlators are computed with the aid of Fig. 1, in which we associate with each line the Green function (9) and with each cross an electron momentum  $\mathbf{p}$  multiplied by the Pauli ma-

trix  $\tau^2$ . Any diagram with a dotted line encompassing a vector vertex is, for simplicity, assumed to be equal to zero, it corresponding to a  $\delta$ -function short-range potential.

As in the theory of second-order phase transitions, we go beyond the self-consistent field method by taking into account the longitudinal and transverse order-parameter fluctuations. Corresponding to the transverse fluctuations in the model in question are the impurity-averaged pairs of electron lines with frequencies of opposite signs: diffusons in the case of momenta having the same direction and almost equal in magnitude and cooperons in the case of almost oppositely-directed electron momenta with different magnitudes. The corresponding correlator has a singularity of the diffusional type<sup>4</sup>:

$$K_g(q) = (|\omega| + Dq^2)^{-1}. \quad (12)$$

Here  $\omega = \omega_1 + \omega_2$ ,  $q^2 = (\mathbf{p}_1 \pm \mathbf{p}_2)^2$  and  $D$  is the diffusion coefficient.

Corresponding to the longitudinal fluctuations are the impurity-averaged pairs of electron lines with low frequencies of the same sign both in the case of a small momentum transfer and in the case of a small total momentum. According to Ref. 5, the corresponding correlator has a singularity of the Ornstein-Zernike type:

$$K_c(q) = (|\tau| + R_0^2 q^2)^{-1}. \quad (13)$$

Here  $\tau$  is the dimensionless parameter characterizing the proximity to the  $M$ - $D$  transition point:  $|\tau| \ll 1$ . According to Ref. 10, the correlation length  $R_0$  vanishes at points far from the  $M$ - $D$  transition point. Near the transition point in the low-frequency region (4)

$$D \propto R_0^2 |\tau|^{1/2}, \quad (14)$$

so that the diffusional correlator has a stronger singularity for the same momenta  $\mathbf{q}$ .

In the high-frequency region (3) the dimensionless quantity  $|\tau|^{1/2}$  in (13) and (14) should be replaced by the metal order parameter  $\bar{\omega}$ , which, according to (10), has in this region the order of magnitude

$$\bar{\omega} \propto \omega^{1/2}. \quad (15)$$

In this case the diffusional and critical correlators have the same form:

$$K_c(\mathbf{q}) \propto (|\omega|^{1/2} + R_0^2 \mathbf{q}^2)^{-1}. \quad (16)$$

Everywhere below we set  $R_0 = 1$  and  $\hbar = 1$ , so that all the quantities having the dimensions of frequency or energy, namely,  $\omega$  and  $\bar{\omega}$ , are assumed to be measured in units of the reciprocal characteristic relaxation time  $1/\tau_p$ , while the quantities having the dimensions of momentum are nondimensionalized by the quantity  $1/R_0$ . Let us, for the purpose of writing down the renormalizing-group equations, consider the diagrams for the four-current correlators, which, on the whole, have the form of a closed quadrangle, and are connected by a certain number of electron-line pairs, each of which gives either a critical or a diffusional long-range correlator. It is clear that an electron-line pair forming a ladder with zero momentum transfer or zero total momentum cannot terminate at a current vertex, since the first ladder dotted line adjoining it will always give a null result in the integration over the momenta. For this reason, the electron-line pairs can end only at the so-called scalar vertices, i.e., those from which the electron lines necessarily diverge and go to different vertices: current or scalar. The direct computation of the diagrams connected by a single pair of electron lines in the end yields a nonsingular correction because of the smallness of the triangular diagrams. For the relevant calculations, see the Appendix in Ref. 10.

In the next approximation, the four-current correlators split up into products of two-current four-electron vertices connected by two pairs of electrons lines, which, after being averaged, furnish correlators of the diffusional and critical types.

Discarding the diagrams containing closed loops, we obtain the nine diagrams shown in Fig. 2. As can be seen from the figure, the four-current vertices can be expressed in terms of two types of two-current correlators:  $\Gamma^{(b)}(1,2,3,4)$  and  $\tilde{\Gamma}^{(b)}(1,2,3,4)$  (see Figs. 3a and 3b). The  $\tilde{\Gamma}^{(b)}(1,2,3,4)$  vertex has current vertices at opposite corners. By definition, the first line enters into a current vertex, but at the next scalar corner the electron momentum changes its direction. At the  $\Gamma^{(b)}(1,2,3,4)$  vertex the line with the number 1 connects two neighboring current vertices, and the line with the number 3 goes around along the electron line through the current vertices in the negative direction. Below we shall also need the vertices  $\tilde{\Gamma}^{(a)}(1,2,3,4)$  and  $\Gamma^{(a)}(1,2,3,4)$  depicted in Figs. 3c and 3d, and differing from  $\tilde{\Gamma}^{(b)}(1,2,3,4)$

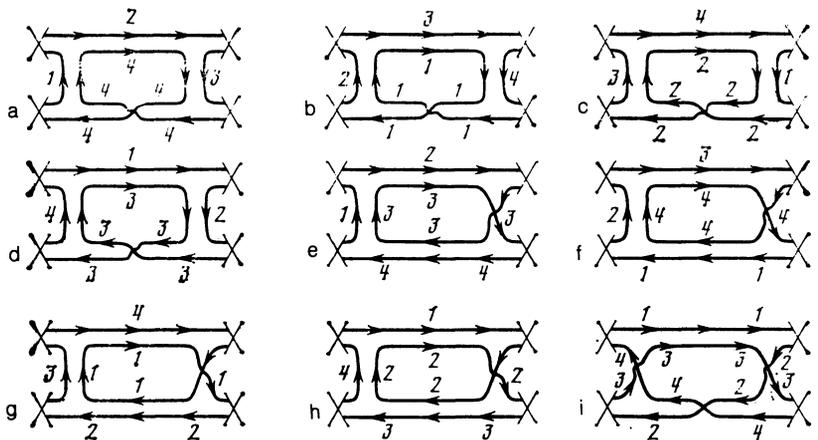


FIG. 2. The relations between the four-current and two-current vertices, as written in second order perturbation theory.

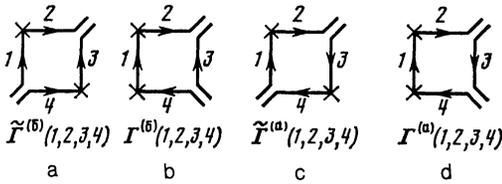


FIG. 3. All the possible two-current vertices, as written in zeroth-order perturbation theory.

and  $\Gamma^{(b)}(1,2,3,4)$  by the fact that the momentum has the "correct" direction for the given direction of traversal. Two symmetry relations follow immediately from the definition:

$$\begin{aligned} \Gamma^{(a)}(1, 2, 3, 4) &= \Gamma^{(a)}(3, 4, 1, 2); \\ \Gamma^{(b)}(1, 2, 3, 4) &= \Gamma^{(b)}(4, 3, 2, 1). \end{aligned} \quad (17)$$

Let us assume that the external frequencies are equal to zero, and we are on the  $M$ - $D$  transition line ( $\tau = 0$ ). Then it is convenient to introduce in the four-dimensional space the logarithmic variable

$$t = \ln(p_m/q), \quad (18)$$

where  $p_m$  is the maximum cutoff momentum and  $q$  is one of the external cutoff momenta, which, by definition, have the same order of magnitude ( $q \ll p_m$ ). Under these assumptions, the renormalization-group equations coincide with the parquet equations, which can be rederived without any difficulty with the aid of second-order perturbation theory:

$$\begin{aligned} & -\frac{\partial}{\partial t} \Gamma_j(+ - - -) \\ &= 2(\Gamma^{(b)}(+ - - -))^2 + 2(\Gamma^{(b)}(- + + +))^2 \\ &+ 2\{\Gamma^{(b)}(+ - + -)\Gamma^{(b)}(+ - + -)\} \\ &+ 2\{\Gamma^{(b)}(- + - +)\Gamma^{(b)}(- + - +)\} \\ &+ \{\Gamma^{(b)}(- + + -)\Gamma^{(b)}(+ - - +)\}. \end{aligned} \quad (19)$$

Here and below the products of the coefficients derived from a pair of critical correlators are denoted by round brackets; the product of the coefficients from two correlators of the diffusional type give a factor, which is represented by curly brackets.

The four-current vertices, which determine the fluctuations (6) in the case of a prescribed current, contain two or three lines with frequencies of the same sign. In this case the greater part of the diagrams is determined by the product of the critical and Cooper correlators, which is indicated by the square brackets:

$$\begin{aligned} & -\frac{\partial}{\partial t} \Gamma_j(+ + - -) = [\Gamma^{(b)}(+ + - -)\Gamma^{(b)}(- - - +)] \\ &+ [\Gamma^{(b)}(- + - -)\Gamma^{(b)}(+ - - +)] \\ &+ [\Gamma^{(b)}(+ + - -)\Gamma^{(b)}(- - - +)] \\ &+ [\Gamma^{(b)}(+ - - +)\Gamma^{(b)}(- + - -)] + [+ \neq -] \\ &+ \{\Gamma^{(b)}(+ + - -)\Gamma^{(b)}(- + - +)\}, \quad (20) \\ & -\frac{\partial}{\partial t} \Gamma_j(+ + + -) = [\Gamma^{(b)}(+ + - -)\Gamma^{(b)}(+ - - +)] \\ &+ (\Gamma^{(b)}(- + + +)\Gamma^{(b)}(+ + + +)) \end{aligned}$$

$$\begin{aligned} & + [\Gamma^{(b)}(+ - + +)\Gamma^{(b)}(+ + + -)] \\ &+ (\Gamma^{(b)}(- + + +)\Gamma^{(b)}(+ + + +)) \\ &+ [\Gamma^{(b)}(+ + + -)\Gamma^{(b)}(+ - + +)] \\ &+ (\Gamma^{(b)}(- + + +)\Gamma^{(b)}(+ + + +)) \\ &+ [\Gamma^{(b)}(+ - + +)\Gamma^{(b)}(+ + + -)] \\ &+ \{\Gamma^{(b)}(+ + - +)\Gamma^{(b)}(+ - + -)\} \\ &+ [\Gamma^{(b)}(+ + - +)\Gamma^{(b)}(- + + +)]. \end{aligned} \quad (21)$$

It is clear that at points far from the transition point (in the metallic phase) the critical vibrations do not make any contribution, so that we should retain in the relations (19)–(21) only the terms enclosed in the curly brackets. Then in four-dimensional space

$$t = \ln(p_m(D/|\omega|)^{1/2}). \quad (22)$$

Exactly the same situation obtains in the low-frequency region (4). In this region the cutoff, which depends on the frequency, arises only from the two correlators of the diffusional type (since, according to (14),  $|\tau| \gg \omega/D$ ); therefore, the terms that remain when the differentiation with respect to the variable (22) (with a given diffusion coefficient (14)) is performed are those enclosed in the curly brackets.

Near the  $M$ - $D$  transition point in the high-frequency region (3) all the singular correlators have one and the same form (16). Therefore, all the coefficients in the relations (19)–(21) are equal, and the variable  $t$  depends on the frequency:

$$t = \ln(p_m|\omega|^{-1/2}). \quad (23)$$

Just as the four-current vertices can be expressed in terms of products of the two-current vertices so the latter can be expressed in terms of the products of themselves and the four-electron scalar vertices. The corresponding system of equations is derived in Ref. 10, so that here we limit ourselves to their graphical derivation. The equation for the two-current vertex  $\tilde{\Gamma}^{(a)}$  can be obtained from Fig. 2i if the right current vertices are replaced by scalar vertices. The equation for the vertex  $\tilde{\Gamma}^{(b)}$  has a similar form if the directions of the third and fourth lines in the diagrams for  $\tilde{\Gamma}^{(a)}$  are reversed:

$$\begin{aligned} -\dot{\tilde{\Gamma}}^{(a)}(1, 2, 4, 3) &= \Gamma^{(b)}(1, 2, 3, 4)\Gamma_3(2, 4, 1, 3), \\ -\Gamma^{(b)}(1, 2, 3, 4) &= \tilde{\Gamma}^{(a)}(1, 2, 4, 3)\Gamma_3(1, 3, 2, 4). \end{aligned} \quad (24)$$

Here the dot denotes differentiation with respect to the variable (18) and the definition of the scalar vertex  $\Gamma_3$  is clear from Fig. 4c. The equations for  $\Gamma^{(a)}$  can be obtained by discarding the two right current vertices in the diagrams in

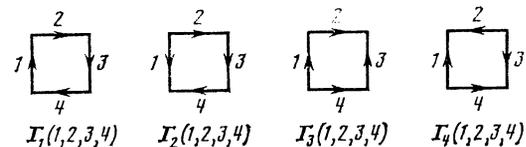


FIG. 4. All the possible scalar vertices, as written in the zeroth-order approximation.

Figs. 2a, 2b, 2f, and 2g and replacing them by scalar ones. The equations for  $\Gamma^{(b)}$  are obtained from the same diagrams by reversing the direction of the electron line connecting the neighboring scalar vertices:

$$\begin{aligned} -\dot{\Gamma}^{(a)}(1, 2, 3, 4) &= \Gamma^{(b)}(1, 2, 2, 4) \Gamma_2(2, 2, 3, 4) \\ &\quad + \Gamma^{(b)}(1, 2, 4, 4) \Gamma_2(4, 2, 3, 4) \\ &\quad + \Gamma^{(b)}(1, 2, 3, 4) \Gamma_3(2, 3, 4, 3) \\ + \Gamma^{(b)}(1, 2, 1, 4) \Gamma_2(1, 2, 3, 4); &\quad (25) \\ -\dot{\Gamma}^{(b)}(1, 2, 3, 4) &= \Gamma^{(b)}(1, 2, 2, 4) \Gamma_4(2, 3, 4, 2) \\ &\quad + \Gamma^{(b)}(1, 2, 4, 4) \Gamma_4(2, 3, 4, 4) \\ &\quad + \Gamma^{(a)}(1, 2, 3, 4) \Gamma_3(3, 2, 3, 4) \\ &\quad + \Gamma^{(b)}(1, 2, 1, 4) \Gamma_4(2, 3, 4, 1). \end{aligned}$$

Thus, the two-current vertex parts satisfy a system of linear equations in which the scalar vertices act as nonlinear sources of metal-order-parameter fluctuations. The equations (24) are independent of (25), so that the vertex parts  $\tilde{\Gamma}^{(a,b)}$  play the role of linear sources for the vertex parts  $\Gamma^{(a)}$  and  $\Gamma^{(b)}$ . The corresponding homogeneous system of equations of  $\Gamma^{(b)}$  and  $\Gamma^{(a)}$  has an independent subsystem in which the first index  $\omega$  is arbitrary and all the rest are identical (+ or -):

$$\begin{aligned} -\dot{\Gamma}^{(a)}(\omega, +++) &= 2(\Gamma^{(b)}(\omega, +++) \Gamma_2) + (\Gamma^{(b)}(\omega, +++) \Gamma_3), & (26) \\ -\dot{\Gamma}^{(b)}(\omega, +++) &= 2(\Gamma^{(b)}(\omega, +++) \Gamma_4) + (\Gamma^{(a)}(\omega, +++) \Gamma_3). \end{aligned}$$

The round brackets denote a cutoff at the value

$$t = \ln(p_m / \max(|\tau|^{1/2}, |\omega|^{1/2})), \quad (27)$$

$$\Gamma_k \equiv \Gamma_k(++++) = \Gamma_k(----).$$

If the second and fourth indices differ in sign, then the independent system is the system of equation for  $\Gamma^{(a,b)}(\omega, 2, 2, 4)$  and  $\Gamma^{(a,b)}(\omega, 2, 4, 4)$ . The corresponding system of four homogeneous equations has a cutoff at the value (27), and is investigated in the Appendix. But if the second and fourth indices coincide and the third differs from them in sign, then all the above-listed systems are independent of the following system:

$$\begin{aligned} -\dot{\Gamma}^{(a)}(\omega, +-+) &= \{\Gamma^{(b)}(\omega, +-+) \Gamma_3(+--+)\}, & (28) \\ -\dot{\Gamma}^{(b)}(\omega, +-+) &= \{\Gamma^{(a)}(\omega, +-+) \Gamma_3(-+--)\}. \end{aligned}$$

The curly brackets denote the product of the diffusional correlators, which, at low frequencies, implies differentiation with respect to the variable (22).

It must be noted that quite recently a similar problem (in the far-metallic region) was considered within the framework of perturbation theory. The first four nonvanishing—in the logarithmic approximation—diagrams for the four-current vertex parts, which in the present paper are shown in Figs. 2e, 2f, 2g, and 2h, were considered. Neglecting the logarithmic corrections to the two-current and scalar four-electron vertices, Kirkpatrick and Dorfman<sup>11</sup> obtain (in the same  $\varepsilon$  approximation) a slightly smaller value for the critical exponent  $\alpha$  ( $\alpha = \varepsilon/2$  instead of  $\alpha = 5/8$ ).

### 3. COMPUTATION OF THE EXPONENT $\alpha$

Let us consider the solutions to the equations (24) and (25) in the asymptotic region  $t \gg 1$ . Using the scale invariance of the nonlinear equations for the scalar vertices, we shall seek their asymptotic solution in the form

$$\Gamma_k(1, 2, 3, 4) = \gamma_k(1, 2, 3, 4)/t. \quad (29)$$

By substituting (29) and going over to the variable  $\ln t$ , we reduce the equations for the two-current vertices to a system of linear equations with constant coefficients  $\gamma_k(1, 2, 3, 4)$ , in terms of which we find all the eigenvalues.

Near the transition point and in the region of high frequencies (3), where all the correlators have one and the same form (16), and the logarithmic integrals are truncated at one and the same value (17), the coefficients  $\gamma$  can be considered to be known from Ref. 10:

$$\begin{aligned} \gamma_1(1, 2, 3, 4) &= \gamma_2(1, 2, 3, 4) = -\gamma_3(1, 2, 3, 4) \\ &= \gamma_4(1, 2, 3, 4) = \gamma(1, 2, 3, 4); \\ \gamma(++++), \dots, \gamma(+++-), \dots, -\gamma(++--), \\ &\dots, \gamma(+--+), \dots = 1. \end{aligned} \quad (30)$$

The dots denote either cyclic permutation of the indices, or the reversal of the sign of each of the frequencies.

The substitution of (29) and (30) into the system (26) shows that this system does not possess growing solutions. Each of the remaining independent subsystems (24), (28), and (A.1) necessarily has a linearly increasing solution. For (24) and (28) this follows directly from the symmetry relation  $\Gamma_3(1, 3, 2, 4) = \Gamma_3(2, 4, 1, 3)$ . For the remaining cases, see the Appendix.

Substitution of the growing solutions into the relations (19)–(21) leads in all the three cases to the growth of the four-current correlators according to the law

$$t^3 = [\ln(|\omega|^{-1/2})]^3,$$

which in  $(4 - \varepsilon)$ -dimensional space furnishes the value of the coefficient

$$\alpha = \varepsilon. \quad (31)$$

(the  $1/f$  law). We should, of course, remember that this result is valid in the narrow frequency range (3) and in the vicinity of the  $M$ - $D$  transition point, where all the components of the metal order parameter fluctuate. In the lowest frequency region (4) and in the vicinity of the  $M$ - $D$  transition point, and also at frequencies  $\omega \ll 1/\tau_p$  and in the metallic-phase region far from the transition point, to bring out the frequency dependence, we need only take account of those terms in the relations (19)–(21) which derive from the diffusional propagators: the cooperons and diffusons (enclosed in the curly brackets). The variable  $t$  then depends on the frequency (in accordance with the relation (22)), and the two-current vertices  $\tilde{\Gamma}^{(b)}$  and  $\Gamma^{(b)}$  can be expressed in terms of the scalar vertices  $\Gamma_k$  with frequencies having alternating signs. Those corrections to these vertices which are determined by the product of correlators of the diffusional type derive from diagrams of the types shown in Figs. 2e, 2f, 2g, and 2h. Replacing the current vertices in them by scalar ones, we obtain the following equations:

$$\begin{aligned}
-D_1 &= 4D_2D_3, & -D_2 &= D_3(D_1 + 2D_2 + D_4), \\
-D_3 &= 4D_4^2, & -D_4 &= 4D_2D_3.
\end{aligned}
\tag{32}$$

Here

$$D_k = \Gamma_k(+--+)=\Gamma_k(-+-+), \quad t = \ln(p_m(D/|\omega|)^{1/2}).$$

This system has the power-function solution (29) with the coefficients

$$\begin{aligned}
\gamma_3(+--+)&=1/4, \\
\gamma_1(+--+)&=\gamma_2(+--+)=\gamma_4(+--+)=\pm 1/4.
\end{aligned}
\tag{33}$$

Substituting these coefficients into Eqs. (24) and (28), we obtain the two-current vertices in the region  $t \gg 1$ , and for any  $\omega = 0 \pm$

$$\begin{aligned}
\Gamma^{(b)}(\omega, +++) &= \Gamma^{(b)}(\omega, -+-) \propto t^{\mu}, \\
\Gamma^{(b)}(++--) &= \Gamma^{(b)}(--++) \propto t^{\mu}.
\end{aligned}
\tag{34}$$

The vertex  $\tilde{\Gamma}^{(b)}(+ - + -)$  turns out to be frequency independent in the low-frequency region of interest to us, since the equation for this vertex does not contain diffusional correlators.

Substitution of the asymptotic expansions (34) into the relations (24), (28), (19) and (20), (21) leads to the following dependences in  $(4 - \varepsilon)$ -dimensional space:

$$\Gamma_j(+--+)\sim\Gamma_j(++++)\sim\Gamma_j(++--) \propto t^{1/4} \rightarrow \omega^{-5\varepsilon/8}, \tag{35}$$

whence  $\alpha = 5\varepsilon/8$ . The formulas (31) and (35) solve the formulated problem in the linear approximation in  $\varepsilon$ .

#### 4. DISCUSSION OF THE RESULTS

It is not difficult to see that the cause of the singularity (2) in the low-frequency region (4) is the interaction of two or more diffusional modes arising from the Cooper channel. This fact can be seen directly from Fig. 2, where each of the nine singular diagrams has two pairs of electron lines with momenta having identical directions. From this we conclude that the proposed mechanism of  $1/f$  noise is shut off in the region of frequencies  $\omega < 1/\tau_\varphi$  at which the Cooper singularity is cut off. As has already been noted, the most severe limitation arises as a result of the scattering on the paramagnetic impurities,<sup>12</sup> when the Cooper singularity and, hence, the singularity (2) are cut off at frequencies of the order of the reciprocal spin-flip relaxation time  $1/\tau_s$ . In this case  $\tau_\varphi \sim \tau_s$  and not less than four orders of magnitude greater than  $\tau_p$ , so that, according to (1), the region of applicability of the relation (2) with  $\alpha = 5/8$  is fairly broad, but is bounded from below by frequencies of the order of  $1/\tau_s$ .

A magnetic field also gives rise to the cutoff effect. According to Ref. 3, for mean free paths of the order of the average distance between the electrons,  $1/\tau_\varphi \sim eH/mc \sim 2 \times 10^7 \text{H (Oe-sec)}^{-1}$ . The effect of the magnetic field turns out to be especially important at  $H > mc/e\tau_s$ , beginning from which low-frequency "nonwhite" noise can be suppressed only by a magnetic field.

The effect whereby the exponent  $\alpha$  increases in the high-frequency region (3), but not at points too close to the  $M$ - $D$  transition point is quite interesting. The latter limita-

tion is due to an increase in the spin-flip scattering in the vicinity of the transition point. The problem of the computation of the quantity  $\tau_\varphi$ , which enters into the inequalities (3) and (4), requires a special investigation.

It is important to note that the above-obtained results are valid in the low-temperature region, where the long-range metal-order-parameter fluctuations are the only source of fluctuations. At finite temperatures and in the frequency region  $\omega < 1/\tau_\varepsilon$  the metal-order-parameter fluctuations are nonlinearity coupled to the temperature fluctuations, and this could lead to an increase in the critical exponent  $\alpha$ .

#### APPENDIX

Assuming that, in the equations (25),  $\omega_2 = \omega_3 \neq \omega_4$ , and then assuming that  $\omega_2 \neq \omega_3 = \omega_4$ , and discarding the last terms with the "external force"  $\tilde{\Gamma}$ , we obtain

$$\begin{aligned}
-\dot{\Gamma}^{(a)}(1, 2, 2, 4) &= \Gamma^{(b)}(1, 2, 2, 4)(B_2 + B_3) \\
&\quad + \Gamma^{(b)}(1, 2, 4, 4)C_2, \\
-\dot{\Gamma}^{(b)}(1, 2, 2, 4) &= \Gamma^{(b)}(1, 2, 2, 4)B_4 + \Gamma^{(b)}(1, 2, 4, 4)C_4 \\
&\quad + \Gamma^{(a)}(1, 2, 2, 4)B_3, \\
-\dot{\Gamma}^{(a)}(1, 2, 4, 4) &= \Gamma^{(b)}(1, 2, 2, 4)C_2 \\
&\quad + \Gamma^{(b)}(1, 2, 4, 4)(B_2 + B_3), \\
-\dot{\Gamma}^{(b)}(1, 2, 4, 4) &= \Gamma^{(b)}(1, 2, 2, 4)C_4 + \Gamma^{(b)}(1, 2, 4, 4)B_4 \\
&\quad + \Gamma^{(a)}(1, 2, 4, 4)B_3.
\end{aligned}
\tag{A.1}$$

Here we have introduced the convenient notation (i.e., one that takes account of the symmetry relations, obtained by the present author in Ref. 10, for the scalar vertices in the case when  $\omega_2 \neq \omega_4$ ):

$$\begin{aligned}
B_2 &= \Gamma_2(2, 2, 2, 4) = \Gamma_2(4, 2, 4, 4), \\
B_3 &= \Gamma_3(2, 2, 2, 4) = \Gamma_3(2, 2, 4, 2) = \Gamma_3(2, 4, 4, 4) \\
&\quad = \Gamma_3(4, 2, 4, 4), \\
B_4 &= \Gamma_4(2, 4, 4, 4) = \Gamma_4(2, 2, 4, 2), \\
C_2 &= \Gamma_2(2, 2, 4, 4) = \Gamma_2(4, 2, 2, 4), \\
C_4 &= \Gamma_4(2, 2, 4, 4) = \Gamma_4(2, 4, 4, 2).
\end{aligned}
\tag{A.2}$$

It is easy to verify that all the terms on the right-hand side (A.1) derive from the product of one critical and one diffusional correlator; therefore, in the case of zero external momenta the dot denotes differentiation with respect to the variable

$$t = \ln[p_m/\max(|\tau|^{1/2}, |\omega|^{1/2})]. \tag{A.3}$$

The system (A.1) admits of a symmetric and an antisymmetric solution  $Z_\alpha^{(\pm)} = \Gamma^{(a)}(1, 2, 2, 4) \pm \Gamma^{(a)}(1, 2, 4, 4)$ , so that, after setting  $B_k = b_k/t$  and  $C_k = c_k/t$ , we find without any difficulty the asymptotic form of the solutions in the region  $t \gg 1$ :

$$Z_\alpha^\pm = t^{\pm\alpha},$$

where

$$\lambda_\pm = -\frac{b_4 \pm c_4}{2} \pm \left[ \frac{b_4 \pm c_4}{2} + b_3(b_3 + b_2 \pm c_2) \right]^{1/2}. \tag{A.4}$$

Near the  $M$ - $D$  transition point, the coefficients  $b_k$  and  $c_k$  can be determined, using the results obtained in Ref. 10:

$$b_2 = -b_3 = b_4 = -c_2 = -c_4 = 1. \quad (\text{A.5})$$

Here the maximum value  $\lambda = 1$  corresponds to the completely symmetric solution  $Z_a^{(+)} = Z_b^{(+)} \propto t$  to the system (A.1). The remaining solutions, which belong to the eigenvalue  $\lambda = 1$ , furnish damped solutions.

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